1.02 Theory and Observations – Earth's Free Oscillations

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1.02.1 Introduction

The study of the Earth's free oscillations is fundamental to seismology, as it is a key part of the theory of the Earth's dynamic response to external or internal forces. Essentially, the same theory is applicable to phenomena as diverse as postseismic relaxation, analysis of seismic surface waves and body waves. The study of free oscillations *per se* is concerned with analyzing and extracting information at very long periods (∼3000–200 s period) since in this range of periods the intrinsic standing wave modes of oscillation are evident in seismic spectra. Such spectra contain important information about the large-scale structure of the Earth. For example, the strongest evidence that the inner core is solid (Dziewonski and Gilbert, 1971) and anisotropic (Woodhouse *et al.*, 1986; Tromp, 1993; Romanowicz and Breger, 2000) comes from the study of free oscillations. Free oscillations provide essential constraints on both the spherically symmetric 'average' Earth, and also on lateral variations in Earth structure due to heterogeneity in temperature, composition, and anisotropy. Modal data are particularly valuable in this regard because, unlike other kinds of seismic data, modal observables depend upon broad averages of the Earth's structural parameters, and are not nearly so affected by limitations of data coverage due to the uneven distribution of seismic events and stations. There is an enormous wealth of information yet to be extracted from long period spectra; one has only to examine almost any portion of a seismic spectrum in detail to realize that current models often do not come close to providing adequate predictions. It is only from such very long period data that it may be possible to obtain direct information on the three-dimensional distribution of density. Even very large scale information on lateral variations in density has the potential to bring unique information to the study of convection and thermal and compositional evolution.

Very long period spectra are also an essential element in the study of earthquakes, as it is only by using data at the longest periods that it is possible to determine the overall moment of very large events. For example, estimates of the moment of the great Sumatra earthquake based on mantle waves, even at periods of several hundred seconds, significantly underestimate the true moment, as the length and duration of the rupture make it possible to gauge the true, integrated moment only by using data at the longest seismic periods (e.g. Park *et al.*, 2005).

Figure 1 shows an example of data and theoretical amplitude spectra computed for the spherically symmetric PREM model (Dziewonski and Anderson, 1981). Modes appear as distinct peaks in the frequency domain. For higher frequencies, the modes are more closely spaced and begin to overlap. The theoretical peaks appear at frequencies very close to the observed peaks. However, the observed peaks are distorted in shape and amplitude due to three-dimensional effects. For example, mode ${}_1S_4$ is split

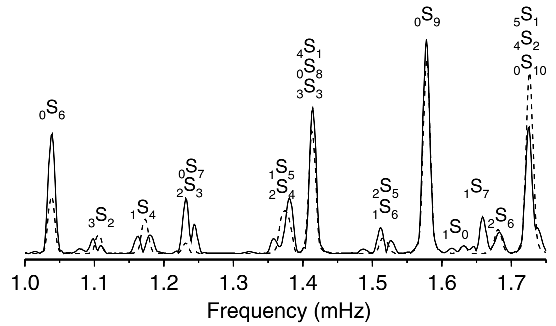

Figure 1 Data (solid line) and PREM synthetic spectrum (dashed line) computed using normal mode summation, for the vertical component recording at station ANMO following the great Sumatra event of 26 December 2004.

into two peaks in the data spectrum, which is not seen in the theoretical spectrum. Not shown here, but equally important in studying both Earth structure and earthquakes, is the 'phase spectrum'. Examples illustrating this are shown in later sections.

Normal mode studies represent the quest to reveal and understand the Earth's intrinsic vibrational spectrum. However, this is a difficult quest, because it is only at the very longest periods ($\geq$500 s, say) that there is the possibility of obtaining data of sufficient duration to make it possible to achieve the necessary spectral resolution. Essentially, the modes attenuate before the many cycles necessary to establish a standing wave pattern have elapsed. Thus, in many observational studies, over a wide range of frequencies, the normal mode representation has the role, primarily, of providing a method for the calculation of theoretical seismograms. Although observed spectra contain spectral peaks, the peaks are broadened by the effects of attenuation in a path-dependent way. Thus, rather than making direct measurements on observed spectra, the analysis needs to be based on comparisons between data and synthetic spectra, in order to derive models able to give improved agreement between data and synthetics.

The use of normal mode theory as a method of synthesis extends well beyond the realm normally thought of as normal mode studies. For example, it has become commonplace to calculate global body wave theoretical seismograms by mode summation in a spherical model, to frequencies higher than 100 mHz (10 s period). Typically, such calculations can be done in seconds on an ordinary workstation, the time, of course, depending strongly upon the upper limit in frequency and on the number of samples in the time series. The advantage of the method is that all seismic phases are automatically included, with realistic time and amplitude relationships. Although the technique is limited (probably for the foreseeable future) to spherically symmetric models, the comparison of such synthetics with data provides a valuable tool for understanding the nature and potential of the observations and for making measurements such as differences in timing between data and synthetics, for use in tomography. Thus, the period range of applications of the normal mode representation extends from several thousand seconds to $\sim$5 s. In between these ends of the spectrum is an enormous range of applications: studies of modes *per se*, surface wave studies, analysis of overtones, and long period body waves, each having relevance to areas such as source parameter estimation and tomography. **Figure 2** shows an example of synthetic and data traces, illustrating this.

There are a number of excellent sources of information on normal modes theory and applications. The comprehensive monograph by Dahlen and Tromp (1998) provides in-depth coverage of the material and an extensive bibliography. An earlier monograph by Lapwood and Usami (1981) contains much interesting and useful information, treated from a fundamental point of view, as well as historical material about early theoretical work and early observations. A review by Takeuchi and Saito (1972) is a good source for the ordinary differential equations for spherical Earth models and methods of solution. Other reviews are by Gilbert (1980), Dziewonski and Woodhouse (1983), and Woodhouse (1996). A review of normal mode observations can be found in Chapter 1.03. Because of this extensive literature we tend in this chapter to expand on some topics that have not found their way into earlier reviews but are nevertheless of fundamental interest and utility.

1.02.2 Hamilton's Principle and the Equations of Motion

To a good approximation, except in the vicinity of an earthquake or explosion, seismic displacements are governed by the equations of elasticity. At long periods self-gravitation also plays an important role. Here we show how the equations of motion arise from Hamilton's principle.

Consider a material which is initially in equilibrium under self-gravitation. Each particle of the material is labeled by Cartesian coordinates x_i ($i = 1, 2, 3$), representing its initial position. The

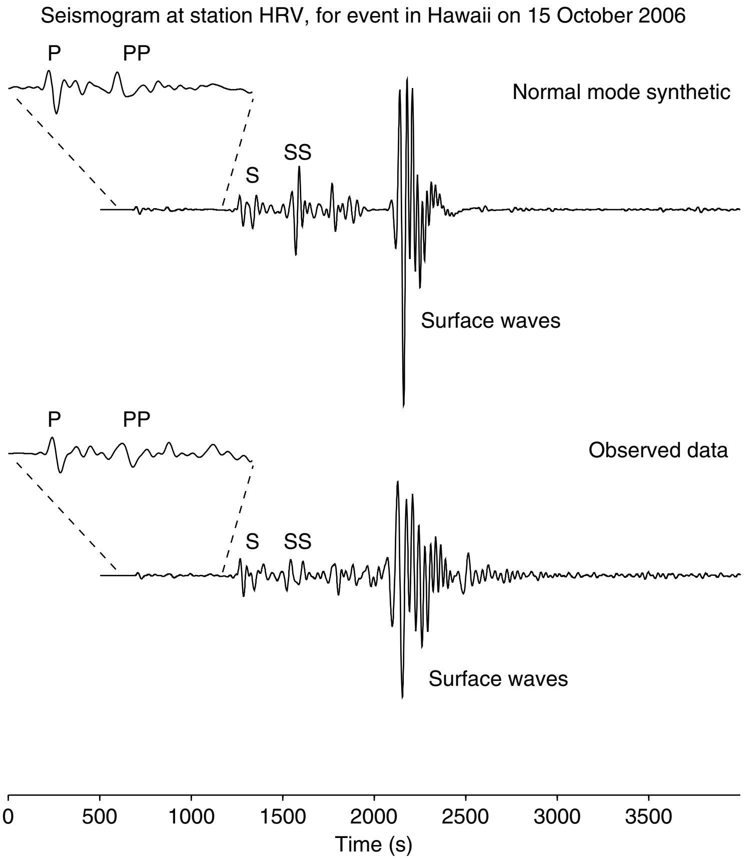

Figure 2 Data and synthetic for a vertical component record at station HRV following the recent event in Hawaii on 15 October 2006. The epicentral distance is 73°. For the calculation of the synthetic seismogram the Harvard/Lamont quick-CMT centroid location and moment tensor parameters were used (G. Ekström, www.globalcmt.org).

material undergoes time-dependent deformation in which the particle initially at x_i moves to $r_i = r_i(\mathbf{x}, t)$ where t is time. A 'hyperelastic' material is defined as one in which there exists an internal energy density function which is a function of the Green strain tensor

$$e_{ij} = \frac{1}{2}\left(r_{k,i} r_{k,j} - \delta_{ij}\right) \qquad [1]$$

where $r_{k,i} \equiv \partial r_k / \partial x_i$, δ_{ij} is the Krönecker delta; summation over repeated indices is assumed. Thus, we introduce the internal elastic energy function, per unit mass, $E(\mathbf{x}, \mathbf{e}, s)$, where s is specific entropy. We shall be concerned only with isentropic deformations, and can henceforth omit the dependence on s. Note that E is represented as a function of the coordinates x_i which label a specific material particle. $E(\mathbf{x}, \mathbf{e})$ is regarded as a given function characterizing the elastic properties of the material. This form of the internal energy function, which forms the basis of finite theories of elasticity, as well as the theory of linear elasticity which we need here, arises from the very general consideration that elastic internal energy should not change as a result of rigid rotations of the material.

The gravitational field is characterized by a potential field $\phi(\mathbf{r})$, which satisfies Poisson's equation

$$\frac{\partial^2 \phi}{\partial r_k \partial r_k} = 4\pi G \rho(\mathbf{r}) \qquad [2]$$

where ρ is the density. We state this equation in terms of the coordinates r_k, as it represents an equation valid in the current configuration of the material.

The Lagrangian governing motion of the elastic-gravitational system (total kinetic energy minus total potential energy) is

$$L = \int\int \left(\frac{1}{2}\rho \dot{r}_k \dot{r}_k - \rho E(\mathbf{x}, \mathbf{e}) - \frac{1}{2}\rho\phi\right) \mathrm{d}^3 r \, \mathrm{d}t \qquad [3]$$

where 'overdot' represents the material time derivative (i.e., the derivative with respect to t at constant $\mathbf{x}$).

The three terms of the integrand represent the kinetic energy, (minus) the elastic energy, and (minus) the gravitational energy, respectively, making use of the fact that the gravitational energy released by assembling a body from material dispersed at infinity is $-\int \frac{1}{2}\rho\phi\, \mathrm{d}^3 r$. The integral in [3] is over the volume occupied by the Earth, which is taken to consist of a number of subregions with internal interfaces and an external free surface. Hamilton's principle requires that L be stationary with respect to variations $\delta r_i(\mathbf{x}, t)$, subject to the constraint that ϕ is determined by [2], and also to the requirement of mass conservation:

$$\rho \mathrm{d}^3 r = \rho^0 \mathrm{d}^3 x \qquad [4]$$

where $\rho^0 = \rho^0(\mathbf{x})$ is the initial density, that is,

$$\rho = \rho^0 / \mathcal{J} \qquad [5]$$

where $\mathcal{J}$ is the Jacobian,

$$\mathcal{J} = \frac{\partial(r_1, r_2, r_3)}{\partial(x_1, x_2, x_3)} \qquad [6]$$

The constraint [2] can be incorporated into the variational principle by introducing a field η which acts as a Lagrange multiplier (e.g., Seliger and Whitham, 1968):

$$L' = \iint \left\{ \frac{1}{2}\rho \dot{r}_k \dot{r}_k - \rho E(\mathbf{x}, \mathbf{e}) - \frac{1}{2}\rho\phi + \eta\left(\frac{\partial^2 \phi}{\partial r_k \partial r_k} - 4\pi G\rho\right)\right\} \mathrm{d}^3 r\, \mathrm{d}t \qquad [7]$$

the term involving η vanishes by [2], and therefore, L' is stationary with respect to variations in η. If we also require that L' be stationary under variations $\delta\phi$, we obtain the Euler–Lagrange equation for η:

$$\frac{\partial^2 \eta}{\partial r_k \partial r_k} = \frac{1}{2}\rho \qquad [8]$$

which can be satisfied by setting $\eta = \phi/8\pi\mathrm{G}$. Thus, we obtain

$$L' = \iint \left\{ \frac{1}{2}\rho \dot{r}_k \dot{r}_k - \rho E(\mathbf{x}, \mathbf{e}) - \rho\phi - \frac{1}{8\pi G}\frac{\partial\phi}{\partial r_k}\frac{\partial\phi}{\partial r_k}\right\} \mathrm{d}^3 r\, \mathrm{d}t \qquad [9]$$

Changing the spatial integration variables, making use of [4], we may also write

$$L' = \iint \left\{ \frac{1}{2}\rho^0 \dot{r}_k \dot{r}_k - \rho^0 E(\mathbf{x}, \mathbf{e}) - \rho^0\phi - \frac{1}{8\pi G}\phi'_{,k}\phi'_{,k}\right\} \mathrm{d}^3 x\, \mathrm{d}t \qquad [10]$$

where ϕ' represents the gravitational potential at the fixed coordinate point $\mathbf{x}$. Notice that the first three terms of the integrand have been transformed by regarding $\mathbf{r}$ to be a function of $\mathbf{x}$ (at each fixed t) through the function $\mathbf{r}(\mathbf{x}, t)$ which defines the deformation. However, the fourth term (which 'could' be treated in the same way) has been transformed by renaming the dummy integration variables r_i to x_i. Hence, the need to introduce ϕ' since ϕ represents $\phi(\mathbf{r}(\mathbf{x}, t))$, which is different from $\phi(\mathbf{x}, t)$. The requirement that L' be stationary with respect to variations $\delta\mathbf{r}(\mathbf{x}, t)$, $\delta\phi(\mathbf{r}, t)$ provides a very succinct, complete statement of the elasto-gravitational dynamical equations.

To obtain the partial differential equations for infinitesimal deformations, we approximate L' in the case that $r_k = x_k + \epsilon u_k(\mathbf{x}, t)$ and $\phi' = \phi^0 + \epsilon\phi^1(\mathbf{x}, t)$, where ϵ is a small parameter. We seek to express the Lagrangian L' in terms of the fields u_i, ϕ^1, to second order in ϵ. We have, to second order in ϵ,

$$\phi = \phi^0 + \epsilon u_i \phi^0_{,i} + \frac{1}{2}\epsilon^2 u_i u_j \phi^0_{,ij} + \epsilon\phi^1 + \epsilon^2 u_i \phi^1_{,i} \qquad [11]$$

We expand $\rho^0 E[\mathbf{x}, \mathbf{e}]$ to second order in strain

$$\rho^0(\mathbf{x})E(\mathbf{x}, \mathbf{e}) = a + t^0_{ij} e_{ij} + \frac{1}{2} c_{ijkl} e_{ij} e_{kl} \qquad [12]$$

As a result of their definitions, as first and second derivatives of $\rho^0(\mathbf{x})E(\mathbf{x}, \mathbf{e})$ with respect to strain, at zero strain, t^0_{ij} and c_{ijkl} possess the symmetries:

$$t^0_{ij} = t^0_{ji} \qquad [13]$$

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} \qquad [14]$$

We use the notation t^0_{ij} since these expansion coefficients represent the initial stress field. The strain tensor [1] is

$$e_{ij} = \frac{1}{2}\epsilon\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\epsilon^2 u_{k,i} u_{k,j} \qquad [15]$$

Thus, the second-order expansion of L' becomes

$$\begin{aligned} L' = \iint \Bigg\{ & -a - \rho_0\phi^0 - \frac{1}{8\pi G}\phi^0_{,i}\phi^0_{,i} \\ & -\epsilon\left(t^0_{ij} u_{i,j} + \rho_0 u_i \phi^0_{,i} + \rho_0\phi^1 + \frac{1}{4\pi G}\phi^0_{,i}\phi^1_{,i}\right) \\ & + \frac{1}{2}\epsilon^2\Big[\rho^0 \dot{u}_k \dot{u}_k - \Lambda_{jilk} u_{i,j} u_{k,l} - \rho^0 u_i u_j \phi^0_{,ij} \\ & - 2\rho^0 u_j \phi^1_{,j} - \frac{1}{4\pi G}\phi^1_{,i}\phi^1_{,i}\Big]\Bigg\} \mathrm{d}^3 x \mathrm{d}t \end{aligned} \qquad [16]$$

where we have introduced

$$\Lambda_{jilk} \equiv \delta_{ik} t^0_{jl} + c_{ijkl} \qquad [17]$$

The Euler–Lagrange equations arising from the requirement that L' be stationary with respect to variations $\delta\mathbf{u}$, $\delta\phi^1$ must hold for each power of ϵ. Thus, the first-order terms give the following equations, which represent the requirement that the initial configuration be in equilibrium under self-gravitation:

$$t^0_{ij,j} = \rho^0\phi^0_{,i} \quad [18]$$

$$\phi^0_{,ii} = 4\pi G\rho^0 \quad [19]$$

The terms in L' that are independent of ϵ do not contribute to the variation, and thus can be omitted. When [18], [19] are satified, there remain only the second-order terms:

$$L'' = \int\int \frac{1}{2}\left[\rho^0\dot{u}_k\dot{u}_k - \Lambda_{jilk}u_{i,j}u_{k,l} - \rho^0 u_i u_j \phi^0_{,ij} - 2\rho^0 u_i\phi^1_{,i} - \frac{1}{4\pi G}\phi^1_{,i}\phi^1_{,i}\right] d^3x\,\mathrm{d}t \quad [20]$$

In [20] we omit the factor ϵ^2, absorbing the small parameter into the definitions of the fields $\mathbf{u}$, ϕ^1.

The Euler–Lagrange equations corresponding to variations $\delta\mathbf{u}$, $\delta\phi^1$ give the equations of motion

$$\rho^0(\ddot{u}_i + \phi^1_{,i} + \phi^0_{,ij}u_j) = (\Lambda_{jilk}u_{k,l})_{,j} \quad [21]$$

$$\phi^1_{,jj} = -4\pi G(\rho^0 u_i)_{,i} \quad [22]$$

The variational principle also leads to certain natural boundary conditions at the free surface and at internal boundaries in the case that $\mathbf{r}(\mathbf{x}, t)$ is required to be continuous at such boundaries – so-called 'welded' boundaries. These are as given below. We also wish to include the case that the model contains fluid regions, having free-slip, boundary conditions at their interfaces with solid regions – so-called 'frictionless' boundaries. The correct treatment of such boundaries introduces complications that, in the interests of giving a concise account, we do not analyze in detail here. Woodhouse and Dahlen (1978) show that it is necessary to include additional terms in the Lagrangian to account for the additional degrees of freedom corresponding to slip (i.e., discontinuous u_i) at such boundaries. The stress boundary conditions are most conveniently stated in terms of the vector t_i defined on the boundary by

$$t_i = \Lambda_{jilk}u_{k,l}n_j - n_i\left(\pi^0 u_k\right)_{;k} + \pi^0 u_{k;i}n_k \ \text{ with } \ \pi^0 = t^0_{jk}n_j n_k \quad [23]$$

where n_i is the Unit normal to the boundary, and where the semicolon notation, for example, $u_{k;i}$, is used to indicate differentiation in the surface: $u_{k;i} = u_{k,i} - n_i n_j u_{k,j}$. The complete set of boundary conditions is

$$\begin{array}{llllll}
\text{Welded:} & \left[t^0_{ij}n_j\right]^+_- = 0, & & [u_i]^+_- = 0, & [t_i]^+_- = 0, & \\
\text{Frictionless:} & \left[t^0_{ij}n_j\right]^+_- = 0, & t^0_{ij}n_j = n_i\pi^0, & [u_k n_k]^+_- = 0, & [t_i]^+_- = 0, & t_i = n_i t_k n_k \\
\text{Free:} & t^0_{ij}n_j = 0, & & t_i = 0, & & \\
\text{All:} & [\phi^0]^+_- = 0, & \left[\phi^0_{,i}\right]^+_- = 0, & [\phi^1]^+_- = 0, & [\phi^1_i + 4\pi G\rho^0 u_i]^+_- = 0, & \\
\text{Infinity:} & \phi^0 \to 0, & & \phi^1 \to 0, & &
\end{array} \quad [24]$$

where $[\ ]^+_-$ represents the discontinuity of the enclosed quantity accross the boundary.

