Chapter 14

Solutions to Exercises

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14.1 SOLUTIONS

Exercise 14.1. Compute the genus and number of boundary components of the diagrams in Figure S14.1.

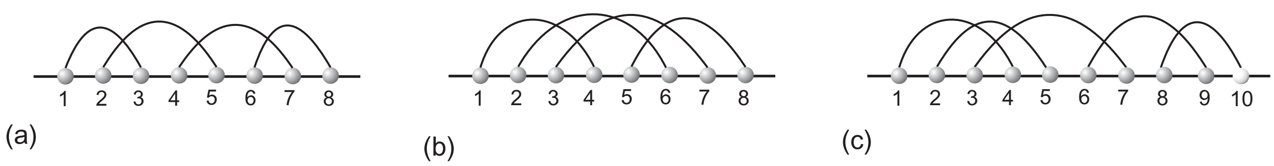

FIGURE S14.1 Exercise 14.1.

Solution.

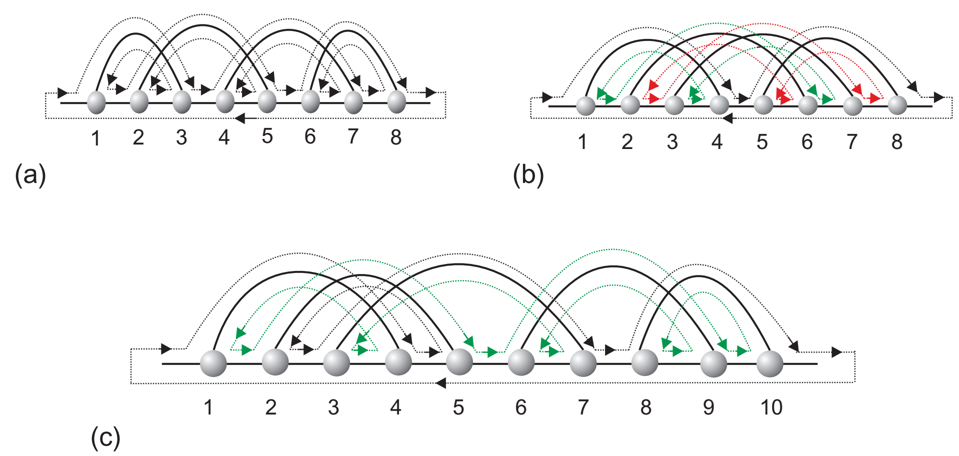

FIGURE S14.2 The boundary components.

(a) Figure S14.2 shows that (a) has only one boundary component and $n = 4$ arcs. We have $2 - 2g - r = 1 - n$, that is, $g = (n - r + 1)/2 = 2$.
(b) (b) has 3 boundary components and $n = 4$ arcs, so $g = (4 - 3 + 1)/2 = 1$.
(c) (c) has 2 boundary components and $n = 5$ arcs, so $g = (5 - 2 + 1)/2 = 2$. □

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Exercise 14.2. Compute the Poincaré dual of the fatgraphs in Figure S14.3.

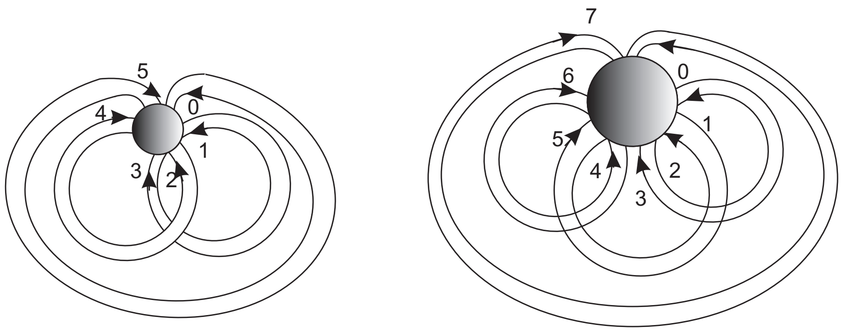

FIGURE S14.3 Exercise 14.2.

Solution. For the fatgraph on the left-hand side, we have $\sigma = (0, 1, 2, 3, 4, 5)$, $\alpha = (0, 5)(1, 3)(2, 4)$. Thus, its Poincaré dual has $\sigma' = \alpha \circ \sigma = (5)(0, 3, 2, 1, 4)$ and $\alpha' = \alpha$. The dual is displayed in Figure S14.4 (left-hand side).

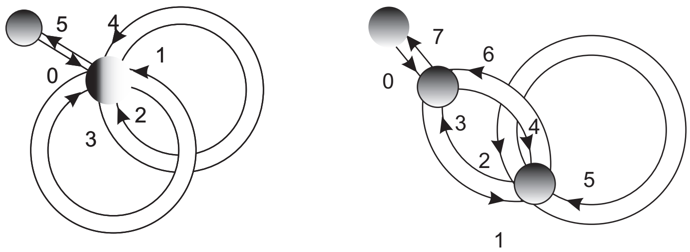

FIGURE S14.4 The duals.

For the fatgraph on the right-hand side, we have $\sigma = (0, 1, 2, 3, 4, 5, 6, 7)$, $\alpha = (0, 7)(1, 3)(2, 5)(4, 6)$. Its Poincaré dual has $\sigma' = \alpha \circ \sigma = (7)(3, 6)(1, 5, 4, 2)$ and $\alpha' = \alpha$. The dual is displayed in Figure S14.4 (right-hand side). □

Exercise 14.3. Find all trisections in the fatgraphs displayed in Figure S14.5.

