Fundamentals of structural analysis
1. Basic elasticity ................................................................. 5
2. Two-dimensional problems in elasticity ................................ 47
3. Torsion of solid sections ..................................................... 69
We consider, in this chapter, the basic ideas and relationships of the theory of elasticity. The treatment is divided into three broad sections: stress, strain, and stress–strain relationships. The third section is deferred until the end of the chapter to emphasize the fact that the analysis of stress and strain, for example, the equations of equilibrium and compatibility, does not assume a particular stress–strain law. In other words, the relationships derived in Sections 1.1–1.14 are applicable to nonlinear as well as linearly elastic bodies.

1.1 STRESS

Consider the arbitrarily shaped, three-dimensional body shown in Fig. 1.1. The body is in equilibrium under the action of externally applied forces \( P_1, P_2, \ldots \) and is assumed to constitute a continuous and deformable material, so that the forces are transmitted throughout its volume. It follows that, at any internal point \( O \), there is a resultant force \( \delta P \). The particle of material at \( O \) subjected to the force \( \delta P \) is in equilibrium, so that there must be an equal but opposite force \( \delta P \) (shown dotted in Fig. 1.1) acting on the particle at the same time. If we now divide the body by any plane \( nn \) containing \( O \), then these two forces \( \delta P \) may be considered as being uniformly distributed over a small area \( dA \) of each face of the plane at the corresponding point \( O \), as in Fig. 1.2. The stress at \( O \) is defined by the equation

\[
\text{Stress} = \lim_{dA \to 0} \frac{\delta P}{dA}
\]  

(1.1)

The directions of the forces \( \delta P \) in Fig. 1.2 are such as to produce tensile stresses on the faces of the plane \( mm \). It must be realized here that, while the direction of \( \delta P \) is absolute, the choice of plane is arbitrary, so that, although the direction of the stress at \( O \) is always in the direction of \( \delta P \), its magnitude depends upon the actual plane chosen, since a different plane has a different inclination and therefore a different value for the area \( dA \). This may be more easily understood by reference to the bar in simple tension in Fig. 1.3. On the cross-sectional plane \( mm \), the uniform stress is given by \( P/A \), while on the inclined plane \( m'm' \) the stress is of magnitude \( P/A' \). In both cases, the stresses are parallel to the direction of \( P \).

Generally, the direction of \( \delta P \) is not normal to the area \( dA \), in which case, it is usual to resolve \( \delta P \) into two components: one, \( \delta P_n \), normal to the plane and the other, \( \delta P_s \), acting in the plane itself (see Fig. 1.2). Note that, in Fig. 1.2, the plane containing \( \delta P \) is perpendicular to \( dA \). The stresses associated with these components are a normal or direct stress defined as

\[
\sigma = \lim_{dA \to 0} \frac{\delta P_n}{dA}
\]  

(1.2)
and a shear stress defined as

\[ \tau = \lim_{\delta A \to 0} \frac{\delta P}{\delta A} \]  

(1.3)

The resultant stress is computed from its components by the normal rules of vector addition, i.e.:

Resultant stress = \( \sqrt{\sigma^2 + \tau^2} \)

Generally, however, as indicated previously, we are interested in the separate effects of \( \sigma \) and \( \tau \).

However, to be strictly accurate, stress is not a vector quantity for, in addition to magnitude and direction, we must specify the plane on which the stress acts. Stress is therefore a tensor, its complete description depending on the two vectors of force and surface of action.
1.2 NOTATION FOR FORCES AND STRESSES

It is usually convenient to refer the state of stress at a point in a body to an orthogonal set of axes \(Oxyz\). In this case we cut the body by planes parallel to the direction of the axes. The resultant force \(\delta P\) acting at the point O on one of these planes may then be resolved into a normal component and two in-plane components, as shown in Fig. 1.4, thereby producing one component of direct stress and two components of shear stress.

The direct stress component is specified by reference to the plane on which it acts, but the stress components require a specification of direction in addition to the plane. We therefore allocate a single subscript to direct stress to denote the plane on which it acts and two subscripts to shear stress, the first specifying the plane, the second direction. Therefore, in Fig. 1.4, the shear stress components are \(\tau_{zx}\) and \(\tau_{zy}\) acting on the \(z\) plane and in the \(x\) and \(y\) directions, respectively, while the direct stress component is \(\sigma_z\).

We may now completely describe the state of stress at a point O in a body by specifying components of shear and direct stress on the faces of an element of side \(d_x, d_y, d_z\), formed at O by the cutting planes as indicated in Fig. 1.5.

The sides of the element are infinitesimally small, so that the stresses may be assumed to be uniformly distributed over the surface of each face. On each of the opposite faces there will be, to a first simplification, equal but opposite stresses.

We now define the directions of the stresses in Fig. 1.5 as positive, so that normal stresses directed away from their related surfaces are tensile and positive; opposite compressive stresses are negative. Shear stresses are positive when they act in the positive direction of the relevant axis in a plane on which the direct tensile stress is in the positive direction of the axis. If the tensile stress is in the
FIGURE 1.4 Components of Stress at a Point in a Body

FIGURE 1.5 Sign Conventions and Notation for Stresses at a Point in a Body
opposite direction, then positive shear stresses are in directions opposite to the positive directions of the appropriate axes.

Two types of external force may act on a body to produce the internal stress system we have already discussed. Of these, surface forces such as \( P_1, P_2, \ldots \), or hydrostatic pressure, are distributed over the surface area of the body. The surface force per unit area may be resolved into components parallel to our orthogonal system of axes, and these are generally given the symbols \( X, Y \), and \( Z \). The second force system derives from gravitational and inertia effects, and the forces are known as body forces. These are distributed over the volume of the body and the components of body force per unit volume are designated \( X, Y \), and \( Z \).

### 1.3 EQUATIONS OF EQUILIBRIUM

Generally, except in cases of uniform stress, the direct and shear stresses on opposite faces of an element are not equal, as indicated in Fig. 1.5, but differ by small amounts. Therefore if, say, the direct stress acting on the \( z \) plane is \( \sigma_z \), then the direct stress acting on the \( z + \delta z \) plane is, from the first two terms of a Taylor’s series expansion, \( \sigma_z + (\partial \sigma_z / \partial z) \delta z \).

We now investigate the equilibrium of an element at some internal point in an elastic body where the stress system is obtained by the method just described.

In Fig. 1.6, the element is in equilibrium under forces corresponding to the stresses shown and the components of body forces (not shown). Surface forces acting on the boundary of the body, although contributing to the production of the internal stress system, do not directly feature in the equilibrium equations.
Taking moments about an axis through the center of the element parallel to the \( z \) axis,

\[
\tau_{xy} \delta y \delta z \frac{\delta x}{2} + \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \frac{\delta x}{\delta x} \right) \delta y \delta z \frac{\delta x}{2} - \tau_{yx} \delta x \delta z \frac{\delta y}{2} \\
- \left( \tau_{yx} + \frac{\partial \tau_{xy}}{\partial y} \delta y \right) \delta x \delta z \delta y \delta z \frac{\delta x}{2} = 0
\]

which simplifies to

\[
\tau_{xy} \delta y \delta z \frac{\delta x}{2} + \frac{\partial \tau_{xy}}{\partial x} \frac{\delta x}{\delta x} \left( \frac{\delta x}{\delta x} \right)^2 - \tau_{yx} \delta x \delta z \frac{\delta y}{2} - \frac{\partial \tau_{xy}}{\partial y} \delta x \delta z \frac{\delta y}{2} = 0
\]

dividing through by \( \delta x \delta y \delta z \) and taking the limit as \( \delta x \) and \( \delta y \) approach zero.

Similarly,

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} \\
\tau_{xz} &= \tau_{zx} \\
\tau_{yz} &= \tau_{zy}
\end{align*}
\]

We see, therefore, that a shear stress acting on a given plane \((\tau_{xy}, \tau_{xz}, \tau_{yz})\) is always accompanied by an equal \textit{complementary shear stress} \((\tau_{yx}, \tau_{zx}, \tau_{zy})\) acting on a plane perpendicular to the given plane and in the opposite sense.

Now, considering the equilibrium of the element in the \( x \) direction,

\[
\left( \sigma_x + \frac{\partial \sigma_x}{\partial x} \delta x \right) \delta y \delta z - \sigma_x \delta y \delta z + \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \delta y \right) \delta x \delta z \\
- \tau_{yx} \delta x \delta z + \left( \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \delta z \right) \delta x \delta y \\
- \tau_{yz} \delta x \delta y + X \delta x \delta y \delta z = 0
\]

which gives

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0
\]

Or, writing \( \tau_{xy} = \tau_{yx} \) and \( \tau_{xz} = \tau_{zx} \) from Eq. (1.4),

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial z} + Y &= 0 \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0
\end{align*}
\]

\[\text{(1.5)}\]

The \textit{equations of equilibrium} must be satisfied at all interior points in a deformable body under a three-dimensional force system.
1.4 PLANE STRESS

Most aircraft structural components are fabricated from thin metal sheet, so that stresses across the thickness of the sheet are usually negligible. Assuming, say, that the $z$ axis is in the direction of the thickness, then the three-dimensional case of Section 1.3 reduces to a two-dimensional case in which $\sigma_z$, $\tau_{xz}$, and $\tau_{yz}$ are all zero. This condition is known as plane stress; the equilibrium equations then simplify to

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y &= 0
\end{aligned}$$

(1.6)

1.5 BOUNDARY CONDITIONS

The equations of equilibrium (1.5)—and also (1.6), for a two-dimensional system—satisfy the requirements of equilibrium at all internal points of the body. Equilibrium must also be satisfied at all positions on the boundary of the body, where the components of the surface force per unit area are $X$, $Y$, and $Z$. The triangular element of Fig. 1.7 at the boundary of a two-dimensional body of unit thickness is then in equilibrium under the action of surface forces on the elemental length AB of the boundary and internal forces on internal faces AC and CB.