In the presence of an applied force distribution $f_i = f_i\,(\mathbf{x}, t)$, per unit volume, the equation of motion [21] becomes

$$\rho^0\left(\ddot{u}_i + \phi^1_{,i} + \phi^0_{,ij}u_j\right) - \left(\Lambda_{jilk}u_{k,l}\right)_{,j} = f_i \quad [25]$$

Taking the Fourier transform in time, we shall also write

$$\rho^0\left(-\omega^2 u_i + \phi^1_{,i} + \phi^0_{,ij}u_j\right) - \left(\Lambda_{jilk}u_{k,l}\right)_{,j} = f_i \quad [26]$$

where ω is the frequency. We shall employ the transform pair:

$$u_i(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} u_i(\mathbf{x}, t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$
$$u_i(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_i(\mathbf{x}, \omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \quad [27]$$

Here, and in subsequent equations we rely on the context to distinguish between time-domain and frequency-domain quantities, adopting the convention that if ω appears in an equation then all functions appearing are the Fourier transforms of the original, time-dependent functions.

Earthquake sources can be modeled by choosing particular force distributions f_i. The problem of determining the 'equivalent body force distribution' to represent (prescribed) slip on an earthquake fault was originally solved by Burridge and Knopoff (1964) using the elastodynamic representation theorem. A very general and, at the same time, simple approach to the problem of determining body force equivalents is that of Backus and Mulcahy (1976). They argue that an earthquake occurs as a result of the failure of the assumed constitutive law, in the linear case Hooke's law, relating stress and strain. This leads them to introduce a symmetric tensor quantity $\Gamma_{ij} = \Gamma_{ij}(\mathbf{x}, t)$, called the 'stress glut', which represents the failure of Hooke's law to be satisfied. Importantly, Γ_{ij} (1) will be zero outside the fault zone; (2) will be zero at times before the earthquake, and (3) will have vanishing time derivative at times after slip has ceased. Thus the 'glut rate', $\dot{\Gamma}_{ij}$, is compact in space and time. The earlier concept of 'stress-free strain', due to Eshelby (1957) is a closely related one. The strain, for slip on a fault, contains δ-function terms at the fault, because the displacement is discontinuous. The stress, on the other hand, is finite on the fault. Thus, there exists a stress-glut – a failure of the stress to satisfy Hooke's law, and a stress-free strain, that is, a component of the strain field that is not reflected in the stress. The existence of a nonvanishing stress-glut, Γ_{ij}, leads us to replace [21] by

$$\rho^0\left(\ddot{u}_i + \phi^1_{,i} + \phi^0_{,ij}u_j\right) = \left(\Lambda_{jilk}u_{k,l} - \Gamma_{ij}\right)_{,j} \qquad [28]$$

that is,

$$\rho^0\left(\ddot{u}_i + \phi^1_{,i} + \phi^0_{,ij}u_j\right) - \left(\Lambda_{jilk}u_{k,l}\right)_{,j} = -\Gamma_{ij,j} \qquad [29]$$

and thus comparing [25] with [29], the equivalent body force distribution is found to be $f_i = -\Gamma_{ij,j}$. Because $\dot{\Gamma}_{ij}$ $(\mathbf{x}, t)$ is compact in space and time, it is appropriate for calculations at long period and long wavelength to replace it by a δ-function in space and time. Defining the 'moment tensor'

$$M_{ij} = \int_V \Gamma_{ij}(\mathbf{x}, \infty)\mathrm{d}^3x = \int_{-\infty}^{\infty}\int_V \dot{\Gamma}_{ij}(\mathbf{x}, t)\mathrm{d}^3x\,\mathrm{d}t \qquad [30]$$

where V is the source volume – the region over which Γ_{ij} is nonzero – a suitable form for Γ_{ij} is $\Gamma_{ij}(\mathbf{x}, t) \approx M_{ij}\delta^3(\mathbf{x}-\mathbf{x}_s)H(t-t_s)$, where $H(t)$ is the Heaviside step function, and where $\mathbf{x}_s, t_s$ are the source coordinates. Thus, $f_i \approx -M_{ij}\partial_j\delta^3(\mathbf{x}-\mathbf{x}_s)H(t-t_s)$. In what follows, we shall consider the more general point source

$$f_i = \left(F_i - M_{ij}\partial_j\right)\delta^3(\mathbf{x}-\mathbf{x}_s)H(t-t_s) \qquad [31]$$

in which M_{ij} is not necessarily symmetric, recognizing that for sources not involving the action of forces external to the Earth, so-called indigenous sources, M_{ij} must be symmetric and F_i must be zero. The solution for a point force F_i is of fundamental theoretical interest since the solution in this case is the Green's function for the problem, which can be used to construct solutions for any force distribution f_i. Nonsymmetric M_{ij} corresponds to a source which exerts a net torque or couple on the Earth.

The hyperelastic constitutive law based on the internal energy function $\mathrm{E}(\mathbf{x}, \mathbf{e})$ needs to be modified to include the effects of energy loss due to such effects as grain boundary sliding and creep. Such 'anelastic' effects lead to dissipation of energy (i.e., conversion of elastic stored energy into heat) and thus to the decay, or 'attenuation' of seismic waves. In addition, they are responsible for such effects as postseismic relaxation, and the theory developed here is in large part applicable to this problem also. A generalization of the constitutive law which retains linearity is the viscoelastic law, which supposes that stress depends not only on the strain at a given instant, but also on the strain history. This can be written as

$$t_{ij}(t) = \int_{-\infty}^{\infty} c_{ijkl}(t-t')u_{k,l}(t')\mathrm{d}t' \qquad [32]$$

where t_{ij} is incremental stress. (In fact, it can be shown that the true increment in stress, at a material particle, includes terms in the initial stress: $t^1_{ij} = c_{ijkl}u_{k,l} + t^0_{ik}u_{j,k} + t^0_{jk}u_{i,k} - t^0_{ij}u_{k,k}$, but this makes no difference to the discussion here.) Thus, the elastic constants become functions of time, relative to a given time t at which the stress is evaluated. Importantly, since stress can depend only upon *past* times, $c_{ijkl}(t)$ must vanish for negative values of t; that is, it must be a 'causal' function of time. In order to recover the strict Hooke's law, $t_{ij} = c_{ijkl}u_{k,l}$ we need $c_{ijkl}(t) = c_{ijkl}\delta(t)$ (we are distinguishing here between c_{ijkl} unadorned, which has the units of stress, and $c_{ijkl}(t)$, which has the units of strees/time). In the frequency domain, using the convolution theorem,

$$t_{ij}(\omega) = c_{ijkl}(\omega)u_{i,j}(\omega) \qquad [33]$$

Because $c_{ijkl}(t)$ is a causal function, its Fourier transform

$$c_{ijkl}(\omega) = \int_0^\infty c_{ijkl}(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \qquad [34]$$

will be analytic, that is, will have no singularities, in the lower half of the complex ω-plane, as the integral (34) will converge unconditionally in the case that ω possesses a negative imaginary part. From [34] $c_{ijkl}(\omega)^* = c_{ijkl}(-\omega^*)$. For our purposes here, the key conclusion is that 'in the frequency domain', it makes virtually no difference to the theory whether the material is hyperelastic, or viscoelastic, as we have simply everywhere to substitute $c_{ijkl}(\omega)$ for c_{ijkl}. In fact, there is even no need to introduce a new notation, but only to remember that now c_{ijkl} can represent a complex quantity depending on ω and analytic in the lower half of the complex ω-plane. When writing equations in the time domain, we have to remember that c_{ijkl} can be a convolution 'operator', acting on the strain. (It may be remarked that the above derivation of the equations of motion, based on Hamilton's principle, is in need of modification if $E(\mathbf{x}, \mathbf{e})$ does not exist. We do not quite know how to do this, but a monograph by Biot (1965) discusses the use of variational principles in the presence of anelastic effects.) Kanamori and Anderson (1977), and references cited therein, is a good source for further information on this topic.

It is often useful to summarize the equations and the boundary conditions by a single simple equation:

$$(\mathcal{H} + \rho^0\partial_t^2)\mathbf{u} = \mathbf{f} \qquad [35]$$

where $\mathcal{H}$ represents the integro-differential operator corresponding to the left side of [25], omitting the term in $\rho^0\ddot{\mathbf{u}}$, in which ϕ^1 is thought of as a functional of $\mathbf{u}$ that is, as the solution of Poisson's equation [22] corresponding to a given $\mathbf{u}(\mathbf{x}, t)$, together with the boundary conditions relating to ϕ^1 in [24]. Thus, $\mathcal{H}\mathbf{u}$ incorporates the solution of [22]. In the attenuating case, $\mathcal{H}$ also includes the time-domain convolutions arising from the viscoelastic rheology.

1.02.3 The Generalized Spherical Harmonics

The reduction of these equations in spherical coordinates is most easily accomplished through the use of the generalized spherical harmonic formalism (Phinney and Burridge, 1973). Here we describe how this formalism is used, giving some key results without derivation. We shall use a standard set of Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) related by

$$\begin{aligned} x_1 &= x = r\sin\theta\cos\phi \\ x_2 &= y = r\sin\theta\sin\phi \\ x_3 &= z = r\cos\theta \end{aligned} \qquad [36]$$

Unit vectors in the coordinate directions are given by

$$\begin{aligned} \hat{\mathbf{r}} &= [\sin\theta\cos\phi, \sin\theta\sin\phi,\ \cos\theta] \\ \hat{\boldsymbol{\theta}} &= [\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta] \\ \hat{\phi} &= [-\sin\phi,\ \cos\phi,\ 0] \end{aligned} \qquad [37]$$

Spherical components of vectors and tensors will be written, for example, as $u_\theta = u_k\hat{\theta}_k, t^0_{r\phi} = t^0_{ij}\hat{r}_i\hat{\phi}_j$.

The prescription provided by the generalized spherical harmonic formalism is first to define the 'spherical contravariant components' of the vectors and tenors that appear, and then to expand their dependence on (θ, ϕ) in terms of complete sets of functions appropriate to the particular component. For a tensor of rank p, having spherical components $s_{i_1 i_2 \cdots i_p}$, spherical contravariant components are defined by

$$s^{\alpha_1\alpha_2\ldots\alpha_p} = C^{\dagger\alpha_1 i_1}C^{\dagger\alpha_2 i_2} \times \cdots \times C^{\dagger\alpha_p i_p}s_{i_1 i_2\ldots i_p} \qquad [38]$$

where indices i_k label the spherical components $i_k \in \{r, \theta, \phi\}$ and where indices α_k take values $\alpha_k \in \{-1, 0, +1\}$; the nonvanishing coefficients $C^{\dagger\alpha i}$ are $C^{\dagger 0r} = 1$, $C^{\dagger\pm 1\theta} = \mp 2^{-1/2}$, $C^{\dagger\pm 1\phi} = 2^{-1/2}\mathrm{i}$. The inverse transformation, from spherical contravariant components to spherical components, is

$$s_{i_1 i_2\ldots i_p} = C_{i_1\alpha_1}C_{i_2\alpha_2} \times \cdots \times C_{i_p\alpha_p}s^{\alpha_1\alpha_1\ldots\alpha_p} \qquad [39]$$

with $C_{i\alpha} = (C^{\dagger\alpha i})^*$. The spherical harmonic basis, $Y_1^{Nm}(\theta, \phi)$ appropriate for each contravariant component is determined by the sum $N = \alpha_1 + \alpha_2 + \cdots + \alpha_p$ of its indices. Thus, we expand

$$\begin{aligned} s^{\alpha_1\alpha_1\cdots\alpha_p} &= \sum_{lm} s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p}(r)\,Y_l^{Nm}(\theta, \phi) \quad \text{with} \\ N &= \alpha_1 + \alpha_1 + \cdots + \alpha_p \end{aligned} \qquad [40]$$

$Y_l^{Nm}(\theta, \phi)$ are the generalized spherical harmonics:

$$Y_l^{Nm}(\theta, \phi) = P_l^{Nm}(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi} = d_{Nm}^{(l)}(\theta)\mathrm{e}^{\mathrm{i}m\phi} \qquad [41]$$

where the 'real' quantities $d_{Nm}^{(l)}(\theta) = P_l^{Nm}(\cos\theta)$ are rotation matrix elements employed in the quantum mechanical theory of angular momentum (Edmonds, 1960); thus, Y_l^{Nm} vanish for N or m outside the range $-l$ to l. In [40] summations over l and m are for integers $l=0,1,\ldots,\infty$, $m=-l,-l+1,\ldots,l$. The spherical harmonic degree l characterizes a group representation of the rotation group; as a consequence, tensor fields that are spherically symmetric have only the term with $l=0$, $N=0$, $m=0$. The property of the rotation matrix elements

$$d_{Nm}^{(l)}(0)=\begin{cases}1, & \text{if } m=N \text{ and } l\geq|N|\\ 0, & \text{otherwise}\end{cases} \qquad [42]$$

is a very useful one, for example, for calculations of source excitation coefficients when it is required to evaluate spherical harmonic expressions for $\theta=0$ (see below). Equation [42] says that regarded as $(2l+1)\times(2l+1)$ matrix, having row index N and column index m, $d_{Nm}^{(l)}(0)$ is the unit matrix. Matrices $d_{Nm}^{(l)}(\theta)$ have symmetries $d_{-N-m}^{(l)}(\theta)=d_{mN}^{(l)}(\theta)=(-1)^{m-N}d_{Nm}^{(l)}(\theta)$, from which follows the relation $Y_l^{Nm}(\theta,\phi)^*=(-1)^{m-N}Y_l^{-N-m}(\theta,\phi)$ where asterick denotes the complex conjugate.

Y_l^{Nm} satisfy the orthogonality relation

$$\int_{-\pi}^{\pi}\int_0^{\pi}Y_{l'}^{Nm'}(\theta,\phi)^*\,Y_l^{Nm}(\theta,\phi)\times\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=\frac{4\pi}{2l+1}\delta_{l'l}\delta_{m'm} \qquad [43]$$

Thus, the expansion coefficients in [40] can be written as

$$s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p}(r)=\frac{2l+1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}Y_l^{Nm}(\theta,\phi)^*s^{\alpha_1\alpha_1\cdots\alpha_p}\times\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \quad \text{with} \quad N=\alpha_1+\alpha_1+\cdots+\alpha_p \qquad [44]$$

Other normalizations for the spherical harmonics have frequently been used in the literature. Completely normalized spherical harmonics can be written as $Y_l^m(\theta,\phi)=\nu_l Y_l^{0m}$, where $\nu_l\equiv\sqrt{(2l+1)/4\pi}$. In this chapter we shall adopt the same conventions as in the paper by Phinney and Burridge (1973) for the generalized spherical harmonics, using the explicit form $\nu_l Y_l^{0m}$ when completely normalized spherical harmonics are needed.

The following is the key property of the generalized spherical harmonics under differentiation:

$$(\pm\partial_\theta+\mathrm{i}\csc\theta\,\partial_\phi)\,Y_l^{Nm}=\sqrt{2}\Omega_{\pm N}^l\,Y_l^{N\mp1m}-N\cot\theta\,Y_l^{Nm} \qquad [45]$$

which leads to the following rule for the expansion coefficients of the gradient of a tensor:

$$\begin{aligned}s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p|0}&=\frac{\mathrm{d}}{\mathrm{d}r}s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p}\\ s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p|\pm}&=r^{-1}\Omega_{\mp N}^l s_{lm}^{\alpha_1\alpha_1\cdots\alpha_p}\\ &\quad-r^{-1}\left(\begin{array}{l}\text{the sum of the terms obtained}\\ \text{from } s_{lm}^{\alpha_1\alpha_1\ldots\alpha_p} \text{ by adding } \pm1 \text{ to each}\\ \text{of } \alpha_1\alpha_1\ldots\alpha_p \text{ in turn, omitting any}\\ \text{terms for which the resulting}\\ \text{index } \alpha_k\pm1\notin\{-1,0,1\}\end{array}\right)\end{aligned} \qquad [46]$$

where $\Omega_N^l\equiv[(l+N)(l-N+1)/2]^{1/2}$ and $N\equiv\alpha_1+\alpha_1+\cdots+\alpha_p$. The notation $s_{lm}^{\alpha_1\alpha_2\cdots\alpha_p|\alpha_{p+1}}$ is used to denote the expansion coefficients of the spherical contravariant components of the tensor having Cartesian components $s_{i_1i_2\cdots i_pi_{p+1}}$. Contraction over contravariant indices is carried out using the metric tensor $g_{\alpha_1\alpha_2}=C_{i\alpha_1}C_{i\alpha_2}$, which has nonvanishing entries $g_{00}=1$, $g_{-+}=g_{+-}=-1$. Thus, for example, the expansion coefficients of the divergence of a tensor $t_i=s_{ij,j}$, say, are given by

$$\begin{aligned}t_{lm}^{\alpha}=-s_{lm}^{\alpha+|-}+s_{lm}^{\alpha\,0|0}-s_{lm}^{\alpha-|+}&=\begin{cases}-s_{lm}^{-+|-}+s_{lm}^{-0|0}-s_{lm}^{--|+}, & \alpha=-1\\ -s_{lm}^{0+|-}+s_{lm}^{00|0}-s_{lm}^{0-|+}, & \alpha=0\\ -s_{lm}^{++|-}+s_{lm}^{+0|0}-s_{lm}^{+-|+}, & \alpha=1\end{cases}\\ &=\begin{cases}-\dfrac{1}{r}\left(\Omega_0^l s_{lm}^{-+}-s_{lm}^{-0}\right)+\dfrac{\mathrm{d}}{\mathrm{d}r}s_{lm}^{-0}-\dfrac{1}{r}\left(\Omega_2^l s_{lm}^{--}-s_{lm}^{0-}-s_{lm}^{-0}\right), & \alpha=-1\\ -\dfrac{1}{r}\left(\Omega_1^l s_{lm}^{0+}-s_{lm}^{-+}-s_{lm}^{00}\right)+\dfrac{\mathrm{d}}{\mathrm{d}r}s_{lm}^{00}-\dfrac{1}{r}\left(\Omega_1^l s_{lm}^{0-}-s_{lm}^{+-}-s_{lm}^{00}\right), & \alpha=0\\ -\dfrac{1}{r}\left(\Omega_2^l s_{lm}^{++}-s_{lm}^{0+}-s_{lm}^{+0}\right)+\dfrac{\mathrm{d}}{\mathrm{d}r}s_{lm}^{+0}-\dfrac{1}{r}\left(\Omega_0^l s_{lm}^{+-}-s_{lm}^{+0}\right), & \alpha=1\end{cases}\end{aligned} \qquad [47]$$

The major advantage of this formalism is that, in a spherically symmetric system, it enables vector and tensor relations to be transformed into relations for spherical harmonic coefficients, by the application of a straightforward set of rules. Importantly, the resulting relations (1) are true for each value of l and m

separately, and (2) are the same for each spherical harmonic order m.

The treatment of aspherical systems requires results for the products of spherical harmonic expansions. It can be shown that

$$
\begin{aligned}
Y_{l_1}^{N_1 m_1} Y_{l_2}^{N_2 m_2} &= (-1)^{N_1+N_2-m_1-m_2} \sum_{l=|l_1-l_2|}^{l_1+l_2} (2l+1) \\
&\times \begin{pmatrix} l & l_1 & l_2 \\ -N_1-N_2 & N_1 & N_2 \end{pmatrix} \\
&\times \begin{pmatrix} l & l_1 & l_2 \\ -m_1-m_2 & m_1 & m_2 \end{pmatrix} Y_l^{N_1+N_2 m_1+m_2} \qquad [48]
\end{aligned}
$$

Where the so-called 'Wigner 3-j symbols' are the (real) quantities arising in the theory of the coupling of angular momentum in quantum mechanics (see Edmonds (1960)). These satisfy

$$
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \quad
\begin{aligned}
&\text{unless:} \\
&|l_2-l_3| \le l_1, |l_3-l_1| \le l_2, |l_1-l_2| \le l_3, \\
&\text{any one of which implies the other two} \\
&|m_1| \le l_1, |m_2| \le l_2, |m_3| \le l_3 \\
&m_1+m_2+m_3 = 0 \qquad [49]
\end{aligned}
$$

The 3-j symbol is symmetric under even permutations of its columns, and either symmetric or antisymmetric under odd permutations, depending upon whether $l_1+l_2+l_3$ is even or odd. This has the consequence that if the sum of the l's is odd and if the m's are zero, the 3-j symbol is 0. Equation [48] leads to the following result for the spherical harmonic coefficients of the product of two tensor fields; suppose that $c_{i_1 i_2 \ldots i_p j_1 j_2 \ldots q} = a_{i_1 i_2 \ldots i_p} b_{j_1 j_2 \ldots j_q}$; then

$$
\begin{aligned}
c_{lm}^{\alpha_1\alpha_2\ldots\alpha_p\beta_1\beta_2\ldots\beta_q} &= (-1)^{N_1+N_2-m} \\
&\times \sum_{l_1 l_2 m_1} \begin{pmatrix} l & l_1 & l_2 \\ -N_1-N_2 & N_1 & N_2 \end{pmatrix} \\
&\times \begin{pmatrix} l & l_1 & l_2 \\ -m & m_1 & m-m_1 \end{pmatrix} a_{l_1 m_1}^{\alpha_1\alpha_2\ldots\alpha_p} b_{l_2 m-m_1}^{\beta_1\beta_2\ldots\beta_q} \qquad [50]
\end{aligned}
$$

The summations here are over all values of l, l_1, m_1; however, it is a 'finite' sum by virtue of the fact that the terms vanish for values outside the ranges specified in [49].