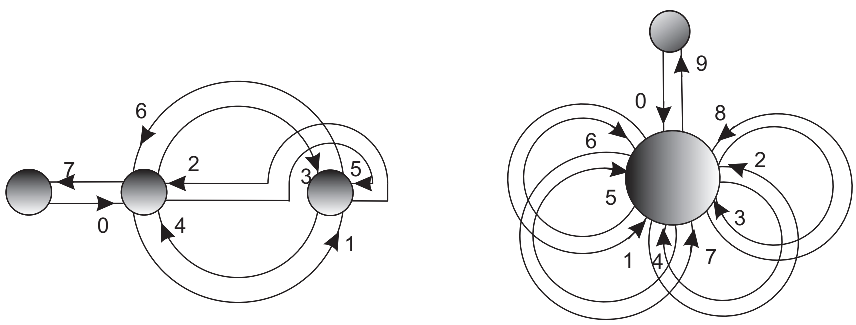

FIGURE S14.5 Exercise 14.3.

□

Solution. The trisections are displayed in Figure S14.6.

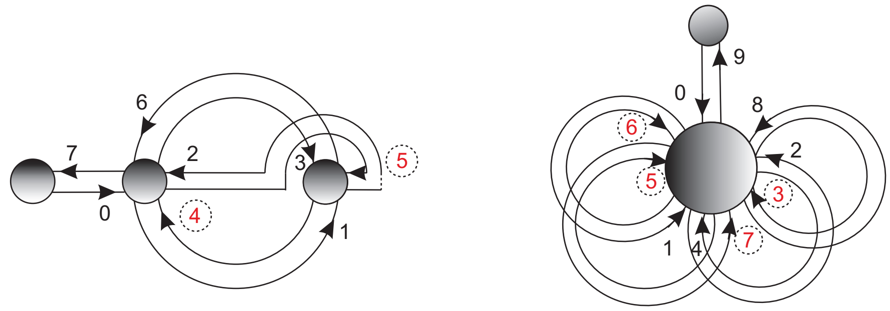

FIGURE S14.6 The trisections.

Exercise 14.4. Show
Corollary 14.1. *Let* τ *denote a shape-preserved sector and* $\mathfrak{m} \in \mathbb{S}^{(k)}_{0,n}$. *Then*

$$\rho\colon (\mathfrak{m}, \tau) \to (\mathfrak{m}', v)$$

is a bijection, where v *is an unlabeled vertex in* $\mathfrak{m}'$ *and* $\mathfrak{m}' \in \mathbb{S}^{(k)}_{0,n+1}$. *In particular, we have*

$$(2k - n - 2)\eta_0(n, k) = (n + 1 - k)\eta_0(n + 1, k)$$

$$\eta_0(n, k) = \binom{2k - (k-1) - 2}{n + 1 - k}\eta_0(k - 1, k) = \binom{k-1}{n+1-k}\text{Cat}(k - 1).$$

□

Proof: The corollary follows by restriction of Rémy's bijection. This bijection implies that the removal of an unlabeled vertex of an $\mathbb{S}^{(k)}_{0,n}$-tree produces a shape-sector. Furthermore, the order of such removals is irrelevant. Therefore, an $\mathbb{S}^{(k)}_{0,n}$-tree can be constructed from an $\mathbb{S}^{(k)}_{0,m-1}$-tree together with $(2k - n + 1)$ shape-preserved sectors. Clearly, the number of $\mathbb{S}^{(k)}_{0,k-1}$-trees equals $\text{Cat}(k - 1)$, where $\text{Cat}(n)$ is the nth Catalan number given by $\frac{1}{n+1}\binom{2n}{n}$. To obtain an $\mathbb{S}^{(k)}_{0,n}$-tree, we need to insert $n + 1 - k$ unlabeled vertices. Choosing $n + 1 - k$ out of $2k - (k - 1) - 2$ shape-preserved sectors from $\mathbb{S}^{(k)}_{0,k-1}$, we derive

$$\eta_0(n, k) = \binom{2k - (k-1) - 2}{n + 1 - k}\eta_0(k - 1, k) = \binom{k-1}{n+1-k}\text{Cat}(k - 1),$$

whence the corollary. □

Exercise 14.5. Show

$$s_g(n) = \sum_{t=1}^{g} \kappa_t^{(g)} \binom{2g + t - 1}{n - (2g + t - 1)}, \tag{S14.1}$$

where $\binom{n}{k} = 0$ if $k < 0$ or $k > n$. □

Proof: We have

$$\sum_{n=2g}^{6g-2} s_g(n) z^n = \sum_{t=1}^{g} \kappa_t^{(g)} z^{2g+t-1}(1 + z)^{2g+t-1} = \sum_{t=1}^{g} \sum_{i=0}^{2g+t-1} \kappa_t^{(g)} \binom{2g + t - 1}{i} z^{2g+t-1+i}.$$

Set $n = 2g + t - 1 + i$. By comparing both sides of the above identity, we obtain the corresponding formula for $s_g(n)$. □

Exercise 14.6. Show that the number $\kappa_t^{(g)}$ is a positive integer.

Proof: The positivity of $\kappa_t^{(g)}$ is clear by definition. We proceed by induction on t: assume that $\kappa_j^{(g)}$ is an integer for $j < t$ and set $n = 2g + t - 1$. By Equation (S14.1), we have

$$s_g(2g + t - 1) = \sum_{j=1}^{t} \kappa_j^{(g)} \binom{2g + j - 1}{t - j},$$

that is,

$$\kappa_t^{(g)} = s_g(2g + t - 1) - \sum_{j=1}^{t-1} \kappa_j^{(g)} \binom{2g + j - 1}{t - j}.$$

Because $s_g(2g + t - 1)$ and $\kappa_j^{(g)}$ are integers for $j < t$, $\kappa_t^{(g)}$ is an integer. □