Summation of forces in the $x$ direction gives

$$X\delta x - \sigma_x\delta y - \tau_{yx}\delta x + X\frac{1}{2}\delta x\delta y = 0$$

which, by taking the limit as $\delta x$ approaches zero, becomes

$$\bar{X} = \sigma_x \frac{dy}{ds} + \tau_{yx} \frac{dx}{ds}$$

FIGURE 1.7 Stresses on the Faces of an Element at the Boundary of a Two-Dimensional Body
The derivatives $dy/ds$ and $dx/ds$ are the direction cosines $l$ and $m$ of the angles that a normal to AB makes with the $x$ and $y$ axes, respectively. It follows that

$$X = \sigma_x l + \tau_{xy} m$$

and in a similar manner

$$Y = \sigma_y m + \tau_{yx} l$$

A relatively simple extension of this analysis produces the boundary conditions for a three-dimensional body, namely,

$$\begin{align*}
X &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\
Y &= \sigma_y m + \tau_{yx} l + \tau_{yz} n \\
Z &= \sigma_z n + \tau_{zx} m + \tau_{xz} l
\end{align*}$$

where $l$, $m$, and $n$ become the direction cosines of the angles that a normal curvature to the surface of the body makes with the $x$, $y$, and $z$ axes, respectively.

### 1.6 Determination of Stresses on Inclined Planes

The complex stress system of Fig. 1.6 is derived from a consideration of the actual loads applied to a body and is referred to a predetermined, though arbitrary, system of axes. The values of these stresses may not give a true picture of the severity of stress at that point, so that it is necessary to investigate the state of stress on other planes on which the direct and shear stresses may be greater.

We restrict the analysis to the two-dimensional system of plane stress defined in Section 1.4.

Figure 1.8(a) shows a complex stress system at a point in a body referred to axes Ox, Oy. All stresses are positive, as defined in Section 1.2. The shear stresses $\tau_{xy}$ and $\tau_{yx}$ were shown to be equal in Section 1.3. We now, therefore, designate them both $\tau_{xy}$. The element of side $\delta x$, $\delta y$ and of unit
thickness is small, so that stress distributions over the sides of the element may be assumed to be uniform. Body forces are ignored, since their contribution is a second-order term.

Suppose that we need to find the state of stress on a plane AB inclined at an angle $\theta$ to the vertical. The triangular element EDC formed by the plane and the vertical through E is in equilibrium under the action of the forces corresponding to the stresses shown in Fig. 1.8(b), where $\sigma_n$ and $\tau$ are the direct and shear components of the resultant stress on AB. Then, resolving forces in a direction perpendicular to ED, we have

$$\sigma_n \text{ED} = \sigma_x \text{EC \cos } \theta + \sigma_y \text{CD \sin } \theta + \tau_{xy} \text{EC \sin } \theta + \tau_{xy} \text{CD \cos } \theta$$

Dividing through by ED and simplifying,

$$\sigma_n = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + \tau_{xy} \sin \theta$$  \hspace{1cm} (1.8)

Now, resolving forces parallel to ED,

$$\tau \text{ED} = \sigma_x \text{EC \sin } \theta - \sigma_y \text{CD \cos } \theta - \tau_{xy} \text{EC \cos } \theta + \tau_{xy} \text{CD \sin } \theta$$

Again, dividing through by ED and simplifying,

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin \theta - \tau_{xy} \cos \theta$$  \hspace{1cm} (1.9)

Example 1.1

A cylindrical pressure vessel has an internal diameter of 2 m and is fabricated from plates 20 mm thick. If the pressure inside the vessel is 1.5 N/mm$^2$ and, in addition, the vessel is subjected to an axial tensile load of 2500 kN, calculate the direct and shear stresses on a plane inclined at an angle of 60$^\circ$ to the axis of the vessel. Calculate also the maximum shear stress.

The expressions for the longitudinal and circumferential stresses produced by the internal pressure may be found in any text on stress analysis$^1$ and are

Longitudinal stress $$(\sigma_x) = \frac{pd}{4t} = 1.5 \times 2 \times 10^3/4 \times 20 = 37.5 \text{ N/mm}^2$$

Circumferential stress $$(\sigma_y) = \frac{pd}{2t} = 1.5 \times 2 \times 10^3/2 \times 20 = 75 \text{ N/mm}^2$$

The direct stress due to the axial load will contribute to $\sigma_x$ and is given by

$$\sigma_x \text{ (axial load)} = 2500 \times 10^3/\pi \times 2 \times 10^3 \times 20 = 19.9 \text{ N/mm}^2$$

A rectangular element in the wall of the pressure vessel is then subjected to the stress system shown in Fig. 1.9. Note that no shear stresses act on the x and y planes; in this case, $\sigma_x$ and $\sigma_y$ form a biaxial stress system.

The direct stress, $\sigma_n$, and shear stress, $\tau$, on the plane AB, which makes an angle of 60$^\circ$ with the axis of the vessel, may be found by first principles by considering the equilibrium of the triangular element ABC or by direct substitution in Eqs. (1.8) and (1.9). Note that, in the latter case, $\theta = 30^\circ$ and $\tau_{xy} = 0$. Then,

$$\sigma_n = 57.4 \cos^2 30^\circ + 75 \sin^2 30^\circ = 61.8 \text{ N/mm}^2$$

$$\tau = (57.4 - 75) |\sin(2 \times 30^\circ)|/2 = -7.6 \text{ N/mm}^2$$
The negative sign for \( \tau \) indicates that the shear stress is in the direction BA and not AB.

From Eq. (1.9), when \( \tau_{xy} = 0 \),

\[
\tau = (\sigma_x - \sigma_y)(\sin \theta)/2
\]  

(i)

The maximum value of \( \tau \) therefore occurs when \( \sin \theta \) is a maximum, that is, when \( \sin \theta = 1 \) and \( \theta = 45^\circ \). Then, substituting the values of \( \sigma_x \) and \( \sigma_y \) in Eq. (i),

\[
\tau_{\text{max}} = (57.4 - 75)/2 = -8.8 \text{ N/mm}^2
\]

Example 1.2

A cantilever beam of solid, circular cross-section supports a compressive load of 50 kN applied to its free end at a point 1.5 mm below a horizontal diameter in the vertical plane of symmetry together with a torque of 1200 Nm (Fig. 1.10). Calculate the direct and shear stresses on a plane inclined at 60° to the axis of the cantilever at a point on the lower edge of the vertical plane of symmetry. See Ex. 1.1.

The direct loading system is equivalent to an axial load of 50 kN together with a bending moment of \( 50 \times 10^3 \times 1.5 = 75,000 \text{ Nmm} \) in a vertical plane. Therefore, at any point on the lower edge of the vertical plane of symmetry,
there are compressive stresses due to the axial load and bending moment that act on planes perpendicular to the axis of the beam and are given, respectively, by Eqs. (1.2) and (16.9); that is,

\[
\sigma_x (\text{axial load}) = \frac{50}{C^2} \times 10^3 = \frac{17.7}{C^2} \times 10^3 = 17.7 N/mm^2
\]

\[
\sigma_y (\text{bending moment}) = 75,000 \times \frac{30}{\pi} \times \frac{60^2}{64} = 3.5 N/mm^2
\]

The shear stress, \( \tau_{xy} \), at the same point due to the torque is obtained from Eq. (iv) in Example 3.1; that is,

\[
\tau_{xy} = 1200 \times 10^3 \times \frac{30}{\pi} \times \frac{60^2}{32} = 28.3 N/mm^2
\]

The stress system acting on a two-dimensional rectangular element at the point is shown in Fig. 1.11. Note that, since the element is positioned at the bottom of the beam, the shear stress due to the torque is in the direction shown and is negative (see Fig. 1.8).

Again, \( \sigma_n \) and \( \tau \) may be found from first principles or by direct substitution in Eqs. (1.8) and (1.9). Note that \( \theta = 30^\circ, \sigma_x = 0, \) and \( \tau_{xy} = -28.3 N/mm^2, \) the negative sign arising from the fact that it is in the opposite direction to \( \tau_{xy} \) in Fig. 1.8.

Then,

\[
\sigma_n = -21.2 \cos^2 30^\circ - 28.3 \sin 60^\circ = -40.4 N/mm^2 (\text{compression})
\]

\[
\tau = (-21.2/2) \sin 60^\circ + 28.3 \cos 60^\circ = 5.0 N/mm^2 (\text{acting in the direction AB})
\]

Different answers are obtained if the plane AB is chosen on the opposite side of AC.

## 1.7 Principal Stresses

For given values of \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \), in other words, given loading conditions, \( \sigma_n \) varies with the angle \( \theta \) and attains a maximum or minimum value when \( d\sigma_n/d\theta = 0 \). From Eq. (1.8),

\[
\frac{d\sigma_n}{d\theta} = -2\sigma_x \cos \theta \sin \theta + 2\sigma_y \sin \theta \cos \theta + 2\tau_{xy} \cos 2\theta = 0
\]

Hence,

\[-(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0\]
or

\[ \tan2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]  \hspace{1cm} (1.10)

Two solutions, \( \theta \) and \( \theta + \pi/2 \), are obtained from Eq. (1.10), so that there are two mutually perpendicular planes on which the direct stress is either a maximum or a minimum. Further, by comparison with Eqs. (1.9) and (1.10), it will be observed that these planes correspond to those on which there is no shear stress. The direct stresses on these planes are called \textit{principal stresses} and the planes themselves, \textit{principal planes}.