As an application of the spherical harmonic formalism, here we consider the expansion of the point force distribution [31] in spherical harmonics. It will be sufficient to locate the source at time $t_s = 0$ and at a point on the positive z-axis, that is, at $x=0, y=0, z=r_s$, where r_s is the source radius. Because $\theta = 0$ is a singular point in the spherical coordinate system, we shall consider the limit as the source approaches the 'pole', $\theta_s = 0$, along the 'meridian', $\phi_s = 0$. In the frequency domain, [31] becomes

$$
f_i = \frac{1}{\mathrm{i}\omega}\left(F_i - M_{ij}\partial_j\right) r^{-2} \csc\theta\, \delta(\theta-\theta_s)\, \delta(\phi)\delta(r-r_s) \qquad [51]
$$

In the limiting process, $\theta_s \to 0$, we shall take F_i, M_{ij} to have constant spherical components F_ϕ, M_{rr}, $\mathrm{M}_{r\theta}$, etc. As $\theta_s \to 0$, the $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{r}}$ directions end up pointing along the x, y, z directions, respectively, of the global Cartesian coordinate system (e.g., see eqn [37]), and thus although $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ are undefined at $\theta = 0$, we can nevertheless interpret the spherical components F_ϕ, $\mathrm{M}_{r\theta}$, etc., as representing the components F_y, M_{zx}, etc., of the point force and moment tensor in the global Cartesian system. Let χ represent the expression $\chi = (\mathrm{i}\omega)^{-1} r^{-2} \csc\theta\, \delta(\theta-\theta_s)\delta(\phi)\delta(r-r_s)$, so that [51] can be written as $f_i = F_i\chi - \partial_i M_{ij}\chi$. The spherical contravariant components of F_i and M_{ij}, using a matrix representation of [39], are given by

$$
\begin{bmatrix} F^- \\ F^0 \\ F^+ \end{bmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix} \begin{bmatrix} F_\theta \\ F_\phi \\ F_r \end{bmatrix}
$$

$$
\begin{aligned}
\begin{pmatrix} M^{--} & M^{-0} & M^{-+} \\ M^{0-} & M^{00} & M^{0+} \\ M^{+-} & M^{+0} & M^{++} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix} \qquad [52] \\
&\times \begin{pmatrix} M_{\theta\theta} & M_{\theta\phi} & M_{\theta r} \\ M_{\phi\theta} & M_{\phi\phi} & M_{\phi r} \\ M_{r\theta} & M_{r\phi} & M_{rr} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{\mathrm{i}}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

The spherical harmonic expansion coefficients of $F^\alpha\chi$ and $M^{\alpha_1\alpha_2}\chi$, for which we use the notation $(F^\alpha\chi)_{lm}$, $(M^{\alpha_1\alpha_2}\chi)_{lm}$, are immediate using [44], integrating out the δ-functions contained in χ. We obtain

$$
\begin{aligned}
(F^\alpha\chi)_{lm} &= \frac{2l+1}{4\pi\mathrm{i}\omega}\frac{\delta(r-r_s)}{r^2} Y_l^{\alpha m}(\theta_s, 0)^* F^\alpha \\
&= \frac{2l+1}{4\pi\mathrm{i}\omega}\frac{\delta(r-r_s)}{r^2} d_{\alpha m}^{(l)}(\theta_s) F^\alpha \\
&= \begin{cases} \frac{2l+1}{4\pi\mathrm{i}\omega}\frac{\delta(r-r_s)}{r^2} F^\alpha, & \text{if } m=\alpha \text{ and } l \ge |\alpha| \\ 0, & \text{otherwise} \end{cases} \qquad [53]
\end{aligned}
$$

where θ_s has been set to 0 in the second line, using [42]. Similarly,

$$
\begin{aligned}
&(M^{\alpha_1\alpha_2}\chi)_{lm} \\
&= \frac{2l+1}{4\pi i\omega}\frac{\delta(r-r_s)}{r^2} Y_l^{\alpha_1+\alpha_2 m}(\theta_s,0)^* M^{\alpha_1\alpha_2} \\
&= \frac{2l+1}{4\pi i\omega}\frac{\delta(r-r_s)}{r^2} d^{(l)}_{\alpha_1+\alpha_2 m}(\theta_s) M^{\alpha_1\alpha_2} \\
&= \begin{cases} \dfrac{2l+1}{4\pi i\omega}\dfrac{\delta(r-r_s)}{r^2} M^{\alpha_1\alpha_2}, & \text{if } m=\alpha_1+\alpha_2 \text{ and } l \ge |\alpha_1+\alpha_2| \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \qquad [54]
$$

To complete the evaluation of the coefficients f_{lm}^{α} corresponding to [51], we need to find the spherical harmonic coefficients of the divergence of the field represented in [54], for which we can employ [47]; we find

$$
\begin{bmatrix} f_{lm}^{-} \\ f_{lm}^{0} \\ f_{lm}^{+} \end{bmatrix} = \begin{bmatrix} (F^{-}\chi)_{lm} + (M^{-+}\chi)_{lm}^{|-} - (M^{-0}\chi)_{lm}^{|0} + (M^{--}\chi)_{lm}^{|+} \\ (F^{0}\chi)_{lm} + (M^{0+}\chi)_{lm}^{|-} - (M^{00}\chi)_{lm}^{|0} + (M^{0-}\chi)_{lm}^{|+} \\ (F^{+}\chi)_{lm} + (M^{++}\chi)_{lm}^{|-} - (M^{+0}\chi)_{lm}^{|0} + (M^{+-}\chi)_{lm}^{|+} \end{bmatrix} \qquad [55]
$$

$$
= \frac{2l+1}{4\pi i\omega} \times \begin{cases}
\begin{bmatrix} M^{--}\Omega_2^l r^{-1} \\ 0 \\ 0 \end{bmatrix} \dfrac{\delta(r-r_s)}{r^2}, & m=-2,\ l \ge 2 \\[2ex]
\begin{bmatrix} F^{-} - M^{-0}\partial_r - 2M^{-0}r^{-1} - M^{0-}r^{-1} \\ M^{0-}\Omega_1^l r^{-1} \\ 0 \end{bmatrix} \dfrac{\delta(r-r_s)}{r^2}, & m=-1,\ l \ge 1 \\[2ex]
\begin{bmatrix} M^{-+}\Omega_0^l r^{-1} \\ F_0 - M^{00}\partial_r - 2M^{00}r^{-1} - M^{-+}r^{-1} - M^{+-}r^{-1} \\ M^{+-}\Omega_0^l r^{-1} \end{bmatrix} \dfrac{\delta(r-r_s)}{r^2}, & m=0 \\[2ex]
\begin{bmatrix} 0 \\ M^{0+}\Omega_1^l r^{-1} \\ F^{+} - M^{+0}\partial_r - 2M^{+0}r^{-1} - M^{0+}r^{-1} \end{bmatrix} \dfrac{\delta(r-r_s)}{r^2}, & m=1,\ l \ge 1 \\[2ex]
\begin{bmatrix} 0 \\ 0 \\ M^{++}\Omega_2^l r^{-1} \end{bmatrix} \dfrac{\delta(r-r_s)}{r^2}, & m=2,\ l \ge 2
\end{cases} \qquad [56]
$$

These are the spherical harmonic coefficients of the force distribution (31) which will be needed in the following sections. The coefficients are zero for $|m| > 2$.

1.02.4 The Green's Function for the Spherically Symmetric Earth

We consider here the case in which the Earth model is spherically symmetric. In this case the equations of motion are separable in spherical coordinates, and thus can be solved by reduction to ordinary differential equations. Since deviations from spherical symmetry are relatively small in the Earth, they can subsequently be treated by perturbation theory. We assume that in the initial equilibrium configuration the stress is hydrostatic, that is,

$$t_{ij}^0 = -p^0\delta_{ij} \qquad [57]$$

Spherical symmetry requires that ρ^0, ϕ^0, p^0 are functions only of r. The gravitational acceleration is

$g_i^0 = \phi_{,i}^0 = g^0(r)\widehat{r}_i$, and the equilibrium equations [18], [19], and the boundary conditions in [24] have the solutions:

$$\begin{aligned} g^0(r) &= \frac{4\pi G}{r^2}\int_0^r \rho^0(r)\mathrm{d}r \\ \phi^0(r) &= -\int_r^\infty g^0(r)\mathrm{d}r \\ p^0(r) &= \int_r^a \rho^0(r)g^0(r)\mathrm{d}r \end{aligned} \tag{58}$$

where a is the radius of the Earth. The equations of motion [25] with [17] and [19] can be put into the form

$$\rho\left[-\omega^2 u_i - u_{k,k}g_i + \varphi_{,i} + (u_k g_k)_{,i}\right] - (C_{ijkl}u_{k,l})_{,j} = f_i \tag{59}$$

$$(\varphi_{,k} + 4\pi G\rho u_k)_{,k} = 0 \tag{60}$$

where we have introduced the effective stiffness tensor

$$C_{ijkl} = c_{ijkl} + p^0(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \tag{61}$$

which has the same symmetries (eqn [14]) as does c_{ijkl}. In [59], [60], and in subsequent equations, we drop the superscripts in ρ^0, g^0, using simply ρ, g for these quantities. We also use the notation φ in place of ϕ^1. The applied force distribution f_i will be taken to be that given in [31] and [51], having the spherical harmonic coefficients [56].

In a spherically symmetric model, the tensor field C_{ijkl} must be a spherically symmetric tensor field, and therefore its spherical harmonic expansion will have terms only of degree $l = 0$. Its spherical contravariant components $C^{\alpha_1\alpha_2\alpha_3\alpha_3} = C_{00}^{\alpha_1\alpha_2\alpha_3\alpha_3}$ must have indices that sum to zero, and must also satisfy the usual elastic tensor symmetries [14]. It is easily seen that there are only five independent components that satisfy these requirements: $C^{0000}, C^{+-00}, C^{+0-0}, C^{++--}, C^{+-+-}$, which must be real (in the nonattenuating case) in order that the spherical components C_{ijkl} are real. Conventionally, these are designated (Love, 1927; Takeuchi and Saito, 1972) as

$$\begin{aligned} C^{0000} &= C(r),\ C^{+-00} = -F(r),\ C^{+0-0} = -L(r) \\ C^{++--} &= 2N(r),\ C^{+-+-} = A(r) - N(r) \end{aligned} \tag{62}$$

the independent spherical components being (using [39])

$$\begin{aligned} &C_{rrrr} = C(r), \quad C_{rr\theta\theta} = C_{rr\phi\phi} = F(r) \\ &C_{r\theta r\theta} = C_{r\phi r\phi} = L(r), \quad C_{\theta\phi\theta\phi} = N(r) \\ &C_{\theta\theta\phi\phi} = A(r) - 2N(r), \quad C_{\theta\theta\theta\theta} = C_{\phi\phi\phi\phi} = A(r) \end{aligned} \tag{63}$$

The mean 'bulk modulus' κ and 'shear modulus' μ can be defined by

$$\begin{aligned} \kappa &= \frac{1}{9}C_{iijj} = \frac{1}{9}(4A + C - 4N + 4F) \\ \mu &= \frac{1}{10}C_{ijij} - \frac{1}{30}C_{iijj} = \frac{1}{15}(A + C + 6L + 5N - 2F) \end{aligned} \tag{64}$$

Other conventional notations are

$$\begin{aligned} &\lambda = \kappa - \frac{2}{3}\mu,\ v_{PV}^2 = C/\rho,\ v_{PH}^2 = A/\rho,\ v_{SV}^2 = L/\rho \\ &v_{SH}^2 = N/\rho,\ \eta = F/(A - 2L) \end{aligned} \tag{65}$$

In the case that the material is 'isotropic', $A = C = \lambda + 2\mu = \kappa + \frac{4}{3}\mu$, $N = L = \mu$, and $\eta = 1$.

Now we seek solutions u_i, φ [59], [60] in terms of spherical harmonic expansions. It is convenient to write

$$u_{lm}^- = \nu_l\Omega_0^l[V_{lm}(r) - \mathrm{i}W_{lm}(r)] \tag{66}$$

$$u_{lm}^0 = \nu_l U_{lm}(r) \tag{67}$$

$$u_{lm}^+ = \nu_l\Omega_0^l[V_{lm}(r) + \mathrm{i}W_{lm}(r)] \tag{68}$$

where $\nu_l = \sqrt{(2l+1/4\pi}$, as the '$U, V, W$' notation is almost universally used in the literature on long period seismology. The expansion corresponding to [40] is then equivalent to the vector spherical harmonic representation (e.g., Morse and Feshbach, 1953):

$$\begin{aligned} u_r &= \sum_{lm} U_{lm}(r)\nu_l Y_l^{0m}(\theta,\phi) \\ u_\theta &= \sum_{lm}\left[V_{lm}(r)\partial_\theta + W_{lm}(r)\csc\theta\,\partial_\phi\right]\nu_l Y_l^{0m}(\theta,\phi) \\ u_\phi &= \sum_{lm}\left[V_{lm}(r)\csc\theta\,\partial_\phi - W_{lm}(r)\partial_\theta\right]\nu_l Y_l^{0m}(\theta,\phi) \end{aligned} \tag{69}$$

We shall abbreviate such vector spherical harmonic expansions using the shorthand

$$u_i: \rightarrow \begin{bmatrix} U_{lm}(r) \\ V_{lm}(r) \\ W_{lm}(r) \end{bmatrix} \tag{70}$$

meaning that the vector field having Cartesian components u_i is expressible in vector spherical harmonics as in [69]. We shall also suppress the suffices l, m and the explicit dependence upon r, writing simply U, V, W.

The spherical harmonic expansion of f_i can also be converted into this vector spherical harmonic notation. Using [56] with [52], we obtain

$$f_i :\to \frac{\nu_l}{\mathrm{i}\omega}\begin{cases}\begin{bmatrix}(rF_r + M_{\theta\theta} + M_{\phi\phi})\delta^{(0)}(r) - M_{rr}\delta^{(1)}(r) \\ \frac{1}{2}(-M_{\theta\theta} - M_{\phi\phi})\delta^{(0)}(r) \\ \frac{1}{2}(-M_{\theta\phi} + M_{\phi\theta})\delta^{(0)}(r)\end{bmatrix}, & m = 0 \\ \begin{bmatrix}\frac{\zeta}{2}(\mp M_{r\theta} + iM_{r\phi})\delta^{(0)}(r) \\ \frac{1}{2\zeta}[(\mp rF_\theta + \mathrm{i}rF_\phi \pm M_{r\theta} - \mathrm{i}M_{r\phi})\delta^{(0)}(r) + (\pm M_{\theta r} - \mathrm{i}M_{\phi r})\delta^{(1)}(r)] \\ \mp\frac{\mathrm{i}}{2\zeta}[(\mp rF_\theta + \mathrm{i}rF_\phi \pm M_{r\theta} - \mathrm{i}M_{r\phi})\delta^{(0)}(r) + (\pm M_{\theta r} - \mathrm{i}M_{\phi r})\delta^{(1)}(r)]\end{bmatrix}, & m = \pm 1 \\ \begin{bmatrix}0 \\ \frac{1}{4\zeta}\sqrt{\zeta^2 - 2}(M_{\theta\theta} - M_{\phi\phi} \mp \mathrm{i}M_{\theta\phi} \mp \mathrm{i}M_{\phi\theta})\delta^{(0)}(r) \\ \mp\frac{\mathrm{i}}{4\zeta}\sqrt{\zeta^2 - 2}(M_{\theta\theta} - M_{\phi\phi} \mp \mathrm{i}M_{\theta\phi} \mp \mathrm{i}M_{\phi\theta})\delta^{(0)}(r)\end{bmatrix}, & m = \pm 2\end{cases} \qquad [71]$$

where $\delta^{(0)}(r) = r^{-3}\delta(r - r_s)$ and $\delta^{(1)}(r) = r^{-2}\delta'(r - r_s)$ and where $\zeta = \sqrt{l(l+1)}$.

Using the spherical harmonic formalism, the vector spherical harmonic representation of the 'radial tractions' is given by

$$C_{ijkl}u_{k,l}\hat{r}_j \to \begin{cases}P = Fr^{-1}(2U - \zeta^2 V) + C\partial_r U \\ S = L(\partial_r V - r^{-1}V + r^{-1}U) \\ T = L(\partial_r W - r^{-1}W)\end{cases} \qquad [72]$$

where we have introduced traction scalars $P = P_{lm}(r)$, $S = S_{lm}(r)$, $T = T_{lm}(r)$. Because the radial tractions are required to be continuous at interfaces between different regions of the model (from [24]), it is usual to treat them as new dependent variables, and to express the derivatives $\partial_r U$, $\partial_r V$, $\partial_r W$ in terms of them:

$$\begin{aligned}\partial_r U &= -r^{-1}FC^{-1}(2U - \zeta^2 V) + C^{-1}P \\ \partial_r V &= r^{-1}(V - U) + L^{-1}S \\ \partial_r W &= r^{-1}W + L^{-1}T\end{aligned} \qquad [73]$$

The vector spherical harmonic expansion of the left side of [59] becomes

$$f_i :\to \begin{cases}-\rho\omega^2 U + 2r^{-2}(A - N)(2U - \zeta^2 V) + \rho(\partial_r g - 2r^{-1}g)U + \zeta^2 r^{-1}\rho g V - \partial_r P + 2r^{-1}F\partial_r U - 2r^{-1}P + r^{-1}\zeta^2 S + \rho\partial_r\varphi \\ -\rho\omega^2 V - r^{-2}A(2U - \zeta^2 V + 2r^{-2}N(U - V)) + r^{-1}\rho(gU + \varphi) - \partial_r S - 3r^{-1}S - r^{-1}\partial_r U \\ -\rho\omega^2 W + r^{-2}N(\zeta^2 - 2)W - \partial_r T - 3r^{-1}T\end{cases} \qquad [74]$$

Thus, ordinary differential equations in r for U, V, W, P, S, T are obtained by equating these to the forcing terms in [71]. In addition, the expansion coefficients of the perturbation in gravitational potential, $\varphi = \varphi_{lm}(r)$, are subject to equations derived from (60). The boundary conditions [24] require that φ and $\partial_r\varphi + 4\pi G\rho U$ are continuous throughout the model, and that φ vanishes at infinity. For a given spherical harmonic degree l the solutions of Laplace's equation tending to zero at infinity are proportional to r^{-l-1}, and for this reason it is useful to define the new dependent variable $\psi = \partial_r\varphi + (l+1)r^{-1}\varphi + 4\pi G\rho U$, so that ψ is continuous throughout the model and vanishes at the surface. This leads to the following coupled equations for $\varphi_{lm}(r)$, $\psi_{lm}(r)$:

$$\begin{aligned}\partial_r\varphi &= \psi - (l+1)r^{-1}\varphi - 4\pi G\rho U \\ \partial_r\psi &= 4\pi G\rho r^{-1}(\zeta^2 V - (l+1)U) + r^{-1}(l-1)\psi\end{aligned} \qquad [75]$$

Thus, the complete boundary-value problem for $\mathbf{u}(\mathbf{x},\omega)$ is to find, for each l, m, solutions $U_{lm}(r)$, $V_{lm}(r)$, $W_{lm}(r)$, $\phi_{lm}(r)$, $P_{lm}(r)$, $S_{lm}(r)$, $T_{lm}(r)$, $\psi_{lm}(r)$, satisfying (1) equality of the expressions in [71] and [74], (2) eqns [73] – in essence definitions of P, S,

T – and (3) eqns [75] governing the self-gravitation. The equations governing W and T are independent of the others, and so the problem naturally separates into the problems for the six functions U, V, ϕ, P, S, ψ, relating to 'spheroidal' motion, and for the two functions W, T, relating to 'toroidal' motion. In a fluid region, shear stresses are required to vanish, resulting in $S = 0$ and $L = N = 0, A = C = F$; the second equation in [72] drops out and the equation arising from the second of [74] can be solved for $V, V = (gU - P/\rho + \varphi)/r\omega^2$, and thus S and V can be eliminated from the equations, resulting in equations for the four remaining variables U, ϕ, P, ψ. The case $l = 0$ leads to purely radial motion, $V = 0$ with $\psi = \varphi/r, \mathrm{d}\varphi/\mathrm{d}r = -4\pi G\rho U$, and the effective equations involve only U, P. The equations are most conveniently stated as matrix differential equations, by rearranging them to give the radial derivatives of either the six, four or two functions to be determined in terms of the functions themselves. Here we define 'stress displacement vectors' as follows:

$$\text{Spheroidal solid:}\quad \mathbf{y}^{\mathrm{Ss}} = \begin{bmatrix} rU \\ r\zeta V \\ r\phi \\ rP \\ r\zeta S \\ r\psi/4\pi G \end{bmatrix}, \quad \text{Spheroidal fluid:}\ \mathbf{y}^{\mathrm{Sf}} = \begin{bmatrix} rU \\ r\phi \\ rP \\ r\psi/4\pi G \end{bmatrix}, \quad \text{Radial:}\ \mathbf{y}^{\mathrm{R}} = \begin{bmatrix} rU \\ rP \end{bmatrix}$$
$$\text{Toroidal solid:}\ \mathbf{y}^{\mathrm{Ts}} = \begin{bmatrix} r\zeta W \\ r\zeta T \end{bmatrix} \qquad [76]$$

In each case, the resulting equations take the form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}r} = \mathbf{A}\mathbf{y} + \mathbf{a}\delta(r - r_s) + \mathbf{b}\delta'(r - r_s) \qquad [77]$$

where the vectors $\mathbf{a} = \mathbf{a}(r, \omega, l, m)$, $\mathbf{b} = \mathbf{b}(r, \omega, l, m)$ can be readily derived from [71], and where the matrices $\mathbf{A} = \mathbf{A}(r, \omega, l)$ can be written in terms of submatrices in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{T} & \mathbf{K} \\ \mathbf{S} & -\mathbf{T}' \end{pmatrix} \qquad [78]$$

where $\mathbf{K}$ and $\mathbf{S}$ are symmetric and where $\mathbf{T}'$ is the transpose of $\mathbf{T}$. The fact that the equations have this special form stems from the fact that they arise from a variational principle, and in fact are a case of Hamilton's canonical equations (Woodhouse, 1974; Chapman and Woodhouse, 1981). However, the usual variational derivations of the equations of motion neglect attenuation, and so it is interesting that this symmetry of the equations remains valid in the attenuating case. It plays an important part in methods of calculation of normal modes (see below), and also enables a complex version of the theory in attenuating media to be developed along the same lines (AL-Attar, 2007). It will later be useful to introduce the partitioned matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{1}$ is the unit matrix of appropriate dimension, so that the matrix

$$\mathbf{\Sigma A} = \begin{pmatrix} \mathbf{S} & -\mathbf{T}' \\ -\mathbf{T} & -\mathbf{K} \end{pmatrix}$$

is symmetric. We shall also make use of the fact (Woodhouse, 1988) that in the non-attenuating case, $-\mathbf{\Sigma A}_\lambda$, by which we shall mean the partial derivative of $\mathbf{A}$ with respect to $\lambda = \omega^2$, is positive semidefinite, i.e. that $\mathbf{y}'(-\mathbf{\Sigma A}_\lambda)\mathbf{y} \geq 0$ for any real column $\mathbf{y}$, as can be verified using the following forms for the submatrices of $\mathbf{A}$. The specific forms of matrices $\mathbf{T}, \mathbf{K}, \mathbf{S}$ are:

$$\mathbf{T}^{\mathrm{Ss}} = \begin{pmatrix} (1 - 2F/C)/r & \zeta F/Cr & 0 \\ -\zeta/r & 2/r & 0 \\ -4\pi G\rho & 0 & -l/r \end{pmatrix}, \quad \mathbf{K}^{\mathrm{Ss}} = \begin{pmatrix} 1/C & 0 & 0 \\ 0 & 1/L & 0 \\ 0 & 0 & 4\pi G \end{pmatrix}$$
$$\mathbf{S}^{\mathrm{Ss}} = \begin{pmatrix} -\rho\omega^2 + 4(\gamma - \rho g r)/r^2 & \zeta(\rho g r - 2\gamma)/r^2 & -\rho(l+1)/r \\ \zeta(\rho g r - 2\gamma)/r^2 & -\rho\omega^2 + [\zeta^2(\gamma + N) - 2N]/r^2 & \rho\zeta/r \\ -\rho(l+1)/r & \rho\zeta/r & 0 \end{pmatrix} \qquad [79]$$

$$\mathbf{T}^{\mathrm{Sf}} = \begin{pmatrix} (g\zeta^2/\omega^2 r - 1)/r & \zeta^2/\omega^2 r^2 \\ -4\pi G\rho & -l/r \end{pmatrix}$$
$$\mathbf{K}^{\mathrm{Sf}} = \begin{pmatrix} 1/C - \zeta^2/\rho\omega^2 r^2 & 0 \\ 0 & 4\pi G \end{pmatrix} \qquad [80]$$
$$\mathbf{S}^{\mathrm{Sf}} = \begin{pmatrix} -\rho\omega^2 + \rho g \times (g\zeta^2/r\omega^2 - 4)/r & \rho(g\zeta^2/r\omega^2 - l - 1)/r \\ \rho(g\zeta^2/r\omega^2 - l - 1)/r & \rho\zeta^2/r^2\omega^2 \end{pmatrix}$$

$$\mathbf{T}^{\mathrm{R}} = ((1-2F/C)/r), \quad \mathbf{K}^{\mathrm{R}} = (1/C)$$
$$\mathbf{S}^{\mathrm{R}} = \left(-\rho\omega^2 + 4(\gamma - \rho g r)/r^2\right) \qquad [81]$$

$$\mathbf{T}^{\mathrm{Ts}} = (2/r), \ \mathbf{K}^{\mathrm{Ts}} = (1/L)$$
$$\mathbf{S}^{\mathrm{Ts}} = \left(-\rho\omega^2 + (\zeta^2 - 2)N/r^2\right) \qquad [82]$$

where $\gamma = A - N - F^2/C$.

Equation [77] leads to solutions $\mathbf{y}(l, m, r)$ which are discontinuous at the source radius r_s. It can be shown that the discontinuity at r_s is given by (Hudson, 1969; Ward, 1980)

$$[\mathbf{y}]_{r=r_s^-}^{r=r_s^+} = \mathbf{s} = \mathbf{a} + \mathbf{A}\mathbf{b} - \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}r} \qquad [83]$$

and thus the boundary-value problem for $\mathbf{y}$ requires the solution of the homogeneous equation $\mathrm{d}\mathbf{y}/\mathrm{d}r = \mathbf{A}\mathbf{y}$ above and below the source, subject to the conditions that the solution (1) is nonsingular at the center of the Earth, (2) has vanishing traction components at the surface, and (3) has the prescribed discontinuity $\mathbf{s}$ at the source radius r_s. The specific forms for the discontinuity vector $\mathbf{s}$, using [71] and [83], are

$$\mathbf{s}^{\mathrm{Ss}} = \frac{\nu_l}{\mathrm{i}\omega r_s^2} \times \begin{cases} \begin{bmatrix} r_s M_{rr}/C \\ 0 \\ 0 \\ -r_s F_r - M_{\theta\theta} - M_{\phi\phi} + 2M_{rr}F/C \\ \zeta(M_{\theta\theta} + M_{\phi\phi})/2 - \zeta M_{rr}F/C \\ 0 \end{bmatrix}, & l \geq 1, m = 0 \\ \begin{bmatrix} 0 \\ r_s(\mp M_{\theta r} + \mathrm{i}M_{\phi r})/2L \\ 0 \\ \zeta(\mp M_{\theta r} + \mathrm{i}M_{\phi r} \pm M_{r\theta} - \mathrm{i}M_{r\phi})/2 \\ r_s(\pm F_\theta - \mathrm{i}F_\phi)/2 - (\mp M_{\theta r} + \mathrm{i}M_{\phi r} \pm M_{r\theta} - \mathrm{i}M_{r\phi})/2 \\ 0 \end{bmatrix}, & l \geq 1, m = \pm 1 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\zeta^2 - 2}(-M_{\theta\theta} + M_{\phi\phi} \pm \mathrm{i}M_{\theta\phi} \pm \mathrm{i}M_{\phi\theta})/4 \\ 0 \end{bmatrix}, & l \geq 2, m = \pm 2 \end{cases} \qquad [84]$$

$$\mathbf{s}^{\mathrm{R}} = \frac{\nu_0}{\mathrm{i}\omega r_s^2} \times \begin{bmatrix} r_s M_{rr}/C \\ -r_s F_r - M_{\theta\theta} - M_{\phi\phi} + 2M_{rr}F/C \end{bmatrix}, \quad l = m = 0 \qquad [85]$$

$$\mathbf{s}^{\mathrm{Ts}} = \frac{\nu l}{\mathrm{i}\omega r_s^2} \times \begin{cases} \begin{bmatrix} 0 \\ \zeta\left(M_{\theta\phi} - M_{\phi\theta}\right)/2 \end{bmatrix}, & l \geq 1, m = 0 \\ \begin{bmatrix} r_s\left(\pm M_{\phi r} + \mathrm{i}M_{\theta r}\right)/2L \\ r_s(\mp F_\phi - \mathrm{i}F_\theta)/2 + (\mp M_{\phi r} - \mathrm{i}M_{\theta r} \pm M_{r\phi} + \mathrm{i}M_{r\theta})/2 \end{bmatrix}, & l \geq 1, m = \pm 1 \\ \begin{bmatrix} 0 \\ \sqrt{\zeta^2 - 2}(\pm \mathrm{i}M_{\theta\theta} \mp iM_{\phi\phi} + M_{\theta\phi} + M_{\phi\theta})/4 \end{bmatrix}, & l \geq 2, m = \pm 2 \end{cases} \qquad [86]$$

where the elastic parameters are those evaluated at the source radius r_s. We consider here only the case that the source is located in a continuous, solid region of the model. Equivalent results (in the case that the moment tensor is symmetric and the material is isotropic) are given by Ward (1980). (As described earlier we are here considering the source to be located at $(\theta = \varepsilon, \phi = 0)$, for some infinitesimal, positive ε; thus, $\left(F_\theta, F_\phi, F_r\right)$ coincide with $\left(F_x, F_y, F_z\right)$ in the global Cartesian frame defined in [36]. Similarly, $\left(M_{\theta\theta}, M_{\theta\phi}, \text{etc.}\right)$ coincide with $\left(M_{xx}, M_{xy}, \text{etc.}\right)$ If the source is located at a general colatitude θ_s and longitude ϕ_s, the results can be applied in a rotated frame in which the (θ, ϕ, r) components map into (South, East, Up), coordinate θ is epicentral distance and coordinate ϕ is azimuth of the receiver at the source, measured anticlockwise from South.)

In the case that there is no forcing f_i, the equations reduce to the homogeneous system $\mathrm{d}\mathbf{y}/\mathrm{d}r = \mathbf{A}\mathbf{y}$, together with the usual boundary conditions at the center of the Earth and at the surface. This is an eigenvalue problem for ω having solutions corresponding to the modes of 'free oscillation'. The eigenvalues will be denoted by ω_k, where k is an index that incorporates the angular order l, the 'overtone number' n, and the mode type: spheroidal or toroidal. Overtone number is an index labeling the eigenfrequencies for a given l and for a given mode type, in increasing order. Since the spherical harmonic order m does not enter into the equations, the modes are 'degenerate', meaning that there are $2l+1$ different eigenfunctions, $m = -l, -l+1, \ldots, l$ corresponding to a given eigenvalue ω_k. The eigenfunctions will be denoted by $\mathbf{s}^{(km)}(\mathbf{x})$. These are the solutions $\mathbf{u}(\mathbf{x})$ given by [69], for different values of m, but with the 'same' scalar eigenfuctions $U(r)$, $V(r)$, $W(r)$, $\varphi(r)$. The set of eigenfuctions, $\mathbf{s}^{(km)}(\mathbf{x})$, for a given eigenfrequency ω_k is said to constitute a 'multiplet'. The eigenfunctions represent the spatial shape of a mode of free oscillation at frequency ω_k, because $\mathbf{s}^{(km)}(\mathbf{x})\mathrm{e}^{i\omega_k t}$ is a solution of the complete dynamical equations in the absence of any forcing. Of course the eigenfunction is defined only up to an overall factor. If the medium is attenuating, the eigenfrequencies will be complex, their (positive) imaginary parts determining the rate of decay of the mode with time. It is conventional to quantify this decay rate in terms of the 'Q' of the multiplet, Q_k, which is defined in such a way that the mode decays in amplitude by a fator $\mathrm{e}^{-\pi/Q_k}$ per cycle. Therefore, $Q_k = \mathrm{Re}\omega_k/2\mathrm{Im}\omega_k$ is typically a large number, indicating that the modes decay by a relatively small fraction in each cycle. The modal multiplets are conventionally given the names ${}_nS_l$ for spheroidals and ${}_nT_l$ for toroidals.

1.02.5 Numerical Solution

The inhomogeneous (i.e., with forcing term f_i) boundary-value problem as formulated above gives a unique solution for each value of ω. The solution in the time domain can then be obtained in the form of an integral in ω, using the inverse Fourier transform [27]. This, in essence, is the basis of several practical methods for calculating theoretical seismogram, for example, the 'reflectivity method' (Fuchs and Muller, 1971) and the direct solution method of Friederich and Dalkolmo (1995). Alternatively, the inverse transform can be evaluated by completing the integration contour in the upper half of the complex ω plane. Then it is found that the solution can be reduced to a sum over residues, each pole of the integrand corresponding to a particular mode of 'free oscillation' of the model. A more usual approach to the normal mode problem is to consider first the free (i.e., unforced) modal solutions of the equations, and then to make use of the orthogonality and completeness properties of the eigenfunctions to obtain solutions of the inhomogeneous (i.e., forced) problem. Here, we have examined first the inhomogenous problem because the demonstration of orthogonality and completeness in the general (attenuating) case is a nontrivial issue, and it is only by virtue of the analysis of the inhomogeneous problem, and the resulting analytic structure of the integrand in the complex ω-plane, that orthogonality and completeness can be demonstrated. It is necessary to show (in the nonattenuating case) that the only singularities of the integrand are simple poles on the real ω-axis. Then the modal sum emerges and completeness is demonstrated by the solution itself. In the attenuating case the situation is more complex, and there are other singularities located on the positive imaginary ω-axis – let us call them the relaxation singularities, as they are associated with decaying exponential functions in the time domain. Thus, while the solution developed here for the inhomogeneous problem remains valid in this case, arguments based on orthogonality and completeness cannot be made. The solution can nevertheless be derived in the form of a sum over residues, and other singularities. While the contribution from relaxation singularities is the main focus of attention in the analysis of postseismic relaxation, they are expected to make negligible contributions for the typical seismic application. However, even in the seismic domain it is necessary to know modal excitations in the attenuating case, and these are difficult to determine, other than by a rather complex application of mode-coupling theory (Tromp and Dahlen, 1990; Lognonne, 1991), which will be difficult to carry out to high frequencies. Using the inhomogenous solution, on the other hand, the 'seismic' modes and their excitations emerge naturally as the contributions from the residues of poles near the real axis, and can be calculated exactly and economically.

In both the attenuating and nonattenuating cases and for both the homogenous and inhomogeneous problems, the integration of the ordinary differential equation presents severe numerical difficulties. One problem is that the equations are such that evanescent – exponentially increasing and decreasing – solutions exist on more than one spatial scale. At moderately high frequencies, when the equations are integrated numerically, the solutions are effectively projected onto the solution having the most rapid exponential increase, and thus even though a linearly independent set of solutions is guaranteed, theoretically, to remain a linearly independent set, it becomes, numerically, a one-dimensional projection. The general solution of this difficulty is to reformulate the equations in terms of minors (i.e., subdeterminants) of sets of solutions (Gilbert and Backus, 1966). The standard method for normal mode calculations (Woodhouse, 1988) is based on the minor formulation of the equations, and uses a novel generalization of Sturm–Liouville theory to bracket modal frequencies, itself a nontrivial issue for the spheroidal modes, as the modes are irregularly and sometimes very closely spaced in frequency, making an exhaustive search difficult and computationally expensive. The program MINOS of Guy Masters, developed from earlier programs of Gilbert, and of Woodhouse (also a development of Gilbert's earlier programs), implements this method, and has been generously made available to the community. The direct solution method for the inhomogeneous problem of Friederich and Dalkolmo (1995) is based on the minor formulation in the non-self-gravitating case, developed for the flat-earth problem by Woodhouse (1980b).

Here we outline some of the key features of the minor approach. Let us be specific by assuming that the model has a solid inner core, a fluid outer core, and a solid mantle. It may, or may not have an ocean. Let us also consider the case of spheroidal oscillations, for which the solution vector is six dimensional in solid regions ($\mathbf{y}^{\mathrm{Ss}}$) and 4-dimensional in fluid regions ($\mathbf{y}^{\mathrm{Sf}}$). Toroidal an radial modes are much simpler. The basic method of solution is to start at the center of the Earth, and to specify that the solution should be nonsingular there. By assuming that the medium is homogeneous and isotropic within a small sphere at the center, it is possible to make use of known analytical solutions in terms of the spherical Bessel functions (Love, 1911; Pekeris and Jarosch, 1958; Takeuchi and Saito, 1972). Thus (in the spheroidal case that we are considering), there is a three-dimensional set of solutions to be regarded as candidates for components of the solution at the center. Using these three solutions as starting solutions, the equations can be integrated toward the surface, for example, using Runge–Kutta techniques. We introduce a 6×3 matrix $\mathbf{Y} = \mathbf{Y}^{\mathrm{Ss}}(r)$ having colums equal to these three solutions, which has 3×3 subpartitions $\mathbf{Q} = \mathbf{Q}^{\mathrm{Ss}}(r)$ and $\mathbf{P} = \mathbf{P}^{\mathrm{Ss}}(r)$, that is,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{P} \end{bmatrix}$$

What is important about $\mathbf{Y}$ is the subspace of six-dimensional space that is spanned by its three columns, a property that is left unchanged if $\mathbf{Y}$ is postmultiplied by any nonsingular 3×3 matrix. Assuming, for the moment, that $\mathbf{Q}$ and $\mathbf{P}$ are nonsingular, and postmultiplying by $\mathbf{Q}^{-1}$ and by $\mathbf{P}^{-1}$ we conclude that both $\begin{bmatrix} \mathbf{1} \\ \mathbf{U} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{V} \\ \mathbf{1} \end{bmatrix}$, where $\mathbf{V}$ and $\mathbf{U}$ are the mutually inverse matrices $\mathbf{V} = \mathbf{Q}\mathbf{P}^{-1}$, $\mathbf{U} = \mathbf{P}\mathbf{Q}^{-1}$ and where $\mathbf{1}$ is the unit matrix, have columns spanning the same three dimensional space as is spanned by the columns of $\mathbf{Y}$. An unexpected property of $\mathbf{U}$, $\mathbf{V}$, stemming from the self-adjointness property of the equations and boundary conditions, is that they are 'symmetric'. It is not difficult to show that by virtue of the particular structure of the differential equations noted in [78] that if $\mathbf{U}$, and (therefore) $\mathbf{V}$ are symmetric at a given radius, then they remain symmetric as the equations are integrated to other radii. To demonstrate this, consider

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}(\mathbf{Q}'\mathbf{P} - \mathbf{P}'\mathbf{Q}) &= \frac{\mathrm{d}}{\mathrm{d}r}(\mathbf{Y}'\boldsymbol{\Sigma}\mathbf{Y}) \\ &= \mathbf{Y}'\mathbf{A}'\boldsymbol{\Sigma}\mathbf{Y} + \mathbf{Y}'\boldsymbol{\Sigma}\mathbf{A}\mathbf{Y} = 0 \end{aligned} \qquad [87]$$

where we have used the fact that $\boldsymbol{\Sigma}\mathbf{A}$ is symmetric and $\boldsymbol{\Sigma}$ is antisymmetric (see discussion following eqn [78]. Thus, if $\mathbf{Q}'\mathbf{P} - \mathbf{P}'\mathbf{Q}$ vanishes at a given radius, it vanishes everywhere in the interval over which the equations are being integrated. But we can write $\mathbf{Q}'\mathbf{P} - \mathbf{P}'\mathbf{Q} = \mathbf{Q}'(\mathbf{P}\mathbf{Q}^{-1} - \mathbf{Q}'^{-1}\mathbf{P}')\mathbf{Q}$, which shows that $\mathbf{P}\mathbf{Q}^{-1}$ is symmetric if $\mathbf{Q}'\mathbf{P} - \mathbf{P}'\mathbf{Q}$ vanishes. It is interesting to note that this argument does not rely on $\mathbf{Q}$ and $\mathbf{P}$ being nonsingular throughout the interval of integration, since $\mathbf{Q}'\mathbf{P} - \mathbf{P}'\mathbf{Q}$ remains finite and continuous. It is a lengthy algebraic exercise to show that $\mathbf{U}$ and $\mathbf{V}$ are symmetric at the centre of the Earth (i.e., when the analytic solutions are used), but nevertheless this can be verified (it can be easily checked numerically).

To continue the narrative, the equations for $\mathbf{Y}^{\mathrm{IC}}$ are being integrated in the inner core, and we arrive at the inner-core boundary. Here, the component of the solution corresponding to the shear traction on the boundary (the fifth element in our notation) is required to vanish. Thus, at the boundary we need to select from the three-dimensional space spanned by the columns of $\mathbf{Y}$ the (in general) two-dimensional subspace of stress–displacement vectors having vanishing fifth elements. This subspace is most easily identified by considering the basis constituted by $\begin{bmatrix} \mathbf{V} \\ \mathbf{1} \end{bmatrix}$, since its first and third columns have vanishing fifth elements, and thus by deleting the middle column, together with the second and fifth rows, as they correspond to variables not needed on the fluid side, we obtain the following rule for transmitting the basis of allowable solutions from the solid to the fluid side of the boundary:

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} v_{11} & v_{13} \\ v_{13} & v_{33} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [88]$$

We can now continue the integration, using the 4×2 matrix $\mathbf{Y}^{\mathrm{OC}}$ in the fluid outer core. We can define $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{V}$, $\mathbf{U}$ similarly, now 2×2, rather than 3×3 matrices, and again $\mathbf{V}$, $\mathbf{U}$, are symmetric. Continuing the integration, we arrive at the outer-core boundary, and again need to consider how to transmit the solution space across the boundary. Elements in rows $1, 2, 3, 4$ on the fluid side need to be continuous with elements in rows $1, 3, 4, 6$ on the solid side. The fifth element on the solid side, the shear traction component $r\zeta S$, has to vanish. Since the horizontal displacement can be anything on the solid side we have to add to the basis to represent solutions having nonvanishing horizontal displacements on the solid side. The easiest way to satisfy these requirements is to consider the basis represented by $\begin{bmatrix} \mathbf{1} \\ \mathbf{U} \end{bmatrix}$. It can be easily verified that the following rule satisfies the requirements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_{11} & 0 & u_{12} \\ 0 & 0 & 0 \\ u_{12} & 0 & u_{22} \end{pmatrix} \qquad [89]$$

The middle column has been inserted to represent the fact that the solution space has to contain vectors with nonvanishing horizontal displacement scalar $(r\zeta V)$ at the base of the mantle.