From Eq. (1.10),

\[ \sin2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{cos}2\theta = \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \]

and

\[ \sin2(\theta + \pi/2) = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \text{cos}2(\theta + \pi/2) = \frac{-\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \]

Rewriting Eq. (1.8) as

\[ \sigma_n = \sigma_x (1 + \cos\theta) + \sigma_y (1 - \cos\theta) + \tau_{xy} \sin\theta \]

and substituting for \( \{\sin\theta, \cos\theta\} \) and \( \{\sin(\theta + \pi/2), \cos(\theta + \pi/2)\} \) in turn gives

\[ \sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]  \hspace{1cm} (1.11)

and

\[ \sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]  \hspace{1cm} (1.12)

where \( \sigma_I \) is the \textit{maximum} or \textit{major principal stress} and \( \sigma_{II} \) is the \textit{minimum} or \textit{minor principal stress}. Note that \( \sigma_I \) is algebraically the greatest direct stress at the point while \( \sigma_{II} \) is algebraically the least. Therefore, when \( \sigma_{II} \) is negative, that is, compressive, it is possible for \( \sigma_{II} \) to be numerically greater than \( \sigma_I \).

The maximum shear stress at this point in the body may be determined in an identical manner. From Eq. (1.9),

\[ \frac{d\tau}{d\theta} = (\sigma_x - \sigma_y) \cos\theta + 2\tau_{xy} \sin\theta = 0 \]

giving

\[ \tan2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \]  \hspace{1cm} (1.13)
It follows that
\[\sin2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \cos2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\]
\[\sin2(\theta + \pi/2) = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \quad \cos2(\theta + \pi/2) = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\]

Substituting these values in Eq. (1.9) gives
\[\tau_{\text{max, min}} = \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad (1.14)\]

Here, as in the case of principal stresses, we take the maximum value as being the greater algebraic value.

Comparing Eq. (1.14) with Eqs. (1.11) and (1.12), we see that
\[\tau_{\text{max}} = \frac{\sigma_I - \sigma_{II}}{2} \quad (1.15)\]

Equations (1.14) and (1.15) give the maximum shear stress at the point in the body in the plane of the given stresses. For a three-dimensional body supporting a two-dimensional stress system, this is not necessarily the maximum shear stress at the point.

Since Eq. (1.13) is the negative reciprocal of Eq. (1.10), the angles \(\theta\) given by these two equations differ by \(90^\circ\) or, alternatively, the planes of maximum shear stress are inclined at \(45^\circ\) to the principal planes.

### 1.8 MOHR’S CIRCLE OF STRESS

The state of stress at a point in a deformable body may be determined graphically by Mohr’s circle of stress.

In Section 1.6, the direct and shear stresses on an inclined plane were shown to be given by
\[\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin2\theta \quad (1.8)\]
and
\[\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin2\theta - \tau_{xy} \cos2\theta \quad (1.9)\]
respectively. The positive directions of these stresses and the angle \(\theta\) are defined in Fig. 1.12(a). Equation (1.8) may be rewritten in the form
\[\sigma_n = \frac{\sigma_x}{2} (1 + \cos2\theta) + \frac{\sigma_y}{2} (1 - \cos2\theta) + \tau_{xy} \sin2\theta\]
or
\[\sigma_n - \frac{1}{2} (\sigma_x + \sigma_y) = \frac{1}{2} (\sigma_x - \sigma_y) \cos2\theta + \tau_{xy} \sin2\theta\]
Squaring and adding this equation to Eq. (1.9), we obtain

$$\left[ \sigma_n - \frac{1}{2} (\sigma_x + \sigma_y) \right]^2 + \tau^2 = \left[ \frac{1}{2} (\sigma_x - \sigma_y) \right]^2 + \tau_{xy}^2$$

which represents the equation of a circle of radius $\frac{1}{2} \sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$ and having its center at the point $[(\sigma_x - \sigma_y)/2, 0]$.

The circle is constructed by locating the points $Q_1 (\sigma_x, \tau_{xy})$ and $Q_2 (\sigma_y, -\tau_{xy})$ referred to axes $O\sigma\tau$, as shown in Fig. 1.12(b). The center of the circle then lies at $C$, the intersection of $Q_1Q_2$ and the $O\sigma$ axis; clearly $C$ is the point $[(\sigma_x - \sigma_y)/2, 0]$ and the radius of the circle is $\frac{1}{2} \sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$, as required. $CQ'$ is now set off at an angle $2\theta$ (positive clockwise) to $CQ_1$, $Q'$ is then the point $(\sigma_n, -\tau)$, as demonstrated next. From Fig. 1.12(b), we see that

$$ON = OC + CN$$

or, since $OC = (\sigma_x + \sigma_y)/2$, $CN = CQ' \cos(\beta - 2\theta)$, and $CQ' = CQ_1$, we have

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + CQ_1 (\cos\beta \cos 2\theta + \sin\beta \sin 2\theta)$$

But,

$$CQ_1 = \frac{CP_1}{\cos\beta} \text{ and } CP_1 = \frac{(\sigma_x - \sigma_y)}{2}$$

Hence,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + CP_1 \tan\beta \sin 2\theta$$

which, on rearranging, becomes

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta$$
as in Eq. (1.8). Similarly, it may be shown that
\[ Q'N = \tau_{xy} \cos 2\theta - \left( \frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta = -\tau \]
as in Eq. (1.9). Note that the construction of Fig. 1.12(b) corresponds to the stress system of Fig. 1.12(a), so that any sign reversal must be allowed for. Also, the \( O\sigma \) and \( O\tau \) axes must be constructed to the same scale or the equation of the circle is not represented.

The maximum and minimum values of the direct stress, that is, the major and minor principal stresses \( \sigma_1 \) and \( \sigma_2 \), occur when \( N \) (and \( Q' \)) coincide with \( B \) and \( A \), respectively. Thus,
\[ \sigma_1 = OC + \text{radius of circle} \]
\[ = \frac{(\sigma_x + \sigma_y)}{2} + \sqrt{CP_1^2 + P_1Q_1^2} \]
or
\[ \sigma_1 = \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]
and, in the same fashion,
\[ \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} - \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]
The principal planes are then given by \( 2\theta = \beta(\sigma_1) \) and \( 2\theta = \beta + \pi(\sigma_2) \).

Also, the maximum and minimum values of shear stress occur when \( Q' \) coincides with \( D \) and \( E \) at the upper and lower extremities of the circle.

At these points, \( Q'N \) is equal to the radius of the circle, which is given by
\[ CQ_1 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \]
Hence, \( \tau_{\text{max,min}} = \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \), as before. The planes of maximum and minimum shear stress are given by \( 2\theta = \beta + \pi/2 \) and \( 2\theta = \beta + 3\pi/2 \), these being inclined at \( 45^\circ \) to the principal planes.

**Example 1.3**

Direct stresses of 160 N/mm\(^2\) (tension) and 120 N/mm\(^2\) (compression) are applied at a particular point in an elastic material on two mutually perpendicular planes. The principal stress in the material is limited to 200 N/mm\(^2\) (tension). Calculate the allowable value of shear stress at the point on the given planes. Determine also the value of the other principal stress and the maximum value of shear stress at the point. Verify your answer using Mohr’s circle. See Ex. 1.1.

The stress system at the point in the material may be represented as shown in Fig. 1.13 by considering the stresses to act uniformly over the sides of a triangular element ABC of unit thickness. Suppose that the direct stress on the principal plane AB is \( \sigma \). For horizontal equilibrium of the element,
\[ \sigma_{AB} \cos \theta = \sigma_x, \ \ BC + \tau_{xy}AC \]
which simplifies to
\[ \tau_{xy} \tan \theta = \sigma - \sigma_x \] (i)
Considering vertical equilibrium gives
\[ \sigma_{AB} \sin \theta = \sigma_y AC + \tau_{xy} BC \]
or
\[ \tau_{xy} \cot \theta = \sigma - \sigma_y \] (ii)

Hence, from the product of Eqs. (i) and (ii),
\[ \tau_{xy}^2 = (\sigma - \sigma_x)(\sigma - \sigma_y) \]

Now, substituting the values \( \sigma_x = 160 \text{ N/mm}^2, \ \sigma_y = -120 \text{ N/mm}^2, \ \text{and} \ \sigma = \sigma_1 = 200 \text{ N/mm}^2 \), we have
\[ \tau_{xy} = \pm 113 \text{ N/mm}^2 \]

Replacing \( \cot \theta \) in Eq. (ii) with \( 1/\tan \theta \) from Eq. (i) yields a quadratic equation in \( \sigma \):
\[ \sigma^2 - \sigma(\sigma_x - \sigma_y) + \sigma_x \sigma_y - \tau_{xy}^2 = 0 \] (iii)

The numerical solutions of Eq. (iii) corresponding to the given values of \( \sigma_x, \ \sigma_y, \ \text{and} \ \tau_{xy} \) are the principal stresses at the point, namely,
\[ \sigma_1 = 200 \text{ N/mm}^2 \]
given
\[ \sigma_2 = -160 \text{ N/mm}^2 \]

Having obtained the principal stresses, we now use Eq. (1.15) to find the maximum shear stress, thus
\[ \tau_{\max} = \frac{200 + 160}{2} = 180 \text{ N/mm}^2 \]

The solution is rapidly verified from Mohr’s circle of stress (Fig. 1.14). From the arbitrary origin O, OP_1, and OP_2 are drawn to represent \( \sigma_x = 160 \text{ N/mm}^2 \) and \( \sigma_y = -120 \text{ N/mm}^2 \). The mid-point C of P_1P_2 is then located. Next, \( OB = \sigma_1 = 200 \text{ N/mm}^2 \) is marked out and the radius of the circle is then CB. OA is the required principal stress. Perpendiculars P_1Q_1 and P_2Q_2 to the circumference of the circle are equal to \( \pm \tau_{xy} \) (to scale), and the radius of the circle is the maximum shear stress.
Example 1.3 MATLAB®
Repeat the derivations presented in Example 1.3 using the Symbolic Math Toolbox in MATLAB®. Do not recreate Mohr’s circle. See Ex. 1.1.