We can now continue the integration through the mantle. In the inhomogenous problem, the next interesting event is when we arrive at the source radius r_s. In the homogeneous problem, on the other hand we can continue the integrations to the ocean floor, applying the same rule as at the inner-core boundary to transmit the solution space into the ocean, and then continue to the surface, where the free surface condition needs to be satisfied. The requirement, of course, is that there be a linear combination of the columns of the 4×2 matrix

$$\mathbf{Y}^{\mathrm{O}} = \begin{bmatrix} \mathbf{Q}^{\mathrm{O}} \\ \mathbf{P}^{\mathrm{O}} \end{bmatrix}$$

(where superscript 'O' is for ocean) that have vanishing surface traction scalar rP and vanishing gravity scalar $r\psi/4\pi G$, that is, vanishing elements 3 and 4. This requires $\det(\mathbf{P}^{\mathrm{O}}) = 0$ at the free surface. In the absence of an ocean, we similarly need $\det(\mathbf{P}^{\mathrm{C}}) = 0$ where $\mathbf{P}^{\mathrm{C}}$ is the 3×3 matrix in the crust.

In the inhomogeneous case, we need to arrange for there to be a prescribed discontinuity $\mathbf{s}^{\mathrm{Ss}}$ at the source depth. Thus, we need to characterize the solution space above, as well as below the source. At the surface of the ocean the solution must have vanishing elements 3 and 4, and thus we choose $\mathbf{Y}^{\mathrm{O}}$ at the free surface to be, simply

$$\mathbf{Y}^{\mathrm{O}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

Now we can integrate the solution downwards, using the same fluid–solid rule at the ocean floor as was employed at the core–mantle boundary, until we reach the source radius r_s from above. If $\begin{bmatrix} \mathbf{1} \\ \mathbf{U}_1^{\mathrm{s}} \end{bmatrix}$ spans the solution space below the source and $\begin{bmatrix} \mathbf{1} \\ \mathbf{U}_2^{\mathrm{s}} \end{bmatrix}$ spans the solution space above the source, we need to solve

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{U}_2^{\mathrm{s}} \end{bmatrix} \mathbf{x}_2 - \begin{bmatrix} \mathbf{1} \\ \mathbf{U}_1^{\mathrm{s}} \end{bmatrix} \mathbf{x}_1 = \mathbf{s}^{\mathrm{Ss}} \qquad [90]$$

for the two 3-vectors $\mathbf{x}_1$, $\mathbf{x}_2$ which represent the multipliers for the columns of each matrix that are needed to satisfy the condition that the solution has the prescribed discontinuity. The solution is easily found to be

$$\begin{aligned} \mathbf{x}_1 &= (\mathbf{U}_2 - \mathbf{U}_1)^{-1}(\mathbf{s}_P^{\mathrm{Ss}} - \mathbf{U}_2\mathbf{s}_Q^{\mathrm{Ss}}) \\ \mathbf{x}_2 &= (\mathbf{U}_2 - \mathbf{U}_1)^{-1}(\mathbf{s}_P^{\mathrm{Ss}} - \mathbf{U}_1\mathbf{s}_Q^{\mathrm{Ss}}) \end{aligned} \qquad [91]$$

where $\mathbf{s}_Q^{\mathrm{Ss}}$, $\mathbf{s}_P^{\mathrm{Ss}}$ are the upper and lower halves of $\mathbf{s}^{\mathrm{Ss}}$, and the solution vectors below and above the source are given by

$$\mathbf{y}_1^{\mathrm{Ss}} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{U}_1\mathbf{x}_1 \end{bmatrix}, \quad \mathbf{y}_2^{\mathrm{Ss}} = \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{U}_2\mathbf{x}_2 \end{bmatrix} \qquad [92]$$

This determines the linear combination of the basis vectors that are needed to satisfy the source discontinuity condition, and hence to determine the solution at any point of the medium and, in particular, at the surface, where it may be required to calculate some seismograms.

There will be singularities in the integrand of the inverse Fourier transform when $\mathbf{U}_2 - \mathbf{U}_1$ is singular. This will occur for frequencies ω for which a solution exists to the homogenous (i.e., unforced) problem, that is, at the frequencies of free oscillation. If $\mathbf{U}_2 - \mathbf{U}_1$ is singular at a particular source radius, it is, therefore, necessarily singular at all radii. To evaluate the inverse transform as a sum over residues, we can write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} I(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \rightarrow \sum_k \mathrm{i}\frac{\lim_{\omega\to\omega_k} \Delta(\omega)I(\omega)}{\Delta'(\omega_k)} \qquad [93]$$

where ω_k is a mode frequency and $\Delta(\omega)$ is that factor in the denominator of the integrand $I(\omega)$ that vanishes at ω_k, assuming that it has a simple zero at ω_k. Thus, we can replace the inverse transform by a sum over residues provided that the singular part of the integrand is replaced by the expression corresponding to it on the right-hand side of [93]. From [91] we find that the necessary replacement is

$$(\mathbf{U}_2 - \mathbf{U}_1)^{-1} \rightarrow \mathrm{i}\left.\frac{\mathrm{adj}(\mathbf{U}_2 - \mathbf{U}_1)}{\partial_\omega \det(\mathbf{U}_2 - \mathbf{U}_1)}\right|_{\omega=\omega_k} = \frac{1}{2\mathrm{i}\omega_k}\mathbf{z}_k\mathbf{z}_k' \qquad [94]$$

where adj represents matrix adjoint – the matrix of cofactors. The second equality defines the column $\mathbf{z}_k$ and its transpose $\mathbf{z}_k'$, and arises from the fact that the adjoint of a singular matrix, assuming that the rank defect is 1, is expressible as a dyad; the factor $-1/2\omega_k$ is included in the definition for convenience, as with this definition of $\mathbf{z}_k$, it can be shown that the column

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{z}_k \\ \mathbf{U}_1\mathbf{z}_k \end{bmatrix} = \begin{bmatrix} \mathbf{z}_k \\ \mathbf{U}_2\mathbf{z}_k \end{bmatrix}$$

is an eigenfunction (i.e., a solution of $\mathrm{d}\mathbf{y}/\mathrm{d}r = \mathbf{A}\mathbf{y}$) and has normalization

$$\int_0^a \mathbf{y}_k'(-\boldsymbol{\Sigma}\mathbf{A}_\lambda)\mathbf{y}_k\,\mathrm{d}r = 1 \qquad [95]$$

where the notation $-\boldsymbol{\Sigma}\mathbf{A}_\lambda$ is that introduced in the discussion following [78]. The eigenfunctions can be found without needing to calculate the derivatives of solutions with respect to ω, as it can be shown that $\det \mathbf{Q}_1 \det \mathbf{Q}_2 \partial_\omega \det(\mathbf{U}_2 - \mathbf{U}_1)$ is independent of r.

This is the basis for the construction of the eigenfunctions from solutions of the minor equations, although it was arrived at differently in Woodhouse (1988). We see that it is necessary to integrate both upwards and downwards in order to obtain $\mathbf{U}_1$ and $\mathbf{U}_2$ (or, rather, the minors from which they can be constructed, see below) at all radii r.

In the nonattenuating case, the eigenfunctions $\mathbf{y}_k$ are real, and [95] reduces to the standard normalization conditions for the scalar eigenfunctions U_k, V_k, W_k (defined in terms of $\mathbf{y}_k$ as in eqn [76],

$$\begin{aligned} \text{spheroidal:} \quad & \int_0^a \rho\left(U_k^2 + \zeta^2 V_k^2\right) r^2 \mathrm{d}r = 1 \\ \text{toroidal:} \quad & \int_0^a \rho\zeta^2 W_k^2 r^2 \mathrm{d}r \end{aligned} \qquad [96]$$

In the attenuating case, on the other hand, the eigenfunctions are complex and the normalization condition [95] includes terms arising from the derivatives of the elastic parameters with respect to ω. In this case [95] determines both the phase and the amplitude of the eigenfunction. Using the replacement [94] in [91], and making use of the definitions of $\mathbf{s}^{\mathrm{Ss}}$, and similarly $\mathbf{s}^{\mathrm{Ts}}$, from [84]–[86] it can be shown that the inhomogeneous solution, now in the time domain, can be written as a sum over residues:

$$\mathbf{y}_{lm}(r,t) = \sum_k -\frac{1}{2\omega_k^2} E_{km} \mathbf{y}_k(r) \mathrm{e}^{\mathrm{i}\omega_k t} \qquad [97]$$

where 'modal excitations' E_{km} are given by

$$E_{km} = \begin{cases} \mathbf{y}_k' \sum \mathbf{s}^{\mathrm{Ss}} \text{ spheroidal} \\ \mathbf{y}_k' \sum \mathbf{s}^{\mathrm{Ts}} \text{ toroidal} \end{cases}$$

$$= \frac{\nu_l}{r_s} \begin{cases} -r_s U_k F_r - r_s \partial_r U_k M_{rr} - U_k(M_{\theta\theta} + M_{\phi\phi}) + \frac{1}{2}\zeta^2 V_k (M_{\theta\theta} + M_{\phi\phi}) \\ \quad + \frac{1}{2}\zeta^2 W_k (M_{\theta\phi} - M_{\phi\theta}) & m = 0 \\ \frac{1}{2}\zeta r_s (V_k \mp \mathrm{i}W_k)(\pm F_\theta - \mathrm{i}F_\phi) + \frac{1}{2}\zeta r_s (\partial_r V_k \mp \mathrm{i}\partial_r W_k)(\pm M_{\theta r} - \mathrm{i}M_{\phi r}) \\ \quad + \frac{1}{2}\zeta (U_k - V_k \pm \mathrm{i}W_k)(\pm M_{r\theta} - \mathrm{i}M_{r\phi}) & l \geq 1, m = \pm 1 \\ \frac{1}{4}\zeta\sqrt{\zeta^2 - 2}(V_k \mp \mathrm{i}W_k)(M_{\phi\phi} - M_{\theta\theta} \pm \mathrm{i}M_{\theta\phi} \pm \mathrm{i}M_{\phi\theta}) & l \geq 2, m = \pm 2 \end{cases} \qquad [98]$$

where [72] has been used to express the radial traction components P_k, S_k, T_k in $\mathbf{y}_k$ in terms of U_k, V_k, W_k and their derivatives. The eigenfunctions are those evaluated at the source radius r_s. In [98] we have combined the results for toroidal and spheroidal modes; of course, for spheroidal modes $W_k = 0$ and for toroidal modes $U_k = V_k = 0$. This result for the modal excitations is equivalent, in the case of a symmetric moment tensor to the forms given in table 1 of Woodhouse and Girnius (1982).

The sum over residues [97] needs to be carried out over all simple poles in the upper half of the complex ω-plane. It will include the oscillatory 'seismic modes' having $\mathrm{Re}\,\omega_k \neq 0$, and also, possibly, 'relaxation modes' having $\mathrm{Re}\,\omega_k = 0$, $\mathrm{Im}\,\omega_k > 0$. The seismic modes will occur in pairs, ω_k, $-\omega_k^*$ because the equations are symmetric under replacement of ω_k by $-\omega_k^*$ followed by complex conjugation (see discussion following eqn [34]). Thus, each seismic mode having $\mathrm{Re}\,\omega_k > 0$ will have a partner, obtained by reflection in the imaginary axis, at $-\omega_k^*$. It is not necessary to consider the modes for which $\mathrm{Re}\,\omega_k < 0$ explicitly, as when the final result for the displacement in the time domain is calculated, it is possible to include them automatically by adding the complex conjugate, in order that the final result should be real. In the attenuating case, the solution is still not necessarily complete, as the constitutive law may introduce a branch cut along the positive imaginary axis, corresponding to a continuous distribution of relaxation mechanisms. In this case, the sum over residues needs to be augmented by an integral around any branch-cut singularities on the positive imaginary axis. In order to include the 'static' response of the medium, it is necessary to analyze the behavior of the integrand at $\omega = 0$. We do not pursue this in detail here but make some general observations. Because the source that is being considered has, in general, nonvanishing net force and net moment, we would obtain secular terms corresponding to translational and rotational rigid motions (degree $l = 1$ spheroidal and toroidal modes having zero frequency). If the force is set to zero and the moment tensor is taken to be symmetric, these modes would not be excited. In this case, the final displacement field, after all seismic modes and relaxation modes have died away, can be found from the residue at zero frequency, and will correspond to the static ($\omega = 0$) solution of the equations in which the elastic parameters are replaced by their values at zero frequency – their so-called 'relaxed' values. Alternatively, the final displacement can be found by considering the fact that if the static terms are omitted the final displacement is zero, because all modes attenuate with time, whereas, in fact, it is the 'initial' displacement that should be zero. Thus, the static terms must be such as to cancel the dynamic terms at zero time. It is not obvious that these two different ways of evaluating the static response will agree (i.e., using the static solution for relaxed values of the moduli, or using the fact that the initial displacement must be zero). We conjecture, but do not claim to prove that these two methods will give the same result. If this is so, it means that, provided all modes are included in the sum, we can include the static response by substituting $e^{i\omega_k t} - 1$ for $e^{i\omega_k t}$ in [97] – that is, by subtracting the value at zero time.

A central role is played in the foregoing theory by the symmetric matrices $\mathbf{U}$ and $\mathbf{V}$, notwithstanding that they possess singularities within the domain of integration. From their definitions $\mathbf{V} = \mathbf{QP}^{-1}$, $\mathbf{U} = \mathbf{PQ}^{-1}$, using the following standard formula for the inverse of a matrix in terms of its cofactors and its determinant, they can be expressed in terms of the various 3×3 or (in the fluid case) or 2×2 subdeterminants of $\mathbf{Y}$. Explicitly, we have

$$\mathbf{U} = \frac{m_2}{m_1}, \quad \mathbf{V} = \frac{m_1}{m_2} \quad (n = 1) \tag{99}$$

$$\mathbf{U} = \frac{1}{m_1}\begin{pmatrix} -m_4 & m_2 \\ m_2 & m_3 \end{pmatrix}, \quad \mathbf{V} = \frac{1}{m_6}\begin{pmatrix} m_3 & -m_2 \\ -m_2 & -m_4 \end{pmatrix}$$
$$m_1 = \det \mathbf{Q}, \quad m_6 = \det \mathbf{P} \quad (n = 2) \tag{100}$$

$$\mathbf{U} = \frac{1}{m_1}\begin{pmatrix} m_{11} & -m_5 & m_2 \\ -m_5 & -m_6 & m_3 \\ m_2 & m_3 & m_4 \end{pmatrix}, \quad \mathbf{V} = \frac{1}{m_{20}}\begin{pmatrix} m_{10} & -m_9 & m_8 \\ -m_9 & -m_{15} & m_{14} \\ m_8 & m_{14} & m_{17} \end{pmatrix}$$
$$m_1 = \det \mathbf{Q}, \quad m_{20} = \det \mathbf{P} \quad (n = 3) \tag{101}$$

where m_k are the minors of the relevant 2×1, 4×2, 3×6 ($n = 1$, 2 or 3) solution matrices $\mathbf{Y}$. We are using a standard way of enumerating these (see Woodhouse, (1988) and Gilbert and Backus, (1966)). We include the $n = 1$ case here, which is relevant to the case of toroidal and radial oscillations, for the sake of completeness. The results still hold in this case, although the matrices $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{V}$, $\mathbf{U}$ reduce in this case to simple numbers, and the minors reduce to the elements of the solution vector itself (the 1×1 subdeterminants of a 2×1 matrix $\mathbf{Y}$). It is well known (Gilbert and Backus, 1966) that differential equations can be derived that are satisfied by the minors, and thus they can be calculated directly, without the need to integrate the equations for particular solution sets $\mathbf{Y}$. Thus, formulas and results involving $\mathbf{V}$, $\mathbf{U}$ can be readily transcribed into formulas involving the minors. Essentially, any one of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{m}$ provides a way of characterizing a subspace of the $2n$-dimensional space of interest ($n = 1$, 2, or 3), in a way that is independent of any specific basis. However, the minors have the practical advantage that they do not become infinite in the domain of integration.

The matrices $\mathbf{U}$ and $\mathbf{V}$ possess another remarkable property which results from the positive semidefiniteness of $-\boldsymbol{\Sigma}\mathbf{A}_\lambda$ in the nonattenuating case. Using this property, it is possible to show that the derivatives $\mathbf{U}_\lambda$ and $\mathbf{V}_\lambda$ also have definiteness properties. For upward integration $\mathbf{U}_\lambda \leq 0$ and $\mathbf{V}_\lambda \geq 0$ where ≥ 0 and ≤ 0 is used as a a shorthand for the relevant semidefiniteness property. To prove this, consider the matrix $\mathbf{P}'\mathbf{Q}_\lambda - \mathbf{Q}'\mathbf{P}_\lambda = -\mathbf{Y}'\boldsymbol{\Sigma}\mathbf{Y}_\lambda$. The radial derivative of this matrix is given by

$$\begin{aligned}\frac{d}{dr}(\mathbf{P}'\mathbf{Q}_\lambda - \mathbf{Q}'\mathbf{P}_\lambda) &= -\frac{d}{dr}(\mathbf{Y}'\boldsymbol{\Sigma}\mathbf{Y}_\lambda) \\ &= -\mathbf{Y}'\mathbf{A}'\boldsymbol{\Sigma}\mathbf{Y}_\lambda - \mathbf{Y}'\boldsymbol{\Sigma}(\mathbf{A}_\lambda\mathbf{Y} + \mathbf{A}\mathbf{Y}_\lambda) \\ &= -\mathbf{Y}'\boldsymbol{\Sigma}\mathbf{A}_\lambda\mathbf{Y} \geq 0\end{aligned} \tag{102}$$

where we have used $d\mathbf{Y}/dr = \mathbf{AY}$, together with its transpose and its derivative with respect to λ. The cancellation of the terms not involving $\mathbf{A}_\lambda$ is due to the symmetry of $\boldsymbol{\Sigma}\mathbf{A}$ and the antisymmetry of $\boldsymbol{\Sigma}$. We also have

$$\begin{aligned}\mathbf{V}_\lambda = \left(\mathbf{QP}^{-1}\right)_\lambda &= \mathbf{Q}_\lambda\mathbf{P}^{-1} - \mathbf{QP}^{-1}\mathbf{P}_\lambda\mathbf{P}^{-1} \\ &= \mathbf{Q}_\lambda\mathbf{P}^{-1} - \mathbf{P}'^{-1}\mathbf{Q}'\mathbf{P}_\lambda\mathbf{P}^{-1} \\ &= \mathbf{P}'^{-1}(\mathbf{P}'\mathbf{Q}_\lambda - \mathbf{Q}'\mathbf{P}_\lambda)\mathbf{P}^{-1}\end{aligned} \tag{103}$$

Thus, [102] shows that if $\mathbf{P}'\mathbf{Q}_\lambda - \mathbf{Q}'\mathbf{P}_\lambda \geq 0$ at some initial point, then it remains positive semidefinite during upward integration. Then, from [103], $\mathbf{V}_\lambda \geq 0$, as we wished to show. Using the analytic solutions at the center of the Earth, it can be shown that $\mathbf{V}_\lambda$ does have the required properties at the starting point of integration, being independent of frequency at the center of the Earth. Also, its semidefiniteness property is preserved on passing from solid to fluid and vice versa. Similarly, it can be shown that for upward integration $\mathbf{U}_\lambda \leqslant 0$.

The diagonal elements of $\mathbf{V}_\lambda$ and $\mathbf{U}_\lambda$, which necessarily share the semidefiniteness properties of the matrices themselves, require, for upward integration, that the diagonal elements of $\mathbf{V}$ and $\mathbf{U}$ are nondecreasing and nonincreasing functions of frequency, respectively. As a function of frequency these diagonal elements behave like the familiar tangent and cotangent functions, having monotonic increase or decrease between their singularities. The singularities in $\mathbf{V}$ at the surface are of particular interest, since the frequencies for which $\mathbf{V}$ is singular (i.e., infinite) at the surface are precisely the frequencies of the normal modes. One particular diagonal element, namely v_{11} in the notation used here, has the additional property, using [88], [89], that it is continuous at solid–fluid and fluid–solid boundaries. In the case of fluid–solid boundaries, this is not so obvious, as both m_{10} and m_{20} vanish on the solid side ($v_{11} = m_{10}/m_{20}$, eqn [101]), but it can be shown that the limit as the boundary is approached from the solid side is, in fact, equal to v_{11} on the fluid side.

The function $\theta_R(r,\lambda) = -\frac{1}{\pi}\cot^{-1}(v_{11})$, which can be made continuous (as a function of r and as a function of λ) through singularities of v_{11}, has the properties that (1) it is independent of frequency at the center of the Earth, (2) it is nondecreasing as a function of λ, and (3) it takes on integer values at the surface at the frequencies $\omega^2 = \lambda$ corresponding the normal modes. This makes it an ideal mode counter, since two integrations of the equations, at frequencies ω_1, ω_2 say, can determine the values $\theta_R(a,\omega_1^2), \theta_R(a,\omega_2^2)$ at the Earth's surface and it is necessary only to find how many integers lie between these values to determine how many modal frequencies lie between ω_1 and ω_2. There is a complication associated with fluid–solid boundaries. As discussed above, both m_{10} and m_{20} vanish on the solid side, even though v_{11} remains continuous. This circumstance leads to singular behavior of θ_R as a function of λ, and we find that it is necessary to increment θ_R by 1 at a fluid solid boundary when the $(2,2)$ element of $\mathbf{s}^{\mathrm{Ss}}$ (eqn [79]) is negative on the solid side, that is, for $\omega^2 > [\zeta^2(A-F^2/C)-2N]/\rho r^2$, in which the elastic constants are those evaluated at the boundary, on the solid side and r is the radius of the boundary. For upward integration (the usual case) this occurs at the core–mantle boundary.

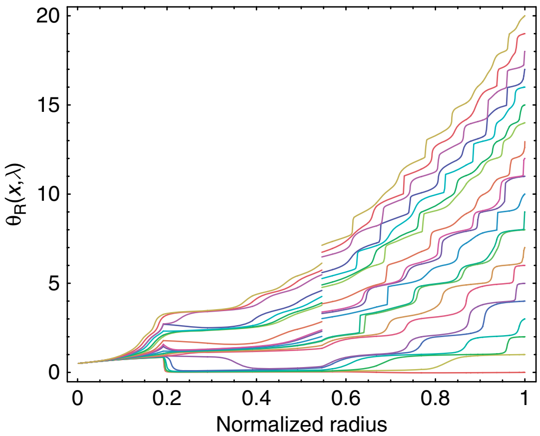

Figure 3 An example of the behavior of $\theta_R(x,\lambda)$ for the spheroidal oscillations of degree $l = 10$. It has been calculated for the true modal frequencies up to ${}_{20}S_{10}$, and thus the surface value of $\theta_R(x,\lambda)$, at normalized radius $x = 1$, takes on successive integer values equal to the overtone number. θ_R can be seen to be independent of frequency at the center of the Earth ($x = 0$), where $\theta_R(x,\lambda) = 1/2$. The nondecreasing property of θ_R as a function of $\lambda = \omega^2$ means that successive curves never cross. Notice that θ_R is not a monotonic function of r. The discontinuity in θ_R at the core–mantle boundary, discussed in the text, affects the mode count for modes higher than ${}_9S_{10}$. In a sense the values of $\theta_R(x,\lambda) = -(1/\pi)\cot^{-1}v_{11}$ are not of physical significance, since v_{11} is a dimensional quantity, and thus the value of $\theta_R(x,\lambda)$ depends on the units employed. In any system of units, however, θ_R takes on integer values at the same values of its arguments. In other words, it is the 'zero crossings' of v_{11}, for which θ_R acts as a counter, that are of primary significance. The results shown here are for $\theta_R(x,\lambda) = -(1/\pi)\cot^{-1}\alpha v_{11}$, with $\alpha = 4\times10^4\,\mathrm{N\,m^{-3}}$.

Figure 3 shows an example of the behavior of $\theta_R(x,\lambda)$ for spheroidal oscillations of degree $l = 10$. The θ_R mode counter can be used to bracket the modal frequencies by a bisection method that seeks values of frequency such that $\theta_R(a,\omega^2)$ takes on values lying between any pair of successive integers in an interval $\left[\theta_R(a,\omega_{\min}^2), \theta_R(a,\omega_{\max}^2)\right]$.

Having bracketed the modal frequencies (for a given l), it is necessary to converge on the zeros of $\det[\mathbf{P}]|_{r=a}$. This can be done in a variety of standard ways, bisection being the ultimately safe method if all else fails. **Figure 4** shows the resulting 'dispersion diagram' for spheroidal modes up to 30 mHz. The crowding and irregularity of the distribution in the left side of the diagram demonstrate the need for the mode-counting scheme. For the toroidal modes, the dispersion diagram is much simpler (**Figure 5**), and so the mode-counting scheme is less critical.

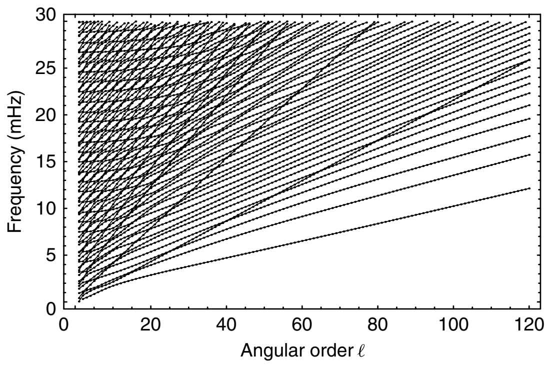

Figure 4 The dispersion diagram for spheroidal modes to 30 mHz.

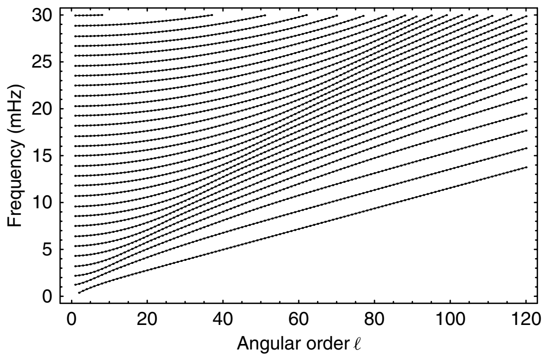

Figure 5 The dispersion diagram for toroidal modes to 30 mHz.

1.02.6 Elastic Displacement as a Sum over Modes

We shall here assume that a catalog of normal mode eigenfrequencies and scalar eigenfunctions has been calculated. Rather than pursuing the inhomogenous problem outlined in earlier sections we shall now adopt the more classical approach, making use of the orthogonality properties of the eigenfunctions to expand the applied force distribution and to find the modal excitations. The argument is strictly valid only in the nonattenuating case. Attenuation is subsequently included by introducing decay constants associated with each mode. Recall that the eigenfrequencies ω_k and the eigenfunctions $\mathbf{s}^{(km)}$ are solutions of the eigenvalue problem

$$\mathcal{H}\mathbf{s}^{(km)} = \rho\omega_k^2\mathbf{s}^{(km)} \qquad [104]$$

It can be shown that the operator $\mathcal{H}$ is self adjoint in the sense

$$\int_V \mathbf{s}'.\mathcal{H}\mathbf{s}\,\mathrm{d}^3x = \int_V \mathbf{s}.\mathcal{H}\mathbf{s}'\,\mathrm{d}^3x \qquad [105]$$

for any differentiable s(x), s′(x) satisfying the boundary conditions for $\mathbf{u}$ in [24], where the volume integration is over the entire Earth model. From this it follows that the eigenfunctions $\mathbf{s}^{(km)}(\mathbf{x})$ form a complete set and that the eigenvalues ω_k^2 are real. We can also assume that they are positive, on the grounds that the model should initially be in stable equilibrium. It is not difficult to show that eigenfunctions belonging to different eigenvalues are orthogonal or, in the case of degeneracy, can be orthogonalized, in the sense

$$\int_V \rho\mathbf{s}^{(k'm')*}\cdot\mathbf{s}^{(km)}\mathrm{d}^3x = 0,\ \text{when}\ k \neq k'\ \text{or}\ m \neq m' \qquad [106]$$

It is straightforward to obtain a formal solution of the equations of motion [35] in terms of a sum of eigenfunctions $\mathbf{s}^{(km)}$. We expand the displacement field $\mathbf{u}(\mathbf{x}, t)$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{km} a_{km}(t)\mathbf{s}^{(km)}(\mathbf{x}) \qquad [107]$$

where $a_{km}(t)$ are to be found. On substituting into [35], multiplying by $\mathbf{s}^{(k'm')*}$ and integrating, making use of the orthogonality relation (106), we obtain

$$\ddot{a}_{km}(t) + \omega_k^2 a_{km}(t) = \omega_k^2 f_{km}(t) \qquad [108]$$

with

$$f_{km}(t) \equiv \frac{1}{\omega_k^2}\frac{\int_V \mathbf{s}^{(km)*}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}, t)\mathrm{d}^3x}{\int_V \rho\mathbf{s}^{(km)*}(\mathbf{x})\cdot\mathbf{s}^{(km)}(\mathbf{x})\mathrm{d}^3x} \qquad [109]$$

The ordinary differential equations [108] for each $a_{km}(t)$ may be solved (e.g., using the method of 'variation of parameters' or Laplace transformation) to give

$$a_{km}(t) = \int_{-\infty}^{t} h_k(t-t')\dot{f}_{km}(t')\mathrm{d}t' \qquad [110]$$

with

$$h_k(t) = 1 - \cos\omega_k t \qquad [111]$$

a result due to Gilbert (1971). As pointed out by Gilbert, this result needs to be modified to account for attenuation by incorporating a decay factor exp $(-\alpha_k t)$ into the cosine term, and thus in place of [111] we write

$$h_k(t) = 1 - \mathrm{e}^{-\alpha_k t}\cos\omega_k t \qquad [112]$$

where α_k is given in terms of the Q of the mode by $\alpha_k = \omega_k/2Q_k$.

Inserting the point source defined in [31], we obtain

$$a_{km}(t) = \frac{1}{\omega_k^2}\left(F_i\, s_i^{(km)}(\mathbf{x}_s)^* + M_{ij}\, s_{i,j}^{(km)}(\mathbf{x}_s)^*\right)h_k(t) \qquad [113]$$

assuming that eigenfunctions are normalized such that $\int_V \rho \mathbf{s}^{(km)*}(\mathbf{x})\cdot\mathbf{s}^{(km)}(\mathbf{x})\mathrm{d}^3x = 1$ .The eigenfunctions are of the form [69] and thus it is a further exercise in spherical harmonic analysis to reduce the excitation coefficients, $E_{km} = \left(F_i s_i^{(km)}(\mathbf{x}_s)^* + M_{ij}\, s_{i,j}^{(km)}(\mathbf{x}_s)^*\right)$, to forms involving the scalar eigenfunctions U_k, V_k, W_k and their derivatives. The result has already been derived in [98], by a different route. Similar formulas are also to be found in Gilbert and Dziewonski (1975), Woodhouse and Girnius (1982), and Dziewonski and Woodhouse (1983). Using [113] in [107], we obtain the following expression for a theoretical seismogram:

$$\mathbf{u}(\mathbf{x},t) = \sum_{km}\frac{1}{\omega_k^2}E_{km}\mathbf{s}^{(km)}(\mathbf{x})(1-\mathrm{e}^{-\alpha_k t}\cos\omega_k t) \qquad [114]$$

The argument of the previous section shows that the correct form of this expression in the attenuating case is

$$\mathbf{u}(\mathbf{x},t) = \sum_{\substack{km \\ \mathrm{Re}\,\omega_k>0}} \mathrm{Re}\left\{\frac{1}{\omega_k^2}E_{km}\mathbf{s}^{(km)}(\mathbf{x})(1-\mathrm{e}^{\mathrm{i}\omega_k t})\right\} \qquad [115]$$

where ω_k is now the complex frequency, and $\mathbf{s}^{km}$ is the complex eigenfunction, having normalization [95]. Additional terms need to be added to [115] if it is desired to include relaxation effects.

This rather simple formula is the key ingredient of many seismological studies, as outlined in the introduction. Mode catalogs for PREM (Dziewonski and Anderson, 1981) exist up to 170 mHz (6 s period). **Figure 2** shows an example of such a synthetic seismogram. For comparison, the observed seismogram is also shown. The direct P and S surface reflected PP and SS body waves are visible in both the synthetic and data seismograms. At later times, the surface waves can be observed. Differences between the synthetic and data seismogram can be attributed to three-dimensional structrue, which is not included in the calculation.

1.02.7 The Normal Mode Spectrum

Here we illustrate some of the properties of different multiplets in the normal mode spectrum. One way to gain understanding of the physical properties is through the use of 'differential kernels' $K(r)$. These are, in essence, the derivatives of the eigenfrequency of a given mode with respect to a structural change at any radius. This takes the form of an integral (cf. the chain rule applied to an infinite number of independent variables). Differential kernels can be defined, for example, corresponding to perturbations in shear modulus and bulk modulus, at fixed density, and these are related to the distribution of elastic shear energy and compressional energy with radius. Similarly, kernels can be defined corresponding to anisotropic perturbations (*see* Chapter 1.16). Theoretical formulas for such kernels can be found in, for example, Backus and Gilbert (1967), and for anisotropic elastic parameters A, C, L, N, F in Dziewonski and Anderson (1981). Such kernels are a special case, in which the perturbation is spherically symmetric, of the kernels that arise when a general aspherical perturbation of the model is considered. This will be further discussed in a later section. Here we shall take relative perturbations in (isotropic) P-velocity, v_P, S-velocity, v_S and density ρ (at constant v_P and v_S) as the independent perturbations, and write

$$\delta\omega_k = \int_0^a \left(K_\rho(r)\frac{\delta\rho(r)}{\rho(r)} + K_\mathrm{P}(r)\frac{\delta v_\mathrm{P}(r)}{v_\mathrm{P}(r)} + K_\mathrm{S}(r)\frac{\delta v_\mathrm{S}(r)}{v_\mathrm{S}(r)}\right)\mathrm{d}r \qquad [116]$$

Our aim here is to use the kernels $K_\mathrm{P}(r)$, $K_\mathrm{S}(r)$ to give insight into the nature of the mode in terms of its traveling-wave content, and into how the P- and S-velocity distributions can be constrained by making observations of a given mode. The kernel $K_\rho(r)$ (for constant v_P and v_S) gives information about how the mode probes the density structure, adding to information about v_P and v_S available from modes but also from travel times.

Another way to gain insight into the physical character of modes is by relating them to traveling body waves and surface waves. The essential quantitative connection between modes and traveling waves is made by equating the horizontal wavelength (or wave number) of the mode with the corresponding horizontal wavelength (or wave number) of a traveling wave. For modes, this wavelength can be derived from the asymptotic properties of the spherical harmonics for large l. A point source at the pole

$\theta = 0$ excites only the modes having low azimuthal order $|m| \leqslant 2$, as we have seen. For fixed m and large l, we have (e.g., Abramowitz and Stegun, 1965)

$$Y_l^{0m}(\theta, \phi) \sim \frac{1}{\pi}\sqrt{\frac{2l+1}{4\pi}}(\sin\theta)^{-1/2}\cos\left[\left(l+\frac{1}{2}\right)\theta + \frac{1}{2}m\pi - \frac{1}{4}\pi\right]e^{im\phi} \qquad [117]$$

where θ plays the role of epicentral distance. Thus, we can identify the horizontal wave number k ($= 2\pi/$ wavelength) to be

$$k = \left(l + \frac{1}{2}\right)\bigg/ a \qquad [118]$$

The angular order l, therefore, is a proxy for wave number k and dispersion diagrams such as those shown in **Figures 4** and **5** can be interpreted, for large l, in the same way as are dispersion relations $\omega(k)$ for surface waves. In particular, we can define phase velocity

$$c(\omega) = \frac{\omega}{k} \qquad [119]$$

and group velocity

$$U(\omega) = \frac{d\omega}{dk} \qquad [120]$$

This defines the relationship between the $\omega - l$ plane and the dispersion properties of Love and Rayleigh waves and their overtones, Love waves corresponding to toroidal modes, and Rayleigh waves to spheroidal modes.

In the case of body waves we may, similarly, identify the horizontal wave number in terms of frequency and 'ray parameter' p (Brune, 1964, 1966). From classical ray theory in the spherical Earth, the horizontal wave number at the Earth's surface for a monochromatic signal traveling along a ray with given ray parameter $p = dT/d\triangle$ is

$$k = \frac{\omega p}{a} \qquad [121]$$

Therefore, using [118],

$$p = \frac{l + \frac{1}{2}}{\omega} \qquad [122]$$

Thus, a mode of angular order l and angular frequency ω is associated with rays having the ray parameter given by [122]. For toroidal modes, these are S-rays, and for spheroidal modes they are both P- and S-rays. It is well known that rays exist only for ranges of depth for which

$$\frac{r}{v_P(r)} \geq p, \quad \text{for P-waves} \qquad [123]$$

$$\frac{r}{v_S(r)} \geq p, \quad \text{for S-waves} \qquad [124]$$

In the diagrams of **Figures 7** and **8** the ranges of depth for which these inequalities are satisfied are indicated in two columns on the right-hand side of each panel. The left column is for P-waves (relevant only for spheroidal multiplets) and the right column for S-waves.

Figure 6 shows the combined dispersion diagrams for spheroidal and toroidal modes at low frequencies ($f \leqslant 3$ MHz), an expanded version of the lower left corner of the dispersion diagrams in **Figures 4** and **5**. Lines connect modes of the same type (spheroidal or toroidal) and the same overtone number n, and define different branches of the dispersion curves. The 'fundamental mode branch' ($n = 0$) contains the modes with the lowest frequency for each l. Modes with $n > 1$ are called 'overtones'. We will make a tour of $\omega - l$ space and use the eigenfunctions and differential kernels defined above to gain insight into the nature of the different types of mode.

Figures 7–9 show eigenfunctions and differential kernels for a number of toroidal and spheroidal modes. By inspecting these diagrams, a few general observations can be made. Moving up along a mode branch (horizontal rows in **Figures 7–9**) will result in eigenfunctions and kernels which are more concentrated toward the surface. This reflects the fact that, for high l, the modes may be interpreted as surface waves. Moving up in overtone number n, for constant l (vertical columns in **Figures 8** and **9**), leads to eigenfunctions that are more oscillatory and to sensitivity kernels that penetrate more deeply.

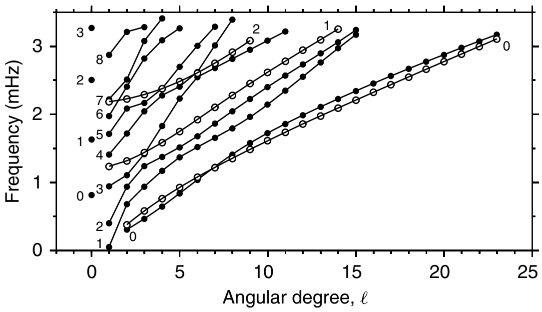

Figure 6 Eigenfrequencies of spheroidal (black circles) and toroidal multiples (white circles) below 3 mHz for the Preliminary Reference Model (PREM, Dziewonski and Anderson, 1981). A multiplet with angular order l consists of $2l + 1$ singlets with azimuthal order $m = -l, -l+1, \ldots, l-1, l$. The branches are labeled by their overtone number n (left spheroidal, right toroidal), the fundamental mode branch is $n = 0$.

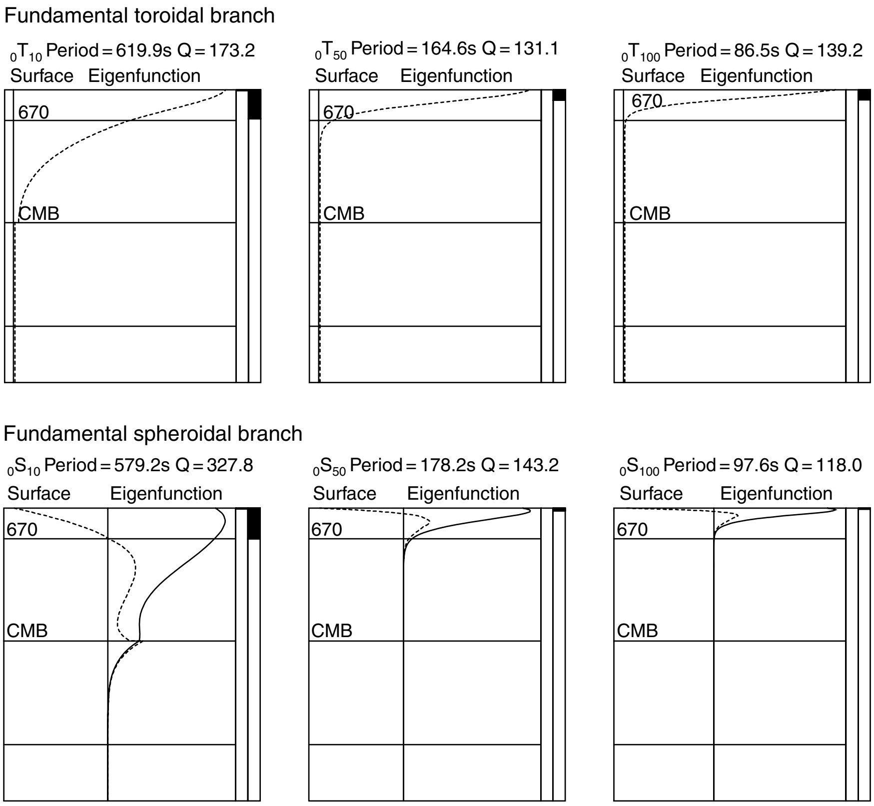

Figure 7 Eigenfunctions of selected toroidal and spheroidal fundamental modes ($n = 0$. For the toroidal modes, the dashed lines show $W(r)$, and for spheroidal modes the dashed lines denote $V(r)$ and the solid lines $U(r)$. At high angular order l the toroidal modes correspond to Love waves and the spheroidal modes to Rayleigh waves.