Using the element shown in Fig. 1.13, derivations of the principal stresses and maximum shear stress are obtained through the following MATLAB file:

```matlab
% Declare any needed symbolic variables
syms sig tau_xy sig_x sig_y theta AB BC AC

% Define known stress values
sig_x = sym(160);
sig_y = sym(-120);
sig_val = sym(200);

% Define relationships between AB, BC, and AC
BC = AB*cos(theta);
AC = AB*sin(theta);

% For horizontal equilibrium of the element
eq1 = sig*AB*cos(theta)-sig_x*BC-tau_xy*AC;

% For vertical equilibrium of the element
eqII = sig*AB*sin(theta)-sig_y*AC-tau_xy*BC;
```

FIGURE 1.14 Solution of Example 1.3 Using Mohr’s Circle of Stress
% Solve eqI and eqII for tau_xy
tau_xyI = solve(eqI,tau_xy);
tau_xyII = solve(eqII,tau_xy);

% Take the square-root of tau_xyI times tau_xyII to get tau_xy
tau_xy_val = sqrt(tau_xyI*tau_xyII);

% Substitute the given value of sig into tau_xy
tau_xy_val = subs(tau_xy_val,sig,sig_val);

% Solve eqI for theta and substitute into eqII
eqI = simplify(eqI/cos(theta));
theta_I = solve(eqI,theta);
eqIII = subs(eqII,theta,theta_I);

% Substitute the value of tau_xy into eqIII and solve for the principle stresses (sig_p)
sig_p = solve(subs(eqIII,tau_xy,tau_xy_val),sig);
sig_I = max(double(sig_p));
sig_II = min(double(sig_p));

% Calculate the maximum shear stress using Eq. (1.15)
tau_max = (sig_I-sig_II)/2;

% Output tau_xy, the principle stresses, and tau_max to the Command Window
disp(['tau_xy = +/- 113.1371 N/mm^2'])
disp(['sig_I = ' num2str(sig_I) 'N/mm^2'])
disp(['sig_II = ' num2str(sig_II) 'N/mm^2'])
disp(['tau_max = ' num2str(tau_max) 'N/mm^2'])

The Command Window outputs resulting from this MATLAB file are as follows:

tau_xy = +/- 113.1371 N/mm^2
sig_I = 200 N/mm^2
sig_II = -160 N/mm^2
tau_max = 180 N/mm^2

1.9 STRAIN

The external and internal forces described in the previous sections cause linear and angular displacements in a deformable body. These displacements are generally defined in terms of strain. \textit{Longitudinal} or \textit{direct strains} are associated with direct stresses \( \sigma \) and relate to changes in length, while \textit{shear strains} define changes in angle produced by shear stresses. These strains are designated, with appropriate suffixes, by the symbols \( \varepsilon \) and \( \gamma \), respectively, and have the same sign as the associated stresses.

Consider three mutually perpendicular line elements OA, OB, and OC at a point O in a deformable body. Their original or unstrained lengths are \( \delta x \), \( \delta y \), and \( \delta z \), respectively. If, now, the body is subjected
to forces that produce a complex system of direct and shear stresses at O, such as that in Fig. 1.6, then the line elements deform to the positions O'A', O'B', and O'C' shown in Fig. 1.15.

The coordinates of O in the unstrained body are \((x, y, z)\) so that those of A, B, and C are \((x + \delta x, y, z)\), \((x, y + \delta y, z)\), and \((x, y, z + \delta z)\). The components of the displacement of O to O' parallel to the x, y, and z axes are \(u\), \(v\), and \(w\). These symbols are used to designate these displacements throughout the book and are defined as positive in the positive directions of the axes. We again employ the first two terms of a Taylor’s series expansion to determine the components of the displacements of A, B, and C. Thus, the displacement of A in a direction parallel to the x axis is \(u + \frac{\partial u}{\partial x} \delta x\). The remaining components are found in an identical manner and are shown in Fig. 1.15.

We now define direct strain in more quantitative terms. If a line element of length \(L\) at a point in a body suffers a change in length \(\Delta L\), then the longitudinal strain at that point in the body in the direction of the line element is

\[
\varepsilon = \lim_{\Delta L \to 0} \frac{\Delta L}{L}
\]

The change in length of the element OA is \((O'A' - OA)\), so that the direct strain at O in the x direction is obtained from the equation

\[
\varepsilon_x = \frac{O'A' - OA}{OA} = \frac{O'A' - \delta x}{\delta x}
\]

Now,

\[
(Q'A')^2 = \left(\delta x + u + \frac{\partial u}{\partial x} \delta x - u\right)^2 + \left(v + \frac{\partial v}{\partial x} \delta x - v\right)^2 + \left(w + \frac{\partial w}{\partial x} \delta x - w\right)^2
\]
or

\[ O'A' = \delta x \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} \]

which may be written, when second-order terms are neglected, as

\[ O'A' = \delta x \left(1 + 2 \frac{\partial u}{\partial x}\right)^{\frac{1}{2}} \]

Applying the binomial expansion to this expression, we have

\[ O'A' = \delta x \left(1 + \frac{\partial u}{\partial x}\right) \] (1.17)

in which squares and higher powers of \( \frac{\partial u}{\partial x} \) are ignored. Substituting for O'A' in Eq. (1.16), we have

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z}
\end{align*}
\]

It follows that

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z}
\end{align*}
\] (1.18)

The shear strain at a point in a body is defined as the change in the angle between two mutually perpendicular lines at the point. Therefore, if the shear strain in the \( xz \) plane is \( \gamma_{xz} \), then the angle between the displaced line elements \( O'A' \) and \( O'C' \) in Fig. 1.15 is \( \pi/2 - \gamma_{xz} \) radians.

Now, \( \cos A'O'C' = \cos(\pi/2 - \gamma_{xz}) = \sin\gamma_{xz} \) and as \( \gamma_{xz} \) is small, \( \cos A'O'C' = \gamma_{xz} \). From the trigonometrical relationships for a triangle,

\[
\cos A'O'C' = \frac{(O'A')^2 + (O'C')^2 - (A'C')^2}{2(O'A')(O'C')} \] (1.19)

We showed in Eq. (1.17) that

\[ O'A' = \delta x \left(1 + \frac{\partial u}{\partial x}\right) \]

Similarly,

\[ (O'C') = \delta z \left(1 + \frac{\partial w}{\partial z}\right) \]

But, for small displacements, the derivatives of \( u, v, \) and \( w \) are small compared with 1, so that, as we are concerned here with actual length rather than change in length, we may use the approximations

\[ O'A' \approx \delta x, \quad O'C' \approx \delta z \]

24 CHAPTER 1 Basic elasticity
Again, to a first approximation,

\[(A'C')^2 = \left( \delta z - \frac{\partial w}{\partial x} \delta x \right)^2 + \left( \delta x - \frac{\partial u}{\partial z} \delta z \right)^2 \]

Substituting for \(O'A', O'C', \) and \(A'C'\) in Eq. (1.19), we have

\[\cos A'O'C' = \frac{(\delta x^2) + (\delta z^2) - [\delta z - (\partial w/\partial x)\delta x]^2 - [\delta x - (\partial u/\partial z)\delta z]^2}{2\delta x\delta z}\]

Expanding and neglecting fourth-order powers gives

\[\cos A'O'C' = \frac{2(\partial w/\partial x)\delta x\delta z + 2(\partial u/\partial z)\delta x\delta z}{2\delta x\delta z}\]

or,

\[
\begin{align*}
\gamma_{xz} & = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\gamma_{xy} & = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{yz} & = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
\end{align*}
\]

Similarly,

\[
\begin{align*}
\gamma_{xz} & = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\gamma_{xy} & = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{yz} & = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
\end{align*}
\]

It must be emphasized that Eqs. (1.18) and (1.20) are derived on the assumption that the displacements involved are small. Normally, these linearized equations are adequate for most types of structural problem, but in cases where deflections are large, for example, types of suspension cable, the full, nonlinear, large deflection equations, given in many books on elasticity, must be employed.

### 1.10 Compatibility Equations

In Section 1.9, we expressed the six components of strain at a point in a deformable body in terms of the three components of displacement at that point, \(u, v,\) and \(w.\) We supposed that the body remains continuous during the deformation, so that no voids are formed. It follows that each component, \(u, v,\) and \(w,\) must be a continuous, single-valued function or, in quantitative terms,

\[u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z)\]

If voids are formed, then displacements in regions of the body separated by the voids are expressed as different functions of \(x, y,\) and \(z.\) The existence, therefore, of just three single-valued functions for displacement is an expression of the continuity or \(compatibility\) of displacement, which we presupposed.

Since the six strains are defined in terms of three displacement functions, they must bear some relationship to each other and cannot have arbitrary values. These relationships are found as follows. Differentiating \(\gamma_{xy}\) from Eq. (1.20) with respect to \(x\) and \(y\) gives

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \frac{\partial v}{\partial x} + \frac{\partial^2}{\partial x \partial y} \frac{\partial u}{\partial y}
\]
or, since the functions of \( u \) and \( v \) are continuous,

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} + \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}
\]

which may be written, using Eq. (1.18), as

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 e_x}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} \quad (1.21)
\]

In a similar manner,

\[
\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 e_z}{\partial y^2} + \frac{\partial^2 e_z}{\partial z^2} \quad (1.22)
\]

\[
\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 e_x}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} \quad (1.23)
\]

If we now differentiate \( \gamma_{xy} \) with respect to \( x \) and \( z \) and add the result to \( \gamma_{xz} \), differentiated with respect to \( y \) and \( x \), we obtain

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{xz}}{\partial y \partial x} = \frac{\partial^2 e_x}{\partial x \partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) + \frac{\partial^2 e_x}{\partial y \partial x} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)
\]

or

\[
\frac{\partial}{\partial x} \left( \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} \right) = \frac{\partial^2 e_x}{\partial x \partial y} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial^2 e_x}{\partial x \partial z} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial^2 e_x}{\partial y \partial x} \left( \frac{\partial u}{\partial z} \right)
\]

Substituting from Eqs. (1.18) and (1.21) and rearranging,

\[
2 \frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.24)
\]

Similarly,

\[
2 \frac{\partial^2 e_y}{\partial x \partial z} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.25)
\]

and

\[
2 \frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (1.26)
\]

Equations (1.21)–(1.26) are the six equations of strain compatibility which must be satisfied in the solution of three-dimensional problems in elasticity.

### 1.11 Plane Strain

Although we derived the compatibility equations and the expressions for strain for the general three-dimensional state of strain, we shall be concerned mainly with the two-dimensional case described in Section 1.4. The corresponding state of strain, in which it is assumed that particles of the body suffer
displacements in one plane only, is known as plane strain. We shall suppose that this plane is, as for plane stress, the \( xy \) plane. Then, \( \varepsilon_z, \gamma_{xz}, \) and \( \gamma_{yz} \) become zero and Eqs. (1.18) and (1.20) reduce to

\[
\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}
\]

and

\[
\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}
\]  

Further, by substituting \( \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \) in the six equations of compatibility and noting that \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy} \) are now purely functions of \( x \) and \( y \), we are left with Eq. (1.21), namely,

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}
\]

as the only equation of compatibility in the two-dimensional or plane strain case.

### 1.12 Determination of strains on inclined planes

Having defined the strain at a point in a deformable body with reference to an arbitrary system of coordinate axes, we may calculate direct strains in any given direction and the change in the angle (shear strain) between any two originally perpendicular directions at that point. We shall consider the two-dimensional case of plane strain described in Section 1.11.

An element in a two-dimensional body subjected to the complex stress system of Fig. 1.16(a) distorts into the shape shown in Fig. 1.16(b). In particular, the triangular element ECD suffers distortion to the shape E’C’D’ with corresponding changes in the length FC and angle EFC. Suppose that the known direct and shear strains associated with the given stress system are \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy} \) (the actual
relationships are investigated later) and we are required to find the direct strain $e_n$ in a direction normal to the plane ED and the shear strain $\gamma$ produced by the shear stress acting on the plane ED.

To a first order of approximation,

$$
\begin{align*}
\text{C'D'} &= CD(1 + e_n) \\
\text{C'E'} &= CE(1 + e_y) \\
\text{E'D'} &= ED(1 + e_{n+\pi/2})
\end{align*}
$$

where $e_n + \pi/2$ is the direct strain in the direction ED. From the geometry of the triangle E'C'D' in which angle E'C'D' = $\pi/2 - \gamma_{xy}$,

$$(\text{E'D'})^2 = (\text{C'D'})^2 + (\text{C'E'})^2 - 2(\text{C'D'})(\text{C'E'}) \cos(\pi/2 - \gamma_{xy})$$

or, substituting from Eqs. (1.29),

$$(\text{ED})^2(1 + e_{n+\pi/2})^2 = (\text{CD})^2(1 + e_x)^2 + (\text{CE})^2(1 + e_y)^2 - 2(\text{CD})(\text{CE})(1 + e_x)(1 + e_y) \sin \gamma_{xy}$$

Noting that $(\text{ED})^2 = (\text{CD})^2 + (\text{CE})^2$ and neglecting squares and higher powers of small quantities, this equation may be rewritten as

$$2(\text{ED})^2 e_{n+\pi/2} = 2(\text{CD})^2 e_x + 2(\text{CE})^2 e_y - 2(\text{CD})(\text{CE}) \gamma_{xy}$$

Dividing through by $2(\text{ED})^2$ gives

$$e_{n+\pi/2} = e_x \sin^2 \theta + e_y \cos^2 \theta - \cos \theta \sin \theta \gamma_{xy}$$

(1.30)

The strain $e_n$ in the direction normal to the plane ED is found by replacing the angle $\theta$ in Eq. (1.30) by $\theta - \pi/2$. Hence,

$$e_n = e_x \cos^2 \theta + e_y \sin^2 \theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

(1.31)

Turning our attention to the triangle C'E'F', we have

$$(\text{C'E'})^2 = (\text{C'F'})^2 + (\text{F'E'})^2 - 2(\text{C'F'})(\text{F'E'}) \cos(\pi/2 - \gamma)$$

(1.32)

in which

$$\text{C'E'} = CE(1 + e_y)$$
$$\text{C'F'} = CF(1 + e_n)$$
$$\text{F'E'} = FE(1 + e_{n+\pi/2})$$

Substituting for C'E', C'F', and F'E' in Eq. (1.32) and writing $\cos(\pi/2 - \gamma) = \sin \gamma$, we find

$$(\text{CE})^2(1 + e_y)^2 = (\text{CF})^2(1 + e_n)^2 + (\text{FE})^2(1 + e_{n+\pi/2})^2$$
$$- 2(\text{CF})(\text{FE})(1 + e_n)(1 + e_{n+\pi/2}) \sin \gamma$$

(1.33)

All the strains are assumed to be small, so that their squares and higher powers may be ignored. Further, $\sin \gamma \approx \gamma$ and Eq. (1.33) becomes

$$(\text{CE})^2(1 + 2e_y) = (\text{CF})^2(1 + 2e_n) + (\text{FE})^2(1 + 2e_{n+\pi/2}) - 2(\text{CF})(\text{FE})\gamma$$
From Fig. 1.16(a), \((CE)^2 = (CF)^2 + (FE)^2\) and the preceding equation simplifies to

\[2(CE)^2 \varepsilon_y = 2(CF)^2 \varepsilon_n + 2(FE)^2 \varepsilon_{n+\pi/2} - 2(CF)(FE)\gamma\]

Dividing through by \(2(CE)^2\) and transposing,

\[\gamma = \frac{\varepsilon_n \sin^2 \theta + \varepsilon_{n+\pi/2} \cos^2 \theta - \varepsilon_y}{\sin \theta \cos \theta}\]

Substitution of \(\varepsilon_n\) and \(\varepsilon_{n+\pi/2}\) from Eqs. (1.31) and (1.30) yields

\[\frac{\gamma}{2} = \frac{(\varepsilon_x - \varepsilon_y)}{2} \sin 2\theta - \frac{\gamma_{xy}}{2} \cos 2\theta \quad (1.34)\]

### 1.13 PRINCIPAL STRAINS

If we compare Eqs. (1.31) and (1.34) with Eqs. (1.8) and (1.9), we observe that they may be obtained from Eqs. (1.8) and (1.9) by replacing \(\sigma_n\) with \(\varepsilon_n\), \(\sigma_x\) by \(\varepsilon_x\), \(\sigma_y\) by \(\varepsilon_y\), \(\tau_{xy}\) by \(\gamma_{xy}/2\), and \(\tau\) by \(\gamma/2\). Therefore, for each deduction made from Eqs. (1.8) and (1.9) concerning \(\sigma_n\) and \(\tau\), there is a corresponding deduction from Eqs. (1.31) and (1.34) regarding \(\varepsilon_n\) and \(\gamma/2\).

Therefore, at a point in a deformable body, there are two mutually perpendicular planes on which the shear strain \(\gamma\) is zero and normal to which the direct strain is a maximum or minimum. These strains are the **principal strains** at that point and are given (from comparison with Eqs. (1.11) and (1.12)) by

\[\varepsilon_1 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.35)\]

and

\[\varepsilon_II = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.36)\]

If the shear strain is zero on these planes, it follows that the shear stress must also be zero; and we deduce, from Section 1.7, that the directions of the principal strains and principal stresses coincide. The related planes are then determined from Eq. (1.10) or from

\[\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad (1.37)\]

In addition, the maximum shear strain at the point is

\[\left(\frac{\gamma}{2}\right)_{\text{max}} = \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \quad (1.38)\]

or

\[\left(\frac{\gamma}{2}\right)_{\text{max}} = \frac{\varepsilon_1 - \varepsilon_II}{2} \quad (1.39)\]

(compare with Eqs. (1.14) and (1.15)).
1.14 MOHR’S CIRCLE OF STRAIN

We now apply the arguments of Section 1.13 to the Mohr’s circle of stress described in Section 1.8. A circle of strain, analogous to that shown in Fig. 1.12(b), may be drawn when $\sigma_x$, $\sigma_y$, etc., are replaced by $e_x$, $e_y$, etc., as specified in Section 1.13. The horizontal extremities of the circle represent the principal strains, the radius of the circle, half the maximum shear strain, and so on.

1.15 STRESS–STRAIN RELATIONSHIPS

In the preceding sections, we developed, for a three-dimensional deformable body, three equations of equilibrium (Eqs. (1.5)) and six strain-displacement relationships (Eqs. (1.18) and (1.20)). From the latter, we eliminated displacements, thereby deriving six auxiliary equations relating strains. These compatibility equations are an expression of the continuity of displacement, which we have assumed as a prerequisite of the analysis. At this stage, therefore, we have obtained nine independent equations toward the solution of the three-dimensional stress problem. However, the number of unknowns totals 15, comprising six stresses, six strains, and three displacements. An additional six equations are therefore necessary to obtain a solution.

So far we have made no assumptions regarding the force–displacement or stress–strain relationship in the body. This will, in fact, provide us with the required six equations, but before these are derived, it is worthwhile considering some general aspects of the analysis.

The derivation of the equilibrium, strain–displacement, and compatibility equations does not involve any assumption as to the stress–strain behavior of the material of the body. It follows that these basic equations are applicable to any type of continuous, deformable body, no matter how complex its behavior under stress. In fact, we shall consider only the simple case of linearly elastic, isotropic materials, for which stress is directly proportional to strain and whose elastic properties are the same in all directions. A material possessing the same properties at all points is said to be homogeneous.