The wave motion of toroidal modes is purely horizontal (see **Figure 7**), and thus these modes are sensitive only to perturbations in S-velocity and density and have no sensitivity to the core (**Figure 8**). The differential kernels tell us how the frequency of the mode will change if we increase the spherical velocity or density at a certain depth. When inspecting **Figure 8**, we find that increasing the shear wave velocity at any depth in the mantle, will always lead to an increase in toroidal mode frequency as the $K_S(r)$ sensitivity kernel is always positive. For density, however, we find that it depends on the depth of the perturbation. For mode ${}_0T_2$, for example, an increase in density in the upper mantle will lead to a decrease in frequency, while an increase in the lower mantle will increase the frequency. When we move from the fundamental to the overtones, we find that the density kernel $K_\rho(r)$ becomes oscillatory around zero (see, e.g., the $n = 5$ overtones). These modes are almost insensitive to smooth variations in density, as the kernels will average to zero. The sensitivity to desity also becomes smaller for larger l along the same branch, which can clearly be seen when progressing from ${}_0T_2$ to ${}_0T_{10}$ along the fundamental mode branch. This agrees well with the interpretation of shorter period toroidal modes in terms of Love waves, which are also dominated by sensitivity to shear wave velocity.

Figure 9 shows examples of sensitivity kernels for the spheroidal modes. The spheroidal modes involve wave motion in both horizontal and vertical directions, and so are sensitive to perturbations in density and to both v_P and v_S. Again, moving to the right along the fundamental mode branch shows that sensitivities become progressively concentrated closer to the surface. Spheroidal modes correspond to Rayleigh waves and at higher l the largest sensitivity is to shear wave velocity, similar to the toroidal modes, except that peak sensitivity is reached at subcrustal depths, making them less sensitive to large variations in shear velocity in the crust than is the case for toroidal modes. The overtones sample

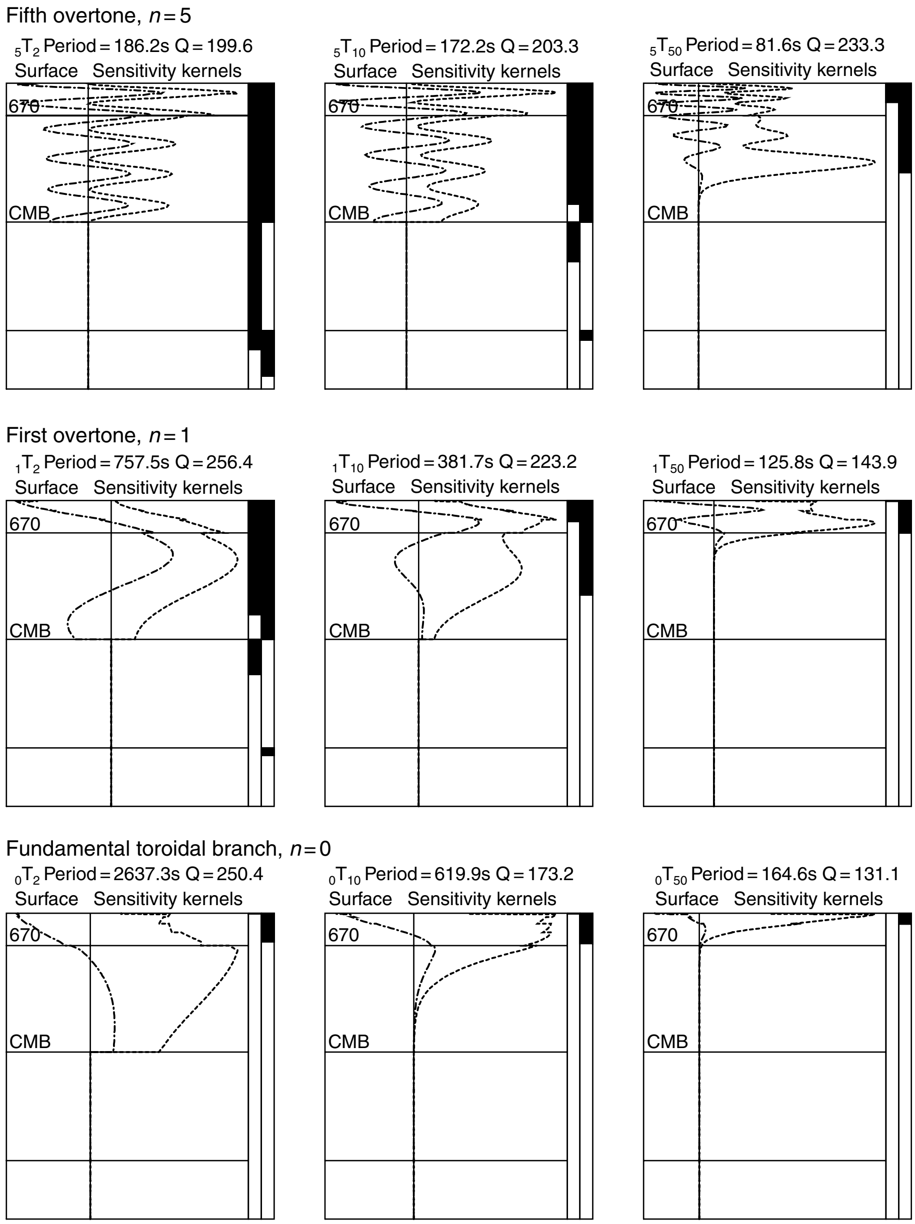

Figure 8 Sensitivity kernels $K_S(r)$ (dashed lines) and $K_\rho(r)$ (dot-dashed lines) of selected toroidal modes.

different families of modes. The mode ${}_5S_{10}$ corresponds to a Stoneley wave (the analog of a Rayleigh wave, but at a fluid–solid interface, rather than at a free surface), traveling along the inner-core boundary. ${}_1S_{10}$ is a mixture of a mantle mode and a Stoneley wave at the core–mantle boundary. The other modes in **Figure 9** show a behavior similar to the toroidal modes. Notice that for ${}_5S_{50}$ the v_p sensitivity decays below the P-wave ray-theoretic turning point (the point at which the shading terminates in the left vertical stripe at the right of the plot), and the v_S sensitivity decays below the S-wave turning point. The fact that, for a given ray parameter, S-waves turn at greater depth than P-waves means that in modeling there is some potential for shallow P-velocity structure to trade off with deep *S*-velocity structure.

Figure 10 shows another family of spheroidal modes, which are characterised by low *l* and high overtone number *n*. These modes are the PKIKP equivalent modes which have strong sensitivity to core structure.

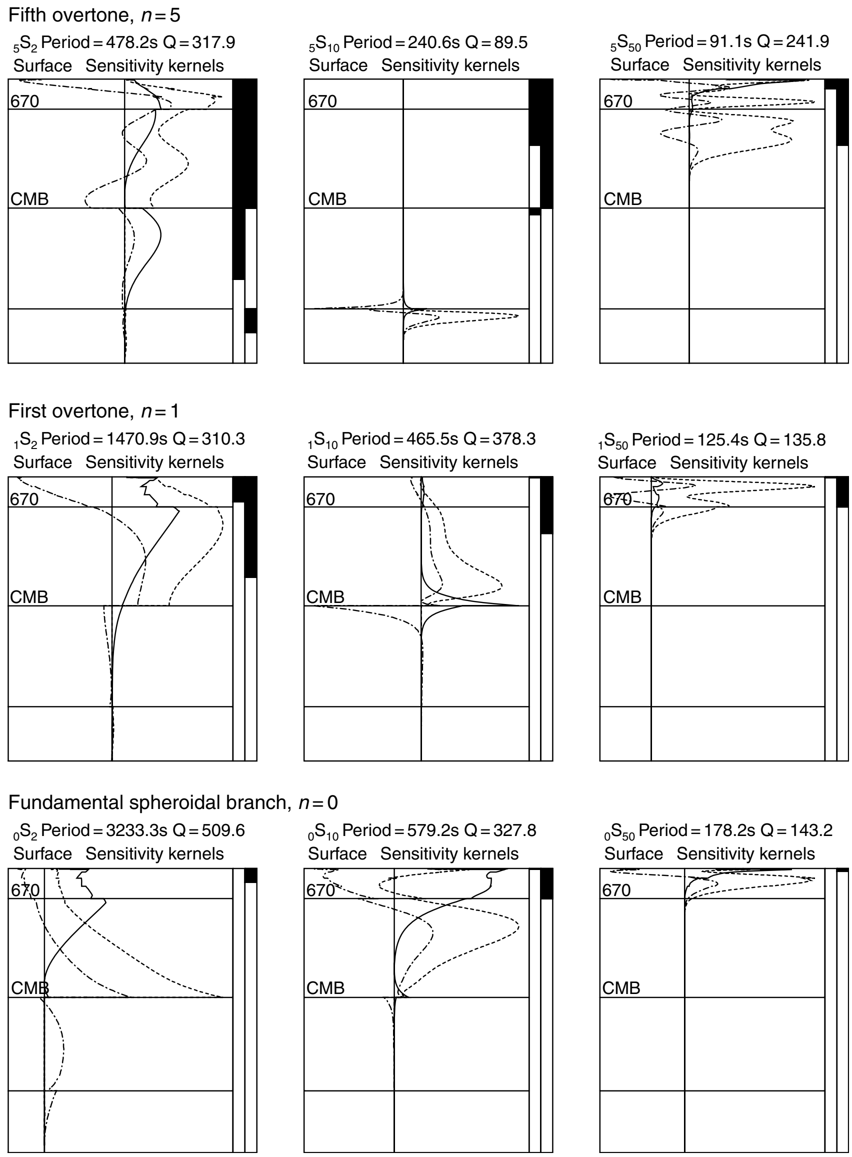

Figure 9 Sensitivity kernels of selected spheroidal modes. $K_S(r)$ dashed, $K_P(r)$ solid, $K_\rho(r)$ dot-dash.

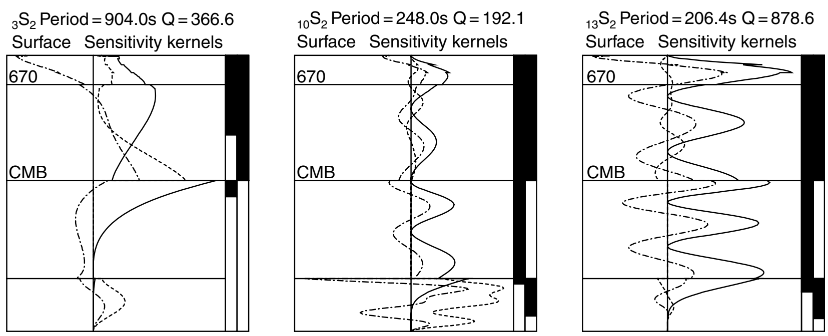

Figure 10 Sensitivity kernels of selected PKIKP equivalent spheroidal modes.

1.02.8 Normal Modes and Theoretical Seismograms in Three-Dimensional Earth Models

The problem of calculating theoretical seismograms and spectra for three-dimensional models is a challenging one. A theoretically straightforward formalism exists for such calculations, based on expanding the wave field in terms of a complete set of (vector) functions. The equations that arise from requiring that the equations of motion be satisfied can be regarded as exact matrix equations. However, they are of infinite dimension, and even when truncated to give a practicable method of solution, the calculations require the manipulation and diagonalization of extremely large matrices. If the expansion is carried out in terms of the eigenfunctions of a spherical model (that is close to reality), the process is simplified, as the off-diagonal terms in the resulting matrices are small, allowing some approximate schemes that are much less cumber-some to be developed.

Let us take as our starting point in considering the form that this theory takes the equation of motion in operator from given in [35]. In the frequency domain

$$(\mathcal{H}-\rho\omega^2)\mathbf{u}=\mathbf{f} \qquad [125]$$

The Earth will be considered to be in a state of steady rotation with angular velocity $\mathbf{\Omega}$ about its center of mass, so that the expression for $\mathcal{H}$ acquires additional terms representing the Coriolis force and the centrifugal potential (Dahlen, 1968):

$$(\mathcal{H}\mathbf{u})_i=\rho\Big(\phi^1_{,i}+\Big[\phi^0-\frac{1}{2}(\Omega_k\Omega_k x_l x_l-\Omega_k\Omega_l x_k x_l)\Big]_{,ij}u_j + 2\mathrm{i}\epsilon_{ijk}\Omega_j u_k\Big)-(\Lambda_{jilk}u_{k,l})_{,j} \qquad [126]$$

where, as previously, ϕ^1 is regarded as a functional of $\mathbf{u}$, by virtue of Poisson's equation [22]. Let $\mathbf{s}^{(km)}$ represent the eigenfunctions of a spherical, nonrotating, nonattenuating reference model, with eigenfrequencies ω_k so that

$$\mathcal{H}_0\mathbf{s}^{(km)}=\rho_0\omega_k^2\mathbf{s}^{(km)} \qquad [127]$$

satisfying the orthogonality relation

$$\int_V \rho_0\mathbf{s}^{(k'm')*}\cdot\mathbf{s}^{(km)}\,\mathrm{d}V=\delta_{k'k}\delta_{m'm} \qquad [128]$$

Where $\mathcal{H}_0$ and ρ_0 are for the reference model, and let us seek a solution of [125] in terms of an expansion

$$\mathbf{u}(\mathbf{x},\omega)=\sum_{km}a_{km}\mathbf{s}^{(km)}(\mathbf{x}) \qquad [129]$$

with coefficients a_{km} to be found. Substituting into [125], and then taking the dot product with $\mathbf{s}^{(k'm')*}$ and integrating, we find

$$\sum_{km}[(k'm'|\mathcal{H}_1-\rho_1\omega^2|km)-(\omega^2-\omega_k^2)\delta_{k'k}\delta_{m'm}]a_{km} = (k'm'|f) \qquad [130]$$

where $\mathcal{H}_1=\mathcal{H}-\mathcal{H}_0$, $\rho_1=\rho-\rho_0$, and where we have introduced the notations $(k'm'|f)=\int_V\mathbf{s}^{(k'm')*}\cdot\mathbf{f}\,dV$, $(k'm'|\mathcal{H}_1-\rho_1\omega^2|km)=\int_V\mathbf{s}^{(k'm')*}(\mathcal{H}_1-\rho_1\omega^2)\mathbf{s}^{(km)}\mathrm{d}V$. Equation [130] can be regarded as a matrix equation, albeit of infinite dimension, in which rows and columns of the matrix on the left side are labeled by (k', m'), (k, m), respectively and in which the rows of the column on the right side are labeled by (k', m'). We can write

$$\mathbf{C}(\omega)\mathbf{a}=\frac{1}{\mathrm{i}\omega}\mathbf{E} \qquad [131]$$

where $\mathbf{C}$ is the matrix having matrix elements $(k'm'|C(\omega)|km)=(k'm'|\mathcal{H}_1-\rho_1\omega^2|km)-(\omega^2-\omega_k^2)\delta_{k'k}\delta_{m'm}$ and where $\mathbf{E}/\mathrm{i}\omega$ is the column vector having elements $(k'm'|f)$, the factor $1/i\omega$ being inserted to reflect the assumed step-function time dependence of the source, so that $\mathbf{E}$, which has elements E_{km} given in [98], is independent of frequency. Hence the formal solution is given by

$$\mathbf{a}=\frac{1}{\mathrm{i}\omega}\mathbf{C}(\omega)^{-1}\mathbf{E} \qquad [132]$$

Recalling that we are thinking of the index k as incorporating angular order l, overtone number n, and the mode type (spheroidal or toroidal), the matrix $\mathbf{C}(\omega)$ consists of blocks of dimension $(2l'+1)\times(2l+1)$, as row index m' takes on values $-l'$ to l' and column index m takes on values $-l$ to l. Thus, each block within $\mathbf{C}(\omega)$ relates two particular multiplets k, k' of the spherical reference model. Because the deviation from the spherical reference model is regarded as small, the matrix elements in $\mathbf{C}$ are small, except for the elements on the diagonal proportional to $\omega^2-\omega_k^2$. These diagonal terms are small for values of ω close to ω_k, for a given multiplet k, but for other diagonal blocks, corresponding to multiplets not close in frequency to multiplet k, they are not small; thus, the matrix is, mostly, diagonally dominant, except for diagonal blocks corresponding to multiplets close in frequency to ω. Of course, the complete solution is to be obtained by substituting [132] into [129] and then evaluating the inverse Fourier transform; thus, we need to consider the behavior of the solution as a function of complex variable ω. In particular, we are interested in the singularities that occur in the complex ω plane, as these will correspond to the modes of the aspherical model, in much the same way as the modes of the spherical model correspond to singularities, in the frequency domain, of the solution of the inhomogenous problem (eqn, [93] and the related discussion). The singularities in the solution [132] will be at the frequencies $\omega = \omega_b$, say, for which there exists a non-trivial solution to the homogeneous problem

$$\mathbf{C}(\omega_b)\mathbf{r}_b = 0 \qquad [133]$$

This is an eigenvalue problem for ω_h and the corresponding right eigenvector $\mathbf{r}_b$. The eigenvalue problem is of a nonstandard form, since the dependence of $\mathbf{C}$ on the eigenvalue parameter $\lambda = \omega^2$ say, is not of the usual form $\mathbf{C}_0 - \lambda\mathbf{1}$. This is because, (1) perturbations in density ρ_1 introduce a more general dependence on ω^2, (2) by virtue of attenuation, the perturbations in the elastic parameters entering into $\mathcal{H}_1$ are dependent upon ω, (3) the terms arising from rotation depend upon ω, rather than on ω^2. The form of the solution as a sum over residues arising from singularities at $\omega = \omega_b$ can be obtained by replacing the inverse Fourier transform by a summation over singularities b, and within the integrand carrying out the replacement (Deuss and Woodhouse, 2004; AL-Attar, 2007; cf. eqn [94]):

$$\mathbf{C}(\omega)^{-1} \to \mathrm{i}\frac{\mathbf{r}_b\mathbf{1}_b}{\mathbf{1}_b\partial_\omega\mathbf{C}(\omega_b)\mathbf{r}_b} \qquad [134]$$

where $\mathbf{1}_b$ is the left-eigenvector (a 'row' rather than a column), the solution of

$$\mathbf{1}_b\mathbf{C}(\omega_b) = 0 \qquad [135]$$

We are assuming here that [133] and [135] determine the right and left eigenvectors $\mathbf{r}_b$, $\mathbf{1}_b$ uniquely, up to multiplying factors, such factors being immaterial for the evaluation of the residue contribution [134]. Thus, from [129] the solution in the time domain can be written as

$$\mathbf{u}(\mathbf{x},t) = 2\mathrm{Re}\sum_{bkm}\frac{1}{\omega_b}\frac{\mathbf{1}_b\cdot\mathbf{E}}{\mathbf{1}_b\partial_\omega\mathbf{C}(\omega_b)\mathbf{r}_b} \times (km|\mathbf{r}_b)\mathbf{s}^{(km)}(\mathbf{x})\mathrm{e}^{i\omega_b t} \qquad [136]$$

where the notation $(km|\mathbf{r}_b)$ represents individual elements of the column $\mathbf{r}_b$. As previously, the contribution from singularities in the left half of the complex ω-plane is incorporated by taking twice the real part, the summation in [136] being taken only for ω_k in the right half-plane.

In order to make use of this theory, it is necessary to obtain expressions for the matrix elements $(k'm'|\mathbf{C}(\omega)|km)$ in terms of the perturbations in elastic parameters, density, etc., and deviations of surfaces of discontinuity from the spherically symmetric reference model. Fairly complete forms for these are given by Woodhouse (1980a), omitting terms in anisotropic parameters and in initial stress. For anisotropic perturbations, *see* Chapter 1.16. The basic method is to expand the perturbations in spherical harmonics, and then to evaluate the integrals of triples of spherical harmonics using the formula, derivable from [43] and [48]:

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}(Y_{l'}^{N'm'})^* Y_{l''}^{N''m''} Y_l^{Nm}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = (-\mathbf{1})^{N'-m'}\begin{pmatrix} l' & l'' & l \\ -N' & N'' & N \end{pmatrix}\begin{pmatrix} l' & l' & l \\ -m' & m'' & m \end{pmatrix} \quad (N' = N'' + N) \qquad [137]$$

The resulting form for the matrix elements can then be written in the form

$$(k'm'|\mathbf{C}(\omega)|km) = \sum_{l''m''}(-\mathbf{1})^{-m'}\begin{pmatrix} l' & l'' & l \\ -m' & m'' & m \end{pmatrix} \times (k'||\mathbf{C}^{(l''m'')}(\omega)||k) \qquad [138]$$

where the so-called 'reduced matrix element' appearing in the right side, itself defined by this equation, is independent of m and m'. This particular form for the dependence of the matrix elements on m and m' is a consequence of the Wigner–Eckart theorem (see Edmonds (1960)). The expressions for the reduced matrix elements take the form of radial integrals involving pairs of scalar eigenfunctions, for multiplets k' and k, and on the $(l''m'')$ component of the spherical harmonic expansion of heterogeneity, together with terms evaluated at boundaries corresponding to the $(l''m'')$ components of the deflections of the boundaries.