Particular cases arise where some of the stress components are known to be zero and the number of unknowns may then be no greater than the remaining equilibrium equations which have not identically vanished. The unknown stresses are then found from the conditions of equilibrium alone and the problem is said to be statically determinate. For example, the uniform stress in the member supporting a tensile load $P$ in Fig. 1.3 is found by applying one equation of equilibrium and a boundary condition. This system is therefore statically determinate.

Statically indeterminate systems require the use of some, if not all, of the other equations involving strain–displacement and stress–strain relationships. However, whether the system be statically determinate or not, stress–strain relationships are necessary to determine deflections. The role of the six auxiliary compatibility equations will be discussed when actual elasticity problems are formulated in Chapter 2.

We now proceed to investigate the relationship of stress and strain in a three-dimensional, linearly elastic, isotropic body.

Experiments show that the application of a uniform direct stress, say $\sigma_x$, does not produce any shear distortion of the material and that the direct strain $e_x$ is given by the equation

$$e_x = \frac{\sigma_x}{E} \quad (1.40)$$
where $E$ is a constant known as the *modulus of elasticity* or *Young’s modulus*. Equation (1.40) is an expression of *Hooke’s law*. Further, $e_x$ is accompanied by lateral strains

$$e_y = -v \frac{\sigma_x}{E}, \quad e_z = -v \frac{\sigma_x}{E}$$

(1.41)

in which $v$ is a constant termed *Poisson’s ratio*.

For a body subjected to direct stresses $\sigma_x$, $\sigma_y$, and $\sigma_z$, the direct strains are, from Eqs. (1.40) and (1.41) and the *principle of superposition* (see Chapter 5, Section 5.9),

$$\begin{align*}
e_x &= \frac{1}{E} [\sigma_x - v(\sigma_y + \sigma_z)] \\
e_y &= \frac{1}{E} [\sigma_y - v(\sigma_x + \sigma_z)] \\
e_z &= \frac{1}{E} [\sigma_z - v(\sigma_x + \sigma_y)]
\end{align*}$$

(1.42)

Equations (1.42) may be transposed to obtain expressions for each stress in terms of the strains. The procedure adopted may be any of the standard mathematical approaches and gives

$$\begin{align*}
\sigma_x &= \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} e_x \\
\sigma_y &= \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} e_y \\
\sigma_z &= \frac{vE}{(1 + v)(1 - 2v)} e + \frac{E}{(1 + v)} e_z
\end{align*}$$

(1.43-1.45)

in which

$$e = e_x + e_y + e_z$$

See Eq. (1.53).

For the case of plane stress in which $\sigma_z = 0$, Eqs. (1.43) and (1.44) reduce to

$$\begin{align*}
\sigma_x &= \frac{E}{1 - v^2} (e_x + v e_y) \\
\sigma_y &= \frac{E}{1 - v^2} (e_y + v e_x)
\end{align*}$$

(1.46-1.47)

Suppose now that, at some arbitrary point in a material, there are principal strains $\varepsilon_I$ and $\varepsilon_{II}$ corresponding to principal stresses $\sigma_I$ and $\sigma_{II}$. If these stresses (and strains) are in the direction of the coordinate axes $x$ and $y$, respectively, then $\tau_{xy} = \gamma_{xy} = 0$ and, from Eq. (1.34), the shear strain on an arbitrary plane at the point inclined at an angle $\theta$ to the principal planes is

$$\gamma = (\varepsilon_I - \varepsilon_{II}) \sin 2\theta$$

(1.48)
Using the relationships of Eqs. (1.42) and substituting in Eq. (1.48), we have

\[
\gamma = \frac{1}{E} [(\sigma_1 - \nu \sigma_2) - (\sigma_2 - \nu \sigma_1)] \sin \theta
\]

or

\[
\gamma = \frac{(1 + \nu)}{E} (\sigma_1 - \sigma_2) \sin \theta
\]

(1.49)

Using Eq. (1.9) and noting that for this particular case \( \tau_{xy} = 0, \sigma_x = \sigma_1, \) and \( \sigma_y = \sigma_2, \)

\[
2\tau = (\sigma_1 - \sigma_2) \sin \theta
\]

from which we may rewrite Eq. (1.49) in terms of \( \tau \) as

\[
\gamma = \frac{2(1 + \nu)}{E} \tau
\]

(1.50)

The term \( E/(1 + \nu) \) is a constant known as the **modulus of rigidity** \( G \). Hence,

\[
\gamma = \frac{\tau}{G}
\]

and the shear strains \( \gamma_{xy}, \gamma_{xz}, \) and \( \gamma_{yz} \) are expressed in terms of their associated shear stresses as follows:

\[
\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}
\]

(1.51)

Equations (1.51), together with Eqs. (1.42), provide the additional six equations required to determine the 15 unknowns in a general three-dimensional problem in elasticity. They are, however, limited in use to a linearly elastic, isotropic body.

For the case of plane stress, they simplify to

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\
\varepsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\
\varepsilon_z &= -\frac{\nu}{E} (\sigma_x - \sigma_y) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{align*}
\]

(1.52)

It may be seen from the third of Eqs. (1.52) that the conditions of plane stress and plane strain do not necessarily describe identical situations. See Ex. 1.1.

Changes in the linear dimensions of a strained body may lead to a change in volume. Suppose that a small element of a body has dimensions \( \delta x, \delta y, \) and \( \delta z. \) When subjected to a three-dimensional stress system, the element sustains a volumetric strain \( \varepsilon \) (change in volume/unit volume) equal to

\[
\varepsilon = \frac{(1 + \varepsilon_x)\delta x (1 + \varepsilon_y)\delta y (1 + \varepsilon_z)\delta z - \delta x \delta y \delta z}{\delta x \delta y \delta z}
\]
Neglecting products of small quantities in the expansion of the right-hand side of this equation yields
\[ e = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad (1.53) \]

Substituting for \( \varepsilon_x \), \( \varepsilon_y \), and \( \varepsilon_z \) from Eqs. (1.42), we find, for a linearly elastic, isotropic body,
\[ e = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z - 2\nu(\sigma_x + \sigma_y + \sigma_z)) \]
or
\[ e = \frac{(1 - 2\nu)}{E} (\sigma_x + \sigma_y + \sigma_z) \]

In the case of a uniform hydrostatic pressure, \( \sigma_x = \sigma_y = \sigma_z = -p \) and
\[ e = -\frac{3(1 - 2\nu)}{E} p \quad (1.54) \]

The constant \( E/(1 - 2\nu) \) is known as the bulk modulus or modulus of volume expansion and is often given the symbol \( K \).

An examination of Eq. (1.54) shows that \( \nu \leq 0.5 \), since a body cannot increase in volume under pressure. Also, the lateral dimensions of a body subjected to uniaxial tension cannot increase, so that \( \nu > 0 \). Therefore, for an isotropic material \( 0 \leq \nu \leq 0.5 \) and for most isotropic materials, \( \nu \) is in the range 0.25–0.33 below the elastic limit. Above the limit of proportionality, \( \nu \) increases and approaches 0.5.

**Example 1.4**

A rectangular element in a linearly elastic, isotropic material is subjected to tensile stresses of 83 and 65 N/mm\(^2\) on mutually perpendicular planes. Determine the strain in the direction of each stress and in the direction perpendicular to both stresses. Find also the principal strains, the maximum shear stress, the maximum shear strain, and their directions at the point. Take \( E = 200,000 \) N/mm\(^2\) and \( \nu = 0.3 \). See Ex. 1.1.

If we assume that \( \sigma_x = 83 \) N/mm\(^2\) and \( \sigma_y = 65 \) N/mm\(^2\), then from Eqs (1.52),
\[ \varepsilon_x = \frac{1}{200,000} (83 - 0.3 \times 65) = 3.175 \times 10^{-4} \]
\[ \varepsilon_y = \frac{1}{200,000} (65 - 0.3 \times 83) = 2.005 \times 10^{-4} \]
\[ \varepsilon_z = \frac{-0.3}{200,000} (83 + 65) = -2.220 \times 10^{-4} \]

In this case, since there are no shear stresses on the given planes, \( \sigma_x \) and \( \sigma_y \) are principal stresses, so that \( \varepsilon_x \) and \( \varepsilon_y \) are the principal strains and are in the directions of \( \sigma_x \) and \( \sigma_y \). It follows from Eq. (1.15) that the maximum shear stress (in the plane of the stresses) is
\[ \tau_{\text{max}} = \frac{83 - 65}{2} = 9 \text{ N/mm}^2 \]
acting on planes at 45° to the principal planes.

Further, using Eq. (1.50), the maximum shear strain is
\[ \gamma_{\text{max}} = \frac{2 \times (1 + 0.3) \times 9}{200,000} \]
so that \( \gamma_{\text{max}} = 1.17 \times 10^{-4} \) on the planes of maximum shear stress.
Example 1.5
At a particular point in a structural member, a two-dimensional stress system exists where \( \sigma_x = 60 \text{ N/mm}^2 \), \( \sigma_y = -40 \text{ N/mm}^2 \), and \( \tau_{xy} = 50 \text{ N/mm}^2 \). If Young’s modulus \( E = 200,000 \text{ N/mm}^2 \) and Poisson’s ratio \( v = 0.3 \), calculate the direct strain in the \( x \) and \( y \) directions and the shear strain at the point. Also calculate the principal strains at the point and their inclination to the plane on which \( \sigma_x \) acts; verify these answers using a graphical method. See Ex. 1.1.

From Eqs. (1.52),

\[
\varepsilon_x = \frac{1}{200,000} \left( 60 + 0.3 \times 40 \right) = 360 \times 10^{-6}
\]

\[
\varepsilon_y = \frac{1}{200,000} \left( -40 - 0.3 \times 60 \right) = -290 \times 10^{-6}
\]

From Eq. (1.50), the shear modulus, \( G \), is given by

\[
G = \frac{E}{2(1 + v)} = \frac{200,000}{2(1 + 0.3)} = 76,923 \text{ N/mm}^2
\]

Hence, from Eqs. (1.52),

\[
\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{50}{76,923} = 650 \times 10^{-6}
\]

Now substituting in Eq. (1.35) for \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy}, \)

\[
\varepsilon_I = 10^{-6} \left[ \frac{360 - 290}{2} + \frac{1}{2} \sqrt{(360 + 290)^2 + 650^2} \right]
\]

which gives

\[
\varepsilon_I = 495 \times 10^{-6}
\]

Similarly, from Eq. (1.36),

\[
\varepsilon_{II} = -425 \times 10^{-6}
\]

From Eq. (1.37),

\[
\tan \theta = \frac{650 \times 10^{-6}}{360 \times 10^{-6} + 290 \times 10^{-6}} = 1
\]

Therefore,

\[
\theta = 45^\circ \text{ or } 225^\circ
\]

so that

\[
\theta = 22.5^\circ \text{ or } 112.5^\circ
\]

The values of \( \varepsilon_I, \varepsilon_{II}, \) and \( \theta \) are verified using Mohr’s circle of strain (Fig. 1.17). Axes \( O\varepsilon \) and \( O\gamma \) are set up and the points \( Q_1(360 \times 10^{-6}, 1/2 \times 650 \times 10^{-6}) \) and \( Q_2(-290 \times 10^{-6}, -1/2 \times 650 \times 10^{-6}) \)
located. The center $C$ of the circle is the intersection of $Q_1Q_2$ and the $O\varepsilon$ axis. The circle is then drawn with radius $CQ_1$ and the points $B(\varepsilon_I)$ and $A(\varepsilon_{II})$ located. Finally, angle $Q_1CB = \theta$ and angle $Q_1CA = 2\theta + \pi$.

### 1.15.1 Temperature effects

The stress–strain relationships of Eqs. (1.43)–(1.47) apply to a body or structural member at a constant uniform temperature. A temperature rise (or fall) generally results in an expansion (or contraction) of the body or structural member so that there is a change in size, that is, a strain.

Consider a bar of uniform section, of original length $L_o$, and suppose that it is subjected to a temperature change $\Delta T$ along its length; $\Delta T$ can be a rise (+ve) or fall (−ve). If the coefficient of linear expansion of the material of the bar is $\alpha$, the final length of the bar is, from elementary physics,

$$L = L_o(1 + \alpha \Delta T)$$

so that the strain, $\varepsilon$, is given by

$$\varepsilon = \frac{L - L_o}{L_o} = \alpha \Delta T$$

Suppose now that a compressive axial force is applied to each end of the bar, such that the bar returns to its original length. The mechanical strain produced by the axial force is therefore just large enough to offset the thermal strain due to the temperature change, making the total strain zero. In general terms, the total strain, $\varepsilon$, is the sum of the mechanical and thermal strains. Therefore, from Eqs. (1.40) and (1.55),

$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T$$

In the case where the bar is returned to its original length or if the bar had not been allowed to expand at all, the total strain is zero and, from Eq. (1.56),

$$\sigma = -E \alpha \Delta T$$
Equations (1.42) may now be modified to include the contribution of thermal strain. Therefore, by comparison with Eq. (1.56),

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] + \alpha \Delta T \\
\varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \nu(\sigma_x + \sigma_z) \right] + \alpha \Delta T \\
\varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \nu(\sigma_x + \sigma_y) \right] + \alpha \Delta T
\end{align*}
\]  

(1.58)

Equations (1.58) may be transposed in the same way as Eqs. (1.42) to give stress–strain relationships rather than strain–stress relationships; that is,

\[
\begin{align*}
\sigma_x &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \varepsilon + \frac{E}{(1 + \nu)} \varepsilon_x - \frac{E}{(1 - 2\nu)} \alpha \Delta T \\
\sigma_y &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \varepsilon + \frac{E}{(1 + \nu)} \varepsilon_y - \frac{E}{(1 - 2\nu)} \alpha \Delta T \\
\sigma_z &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \varepsilon + \frac{E}{(1 + \nu)} \varepsilon_z - \frac{E}{(1 - 2\nu)} \alpha \Delta T
\end{align*}
\]  

(1.59)

For the case of plane stress in which \(\sigma_z = 0\), these equations reduce to

\[
\begin{align*}
\sigma_x &= \frac{E}{(1 - \nu^2)} (\varepsilon_x + \nu \varepsilon_y) - \frac{E}{(1 - \nu)} \alpha \Delta T \\
\sigma_y &= \frac{E}{(1 - \nu^2)} (\varepsilon_y + \nu \varepsilon_x) - \frac{E}{(1 - \nu)} \alpha \Delta T
\end{align*}
\]  

(1.60)

Example 1.6

A composite bar of length \(L\) has a central core of copper loosely inserted in a sleeve of steel; the ends of the steel and copper are attached to each other by rigid plates. If the bar is subjected to a temperature rise \(\Delta T\), determine the stress in the steel and in the copper and the extension of the composite bar. The copper core has a Young’s modulus \(E_c\), a cross-sectional area \(A_c\), and a coefficient of linear expansion \(\alpha_c\); the corresponding values for the steel are \(E_s\), \(A_s\), and \(\alpha_s\). See Ex. 1.1.

Assume that \(\alpha_c > \alpha_s\). If the copper core and steel sleeve are allowed to expand freely, their final lengths would be different, since they have different values of the coefficient of linear expansion. However, since they are rigidly attached at their ends, one restrains the other and an axial stress is induced in each. Suppose that this stress is \(\sigma_x\). Then, in Eqs. (1.58), \(\sigma_z = 0\) or \(\sigma_x = \sigma_z = 0\); the total strain in the copper and steel is then, respectively,

\[
\begin{align*}
\varepsilon_c &= \frac{\sigma_c}{E_c} + \alpha_c \Delta T \\
\varepsilon_s &= \frac{\sigma_s}{E_s} + \alpha_s \Delta T
\end{align*}
\]  

(i)  

(ii)
The total strain in the copper and steel is the same, since their ends are rigidly attached to each other. Therefore, from compatibility of displacement,

\[
\frac{\sigma_c}{E_c} + \alpha_c \Delta T = \frac{\sigma_s}{E_s} + \alpha_s \Delta T
\]

No external axial load is applied to the bar, so that

\[
\sigma_cA_c + \sigma_sA_s = 0
\]

that is,

\[
\sigma_s = -\frac{A_c}{A_s} \sigma_c
\]

Substituting for \(\sigma_s\) in Eq. (iii) gives

\[
\sigma_c \left( \frac{1}{E_c} + \frac{A_c}{A_s E_s} \right) = \Delta T (\alpha_s - \alpha_c)
\]

from which

\[
\sigma_c = \frac{\Delta T (\alpha_s - \alpha_c) A_c E_s}{A_s E_s + A_c E_c}
\]

Also \(\alpha_c > \alpha_s\), so that \(\sigma_c\) is negative and therefore compressive. Now substituting for \(\sigma_c\) in Eq. (iv),

\[
\sigma_s = -\frac{\Delta T (\alpha_s - \alpha_c) A_c E_s}{A_s E_s + A_c E_c}
\]

which is positive and therefore tensile, as would be expected by a physical appreciation of the situation.

Finally, the extension of the compound bar, \(\delta\), is found by substituting for \(\sigma_c\) in Eq. (i) or for \(\sigma_s\) in Eq. (ii). Then,

\[
\delta = \Delta T L \left( \frac{\alpha_c A_c E_c + \alpha_s A_s E_s}{A_s E_s + A_c E_c} \right)
\]

### 1.16 EXPERIMENTAL MEASUREMENT OF SURFACE STRAINS

Stresses at a point on the surface of a piece of material may be determined by measuring the strains at the point, usually by electrical resistance strain gauges arranged in the form of a rosette, as shown in Fig. 1.18. Suppose that \(\varepsilon_1\) and \(\varepsilon_II\) are the principal strains at the point, then if \(\varepsilon_{x}, \varepsilon_{y}, \) and \(\varepsilon_{xy}\) are the measured strains in the directions \(\theta, (\theta + \alpha), (\theta + \alpha + \beta)\) to \(\varepsilon_1\), we have, from the general direct strain relationship of Eq. (1.31),

\[
\varepsilon_{x} = \varepsilon_1 \cos^2 \theta + \varepsilon_II \sin^2 \theta
\]

since \(\varepsilon_{x}\) becomes \(\varepsilon_1\), \(\varepsilon_{y}\) becomes \(\varepsilon_II\), and \(\gamma_{xy}\) is zero, since the \(x\) and \(y\) directions have become principal directions. Rewriting Eq. (1.61), we have

\[
\varepsilon_{x} = \varepsilon_1 \left( \frac{1 + \cos 2\theta}{2} \right) + \varepsilon_II \left( \frac{1 - \cos 2\theta}{2} \right)
\]
or

\[ \varepsilon_a = \frac{1}{2} (\varepsilon_I + \varepsilon_{II}) + \frac{1}{2} (\varepsilon_I - \varepsilon_{II}) \cos 2\theta \]  

(1.62)

Similarly,

\[ \varepsilon_b = \frac{1}{2} (\varepsilon_I + \varepsilon_{II}) + \frac{1}{2} (\varepsilon_I - \varepsilon_{II}) \cos (\theta + \alpha) \]  

(1.63)

and

\[ \varepsilon_c = \frac{1}{2} (\varepsilon_I + \varepsilon_{II}) + \frac{1}{2} (\varepsilon_I - \varepsilon_{II}) \cos (\theta + \alpha + \beta) \]  

(1.64)

Therefore, if \( \varepsilon_a, \varepsilon_b, \) and \( \varepsilon_c \) are measured in given directions, that is, given angles \( \alpha \) and \( \beta \), then \( \varepsilon_I, \varepsilon_{II}, \) and \( \theta \) are the only unknowns in Eqs. (1.62)–(1.64).

The principal stresses are now obtained by substitution of \( \varepsilon_I \) and \( \varepsilon_{II} \) in Eqs. (1.52).