This is a fairly complete theory for the oscillations of a general Earth model. Apart from the treatment of aspherical boundary perturbations, which involves a linearization of the boundary conditions, it is in principle an exact theory (Woodhouse, 1983), provided that coupling between all multiplets is taken into account – that is, provided that the eigenvalue problem for ω_b includes all the blocks of the full matrix $\mathbf{C}(\omega)$. Of course, the theory cannot be applied in its full form, owing to the need to manipulate infinite-dimensional matrices, and so a number of approximate schemes have been developed. The simplest, and up to now most widely applied method is to reduce the eigenvalue problem for ω_b to that for a single diagonal block – the so-called 'self-coupling' approximation. In this case we focus on a single multiplet, k, and reduce $\mathbf{C}(\omega)$ to the $(2l+1)\times(2l+1)$ block corresponding to multiplet k. This is a justifiable approximation for the calculation of $\mathbf{u}(\mathbf{x},\omega)$ for frequencies near ω_k if the mode can be considered 'isolated', which is to say that there are no other modes nearby having significant coupling terms. A precise statement of the conditions that need to be satisfied for a mode to be considered isolated has not, to our knowledge, been worked out but, roughly speaking, it is necessary for the ratio $(k'm'|\mathbf{C}(\omega)|km)/(\omega_k^2-\omega_k'^2)$ for all other multiplets k' to be small, for ω near the frequency ω_k of the target multiplet. In the self-coupling approximation the dependence of $\mathbf{C}(\omega)$ can be linearized for frequencies near ω_k: $\mathrm{C}(\omega)\approx \mathrm{C}(\omega_\mathrm{k})+\mathrm{C}'(\omega_k)\delta\omega$, and a $(2l+1)\times(2l+1)$ matrix eigenvalue problem is obtained for $\delta\omega$:

$$[\mathbf{C}(\omega_k)+\mathbf{C}'(\omega_k)\delta\omega]\mathbf{r}=0 \qquad [139]$$

having eigenvalues $\delta\omega_b$, say. This can also be written, to zeroth order, as

$$\mathbf{H}^{(k)}\mathbf{r}=\delta\omega\mathbf{r} \qquad [140]$$

where the $(2l+1)\times(2\mathrm{l}+1)$ matrix $\mathbf{H}^{(k)}$, called the 'splitting matrix' of the target multiplet, k, has elements $(km'|\mathbf{H}^{(k)}|km)=(km'|\mathbf{C}(\omega_k)|km)/2\omega_k$. The contribution to the right-hand side of [136] can be written (Woodhouse and Girnius, 1982) as

$$\begin{aligned}\mathbf{u}_k(\mathbf{x},t)=\mathbf{Re}\sum_{m'm}-\frac{1}{\omega_k^2}\left(\exp(\mathrm{i}\mathbf{H}^{(k)}t)\right)_{mm'}\\ \times E_{km'}\mathbf{s}^{(km)}(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega_k t}\end{aligned} \qquad [141]$$

the matrix exponential arising from the identity

$$\sum_{b=1}^{2l+1}\frac{\mathbf{r}_b\mathbf{l}_b}{\mathbf{1}_b\cdot\mathbf{r}_b}\exp(\mathrm{i}\delta\omega_b t)=\exp(\mathrm{i}\mathbf{H}t) \qquad [142]$$

Equation [141], which is directly comparable to the result for the spherical reference model in [115], has the simple interpretation that at time $t=0$ the modes are excited as they would be in the reference model, as the matrix exponential is initially equal to the unit matrix. (The static, time-independent terms are not included, as we are considering an approximation valid only in the spectral neighborhood of ω_k, the frequency of the target multiplet.) With time, the effective excitation $\exp(i\mathbf{H}^{(k)}t)\mathbf{E}_k$ evolves on a slow timescale characterized by the incremental eigenfrequencies $\delta\omega_b$, the eigenvalues of $\mathbf{H}^{(\mathrm{k})}$. In the frequency domain, this leads to 'splitting' of the degenerate eigenfrequency ω_k into $2l+1$ 'singlets' – hence the name 'splitting matrix' for $\mathbf{H}^{(\mathrm{k})}$.

It is straightforward to set up the inverse problem of estimating the splitting matrix for isolated multiplets using data spectra for many events. This is simplified by recognizing that the $(2l+1)\times(2l+1)$ matrix $\mathbf{H}^{(k)}$ is equivalent to a certain function on the sphere, known as the 'splitting function' (Woodhouse and Giardini, 1985). It can be shown that for scalar perturbations from the reference model, such as ρ_1, κ_1, μ_1, $\mathbf{H}^{(k)}$ is expressible in terms of coefficients $c_{l''m''}$ which represent the spherical harmonic expansion coefficients of even degree l'', and up to finite spherical harmonic degree $l''\leqslant 2l$ by the expression

$$\begin{aligned}(km'|\mathbf{H}^{(k)}|km)=\Omega\beta_k m\delta_{m'm}+\omega_k\sum_{\substack{l''=0\\ l''\text{even}}}^{2l}\sum_{m''=-l''}^{l''}(-\mathbf{1})^{m'}\\ \times\left(\frac{2l''+1}{4\pi}\right)^{1/2}(2l+1)\begin{pmatrix} l & l'' & l\\ 0 & 0 & 0\end{pmatrix}\\ \times\begin{pmatrix} l & l'' & l\\ -m' & m'' & m\end{pmatrix}c_{l''m''}^{(k)}\end{aligned} \qquad [143]$$

where the first term is the effect of Coriolis forces (Dahlen, 1968), β_k being the (known) rotational splitting parameter for the multiplet. Thus, the inverse problem for $\mathbf{H}^{(k)}$ is equivalent to the estimation of $c_{l''m''}$. The function on the sphere

$$\eta(\theta,\phi)=\sum_{\substack{l''=0\\ l''\text{even}}}^{2l}\sum_{m''=-l''}^{l''}c_{l''m''}^{(k)}\,\nu_l\,Y_{l''}^{0m''}(\theta,\phi) \qquad [144]$$

can be interpreted, at least for high-l modes, as the even degree expansion of $\delta\omega_{\mathrm{local}}/\omega_k$, in which $\delta\omega_{\mathrm{local}}$ is the eigenfrequency that a spherically symmetric model would possess if its radial structure were the same as the structure beneath the point (θ,ϕ) (Jordan, 1978). Only even degrees are present by virtue of the fact that the first 3-j symbol in [143] vanishes for odd values of l''. The splitting function leads to a two-stage inversion for three-dimensional structure in which stage 1 is to find the structure coefficients $c_{l''m''}^{(k)}$ that bring data and theoretical spectra into agreement, using as many events and stations as are available, and the stage 2 is to determine the structural perturbations needed to match the inferred values of $c_{l''m''}^{(k)}$. Stage 1 of the procedure is nonlinear, owing to the fact that the relation between the synthetics and $c_{l''m''}^{(k)}$ involves the exponential exp $\mathrm{i}\mathbf{H}^{(k)}t$.

Stage 2, on the other hand, is linear; $c^{(k)}_{l'm'}$ is related to three-dimensional structural perturbations by integrals involving known differential kernels. This is a similar procedure to that commonly employed in surface wave studies, in which one first determines two-dimensional maps of phase velocity, over a range of frequencies, and then uses these to infer the three-dimensional structure perturbations needed to explain the inferred phase velocity maps. The spectral fitting approach using splitting function coefficients $c^{(k)}_{l'm'}$ has been widely applied (e.g., Giardini *et al.*, 1987, 1988; Ritzwoller *et al.*, 1988; He and Tromp, 1996; Resovsky and Ritzwoller, 1995, 1998; Romanowicz and Breger, 2000; Masters *et al.*, 2000).

Figure 11 shows an example of data and theoretical spectra using the self-coupling approximation, and splitting function coefficients estimated from a large collection of data. In the left panel the splitting effects only of rotation and ellipticity are taken into account, whereas in the right panel the estimated splitting function has been used to calculate the synthetic spectra. The distribution of singlets and their excitations is known only by virtue of the inversion itself. There is no possibility here of resolving the singlets in individual spectra, but by modeling a large collection of spectra, for many events and stations, the underlying singlet distribution is unmasked. This example illustrates the fact that there are large differences between data and synthetics prior to modeling, indicating that long period spectra represent a rich source of information about the Earth's three-dimensional structure.

The fact that in the self-coupling approximation seismic spectra depend only upon the 'even' spherical harmonic degrees of heterogeneity points to a shortcoming of the theory. Since spherical harmonics of even degree are symmetric under point reflection in the center of the Earth, self-coupling theory predicts that the seismic spectra depend only upon the average structure between pairs of antipodal points. Thus, the interaction, or coupling, of modes must be a key effect for understanding wave phenomena that do not have this symmetry property. The theory can be straightforwardly extended to include the coupling of groups of modes. The resulting method is known as 'quasi-degenerate perturbation theory' (Dahlen, 1969; Luh, 1973, 1974; Woodhouse, 1980a), or 'group coupling'. A small group of multiplets $\{k_1, k_2, k_3, \ldots\}$, close in frequency, is selected, and the eigenvalue problem is reduced to that for the matrix obtained from $\mathbf{C}(\omega)$ by selecting only the blocks corresponding to the chosen multiplets. This problem can then be linearized in $\delta\omega$, relative to a fiducial frequency in the chosen band, in much the same way as in the case of self-coupling, outlined above, the resulting matrix eigenvalue problem being of dimension $(2l_1+1)+(2l_2+1)+(2l_3+1)+\cdots$. The selected group of modes is said to form a 'super-multiplet'. Resovsky and Ritzwoller (1995)

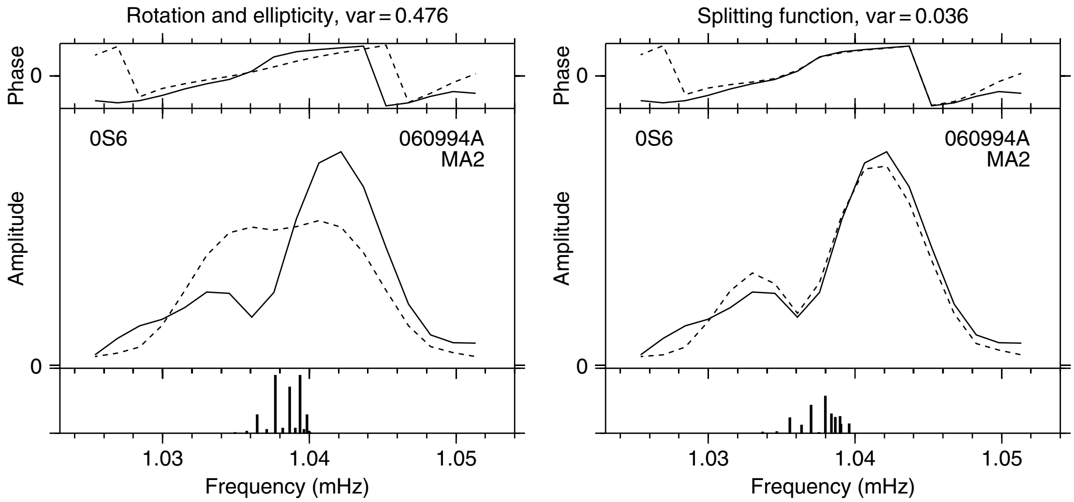

Figure 11 Data and synthetic spectra for $_0S_6$ for an earthquake in Bolivia. The time window is 5–70 h. The solid lines are the observed phase (top) and amplitude spectrum (middle), and dashed lines are for the synthetic spectra. In the left diagram only splitting due to rotation and ellipticity is taken into account, in the right diagram the estimated splitting function is used, reducing the misfit variance ration for this record from 0.476 to 0.036. The distribution of singlets contributing to the synthetic spectra and their relative excitation amplitudes are indicated at the bottom of each panel. Phase is in the interval $(-\pi,\pi]$.

have generalized the notion of the splitting function and structure coefficients to include coupling between pairs of multiplets, so that $c_{l''m''}^{(k)}$ becomes $c_{l''m''}^{(k_1,k_2)}$, and have made estimates of such coefficients from seismic spectra (see, e.g., Resovsky and Pestana (2003)).

The splitting function approach has been used in inversions for tomographic velocity models (Li *et al.*, 1991; Resovsky and Ritzwoller, 1999; Ishii and Tromp, 1999). Some recent tomographic shear wave velocity models such as S20RTS (Ritsema *et al.*, 1999), make use of splitting functions in addition to body wave, surface wave, and overtone data to provide improved constraints on the low degree structure. Splitting functions have also been used in the discovery of inner-core anisotropy (Woodhouse *et al.*, 1986) and have provided constraints on the possible rotation of the inner core (Sharrock and Woodhouse, 1998; Laske and Masters, 1999).

The 'self-coupling' and 'group coupling' techniques depend upon the assumption that further cross-coupling is not needed to approximate the complete solution, which as we have shown includes coupling between all multiplets. Of course, full coupling calculations cannot be done for a truly infinite set of modes, but it is feasible at low frequencies to include coupling between all multiplets below a specified frequency. We shall call this 'full coupling'. Deuss and Woodhouse (2001) have compared the different approximations used in computing normal mode spectra, and have found that 'self-coupling' and 'group coupling' can be a poor approximation to 'full coupling', indicating that a more complete version of the theory will need to be used in the future as it is desired to constrain the three-dimensional distribution of parameters, such as density, attenuation, and mantle anisotropy, on which the spectra depend more subtly.

Figure 12 shows a comparison between data and spectra calculated using the 'self-coupling' with those resulting from a 'full coupling' calculation in which the coupling of all 140 modes up to 3 mHz has been included (see Deuss and Woodhouse, (2001) for details of the calculation). The spheroidal modes are clearly seen, and there is also signal for toroidal mode ${}_0T_{10}$ on the vertical component, which is due to Coriolis coupling. There is reasonable agreement between the data and full coupling synthetics, but the differences between data and synthetics are comparable to the difference between the 'self-coupling' and 'full coupling' synthetics. It may be expected that 'group coupling' would be justified, and that coupling among wide bands of modes can be ignored. However, coupling on groups still shows significant differences compared to 'full coupling' (see **Figure 13**).

In principle, normal mode spectra can be inverted directly to derive tomographic models, avoiding the intermediate step of estimating the splitting function coefficients (Li *et al.*, 1991; Hara and Geller, 2000; Kuo and Romanowicz, 2002). This leads to a one-step inversion procedure in which model parameter

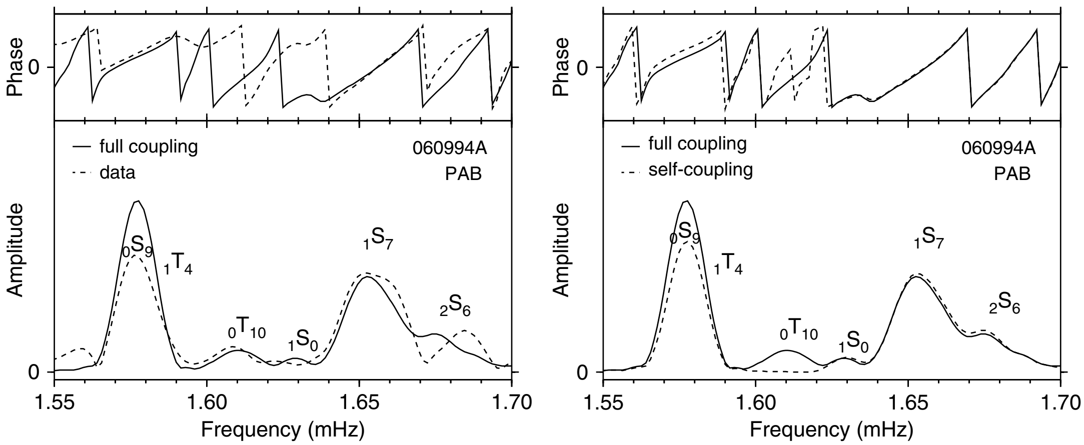

Figure 12 Data and synthetics for modes ${}_0S_9$, ${}_1T_4$, ${}_0T_{10}$, ${}_1S_0$, ${}_1S_7$ and ${}_2S_6$ at station PAB for an earthquake in Bolivia. The time window is 5–45 h. The differences between full coupling and self-coupling are similar to the differences between the data and full coupling and are of the same order as the differences that one would attempt to model. This indicates that full coupling is essential in future attempts to model inhomogeneous structure. From Deuss A and Woodhouse JH (2001) Theoretical free oscillation spectra: The importance of wide band coupling. *Geophysical Journal International* 146: 833–842.

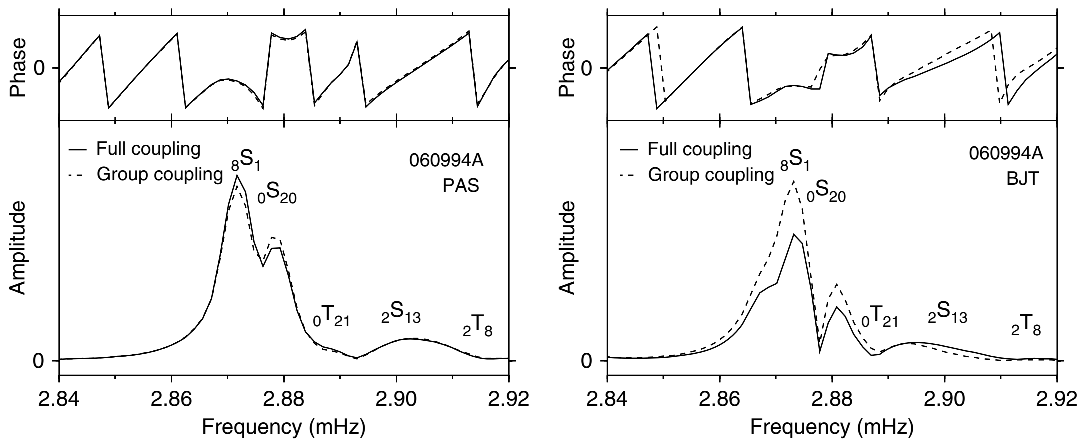

Figure 13 Synthetics for modes $_2S_{13}$, $_0T_{21}$, $_0S_{20}$, $_8S_1$, $_2T_8$ for stations PAS and BJT of the Bolivia event of 9 June 1994, from Deuss and Woodhouse (2001). The time window is 5–80 h. All modes in the frequency interval shown were allowed to couple for the group coupling calculations; the full coupling calculation includes all modes up to 3 mHz.

adjustments that enable the data and theoretical spectra to be brought into agreement are sought directly. This scheme has the advantage that the full coupling approach can be used for the solution of the forward problem and for calculations of the derivatives needed for formulating the inverse problem. Of course, it has the disadvantage that a nonlinear inverse problem needs to be solved within a large model space, rather than being able to restrict the nonlinear stage of inversion to the much smaller number of parameters represented by splitting functions. The calculations are also much more burdensome in terms of computer time and memory. This means that the splitting function technique is still largely the preferred method in spectral fitting studies; however, to investigate the large regions of the spectrum where wide-band coupling is expected to be significant, a final stage inversion involving full coupling will be needed.

Significant theoretical work has been directed toward developing methods able to give accurate splitting and coupling results using a practicable amount of computer time and memory (e.g., Lognonne and Romanowicz, 1990; Park, 1990; Lognonne, 1991). Deuss and Woodhouse (2004) have developed a technique for solving the full coupling generalized eigenvalue problem [133], [135] by an iterative technique, not requiring the eigenvalue decomposition of very large matrices, which is well suited to the accurate modeling of small spectral segments. The first iteration of this technique is similar to the 'subspace projection method' of Park (1990), which similarly aims to approximate full coupling effects while avoiding the need to find the eigenvectors and eigenvalues of very large matrices.

1.02.9 Concluding Discussion

The normal mode formalism provides a well-developed theoretical framework for the calculation of theoretical seismograms in both spherically symmetric and three-dimensional Earth models. For spherically symmetric models, the ability to simply and quickly calculate complete theoretical seismograms plays an important role in the formulation and solution of many seismological problems involving both surface waves and long period body waves. In the three-dimensional case, the theory of mode coupling is too cumbersome to be applied in full, but it enables a number of useful approximations to be developed and tested. The increasing capacity in high-performance computing means that it becomes possible to develop and test increasingly more complete implementations of the fully coupled theory. Progress on fully numerical solutions for seismic wave fields in realistic three-dimensional spherical models (Komatitsch and Tromp, 2002a, 2002b), while it is providing a new and invaluable tool in many areas of global seismology, has not yet made it possible to calculate accurate very long period spectra. In part, this is because a way has not (yet?) been found to fully implement self-gravitation in the spectral element method, and in part because the small time step needed in finite difference and spectral element calculations leads to very long execution times; also, there are very stringent limits on the tolerable amount of numerical dispersion in the solution.

Long period modal spectra constitute a rich source of information on long wavelength heterogeneity, studies to date, we believe, having only scratched the surface. To realize the potential of this information will require large-scale coupling calculations or, possibly, other methods for calculating very long period wave fields yet to be developed. This will make it possible to bring modal spectral data increasingly to bear on furthering our understanding of the Earth's three-dimensional structure.

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