Thus,

\[ \varepsilon_I = \frac{1}{E} (\sigma_I - v\sigma_{II}) \]  

(1.65)

and

\[ \varepsilon_{II} = \frac{1}{E} (\sigma_{II} - v\sigma_I) \]  

(1.66)

Solving Eqs. (1.65) and (1.66) gives

\[ \sigma_I = \frac{E}{1 - v^2} (\varepsilon_I + v\varepsilon_{II}) \]  

(1.67)

and

\[ \sigma_{II} = \frac{E}{1 - v^2} (\varepsilon_{II} + v\varepsilon_I) \]  

(1.68)
A typical rosette would have $\alpha = \beta = 45^\circ$, in which case the principal strains are most conveniently found using the geometry of Mohr’s circle of strain. Suppose that the arm $a$ of the rosette is inclined at some unknown angle $\theta$ to the maximum principal strain, as in Fig. 1.18. Then, Mohr’s circle of strain is as shown in Fig. 1.19; the shear strains $\gamma_a$, $\gamma_b$, and $\gamma_c$ do not feature in the analysis and are therefore ignored. From Fig. 1.19,

\[
OC = \frac{1}{2}(\varepsilon_a + \varepsilon_c)
\]
\[
CN = \varepsilon_a - OC = \frac{1}{2}(\varepsilon_a - \varepsilon_c)
\]
\[
QN = CM = \varepsilon_b - OC = \varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)
\]

The radius of the circle is $CQ$ and

\[
CQ = \sqrt{CN^2 + QN^2}
\]

Hence,

\[
CQ = \sqrt{\left[\frac{1}{2}(\varepsilon_a - \varepsilon_c)\right]^2 + \left[\varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)\right]^2}
\]

which simplifies to

\[
CQ = \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}
\]

Therefore, $\varepsilon_I$, which is given by

\[
\varepsilon_I = OC + \text{radius of circle}
\]
is
\[
\varepsilon_1 = \frac{1}{2} (\varepsilon_a + \varepsilon_c) + \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}
\] (1.69)

Also,
\[
\varepsilon_{II} = OC - \text{radius of circle}
\]
that is,
\[
\varepsilon_{II} = \frac{1}{2} (\varepsilon_a + \varepsilon_c) - \frac{1}{\sqrt{2}} \sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}
\] (1.70)

Finally, the angle \(\theta\) is given by
\[
\tan 2\theta = \frac{QN}{CN} = \frac{\varepsilon_b - \frac{1}{2} (\varepsilon_a + \varepsilon_c)}{\frac{1}{2} (\varepsilon_a - \varepsilon_c)}
\]
that is,
\[
\tan 2\theta = \frac{2\varepsilon_b - \varepsilon_a - \varepsilon_c}{\varepsilon_a - \varepsilon_c}
\] (1.71)

A similar approach may be adopted for a 60° rosette.

**Example 1.7**

A bar of solid circular cross-section has a diameter of 50 mm and carries a torque, \(T\), together with an axial tensile load, \(P\). A rectangular strain gauge rosette attached to the surface of the bar gives the following strain readings: \(\varepsilon_a = 1000 \times 10^{-6}\), \(\varepsilon_b = -200 \times 10^{-6}\), and \(\varepsilon_c = -300 \times 10^{-6}\), where the gauges \(a\) and \(c\) are in line with, and perpendicular to, the axis of the bar, respectively. If Young’s modulus, \(E\), for the bar is 70,000 N/mm\(^2\) and Poisson’s ratio, \(\nu\), is 0.3; calculate the values of \(T\) and \(P\). See Ex. 1.1.

Substituting the values of \(\varepsilon_a\), \(\varepsilon_b\), and \(\varepsilon_c\) in Eq. (1.69),
\[
\varepsilon_1 = \frac{10^{-6}}{2} (1000 - 300) + \frac{10^{-6}}{\sqrt{2}} \sqrt{(1000 + 200)^2 + (-200 + 300)^2}
\]
which gives
\[
\varepsilon_1 = 1202 \times 10^{-6}
\]

Similarly, from Eq. (1.70),
\[
\varepsilon_{II} = -502 \times 10^{-6}
\]

Now, substituting for \(\varepsilon_1\) and \(\varepsilon_{II}\) in Eq. (1.67),
\[
\sigma_1 = 70,000 \times 10^{-6} \left[1202 - 0.3(502)\right]/[1 - (0.3)^2] = 80.9 \text{ N/mm}^2
\]
Similarly, from Eq. (1.68),
\[
\sigma_{II} = -10.9 \text{ N/mm}^2
\]
Since \( \sigma_y = 0 \), Eqs. (1.11) and (1.12) reduce to

\[
\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2} \sqrt{\sigma_x^2 + 4 \tau_{xy}^2}
\]

(i)

and

\[
\sigma_{II} = \frac{\sigma_x}{2} - \frac{1}{2} \sqrt{\sigma_x^2 + 4 \tau_{xy}^2}
\]

(ii)

respectively. Adding Eqs. (i) and (ii), we obtain

\[
\sigma_1 + \sigma_{II} = \sigma_x
\]

Then,

\[
\sigma_x = 80.9 - 10.9 = 70 \text{ N/mm}^2
\]

Therefore,

\[
\sigma_x = 80.9 - 10.9 = 70 \text{ N/mm}^2
\]

For an axial load \( P \),

\[
\sigma_x = 70 \text{ N/mm}^2 = \frac{P}{A} = \frac{P}{\pi \times 50^2/4}
\]

from which

\[
P = 137.4 \text{kN}
\]

Substituting for \( \sigma_x \) in either of Eq. (i) or (ii) gives

\[
\tau_{xy} = 29.7 \text{ N/mm}^2
\]

From the theory of the torsion of circular section bars (see Eq. (iv) in Example 3.1),

\[
\tau_{xy} = 29.7 \text{ N/mm}^2 = \frac{T r}{J} = \frac{T \times 25}{\pi \times 50^4/32}
\]

from which

\[
T = 0.7 \text{kNm}
\]

Note that \( P \) could have been found directly in this particular case from the axial strain. Thus, from the first of Eqs. (1.52),

\[
\sigma_x = E \varepsilon_x = 70000 \times 1000 \times 10^{-6} = 70 \text{ N/mm}^2
\]

as before.

---

**Example 1.7 MATLAB**

Repeat the derivations presented in Example 1.7 using the Symbolic Math Toolbox in MATLAB. To obtain the same results as Example 1.7, set the number of significant digits used in calculations to four. See Ex. 1.1.

Calculations of \( T \) and \( P \) are obtained through the following MATLAB file:

```matlab
% Declare any needed symbolic variables
syms e_a e_b e_c E v sig_y sig_x A J r T tau_xy
```
% Set the significant digits
digits(4);

% Define known variable values
e_a = sym(1000*10^(-6));
e_b = sym(-200*10^(-6));
e_c = sym(-300*10^(-6));
E = sym(70000);
v = sym(0.3);
sig_y = sym(0);
A = sym(pi*50^2/4,'d');
J = sym(pi*50^4/32,'d');
r = sym(25);

% Evaluate Eqs (1.69) and (1.70)
e_I = vpa((e_a + e_c)/2) + vpa(sqrt(((e_a-e_b)^2 + (e_c - e_c)^2))/vpa(sqrt(2)));
e_II = vpa((e_a + e_c)/2) - vpa(sqrt(((e_a-e_b)^2 + (e_c - e_c)^2))/vpa(sqrt(2)));

% Substitute e_I and e_II into Eqs (1.67) and (1.68)
sig_I = vpa(E*(e_I+v*e_II)/(1-v^2));
sig_II = vpa(E*(e_II+v*e_I)/(1-v^2));

% Evaluate Eqs (1.11) and (1.12)
eqI = vpa(-sig_I + (sig_x+sig_y)/2 + sqrt((sig_x-sig_y)^2 + 4*tau_xy^2)/2);
eqII = vpa(-sig_II + (sig_x+sig_y)/2 - sqrt((sig_x-sig_y)^2 + 4*tau_xy^2)/2);

% Add eqI and eqII and solve for the value of sig_x
sig_x_val = vpa(solve(eqI+eqII,sig_x));

% Calculate the axial load (P in kN)
P = vpa(sig_x_val*A/1000);
P = round(double(P)*10)/10;

% Substitute sig_x back into eqI and solve for tau_xy
tau_xy_val = sym(max(double(solve(subs(eqI,sig_x,sig_x_val),tau_xy))));

% Calculate the applied torsion (T in kN-m) using Eq. (iv) in Example 3.1
T = vpa(tau_xy_val*J/r/1000/1000);
T = round(double(T)*10)/10;

% Output values for P and T to the Command Window
disp(["P = '",num2str(P) , 'kN"])
disp(["T = '",num2str(T) , 'kN m"])
**References**


**Additional Reading**


**PROBLEMS**

**P.1.1.** A structural member supports loads that produce, at a particular point, a direct tensile stress of 80 N/mm² and a shear stress of 45 N/mm² on the same plane. Calculate the values and directions of the principal stresses at the point and also the maximum shear stress, stating on which planes this acts.

**Answer**

\[ \sigma_I = 100.2 \text{ N/mm}^2, \quad \theta = 24^\circ 11' \]
\[ \sigma_{II} = -20.2 \text{ N/mm}^2, \quad \theta = 114^\circ 11' \]
\[ \tau_{max} = 60.2 \text{ N/mm}^2 \text{ at } 45^\circ \text{ to principal planes} \]

**P.1.2.** At a point in an elastic material, there are two mutually perpendicular planes, one of which carries a direct tensile stress of 50 N/mm² and a shear stress of 40 N/mm², while the other plane is subjected to a direct compressive stress of 35 N/mm² and a complementary shear stress of 40 N/mm². Determine the principal stresses at the point, the position of the planes on which they act, and the position of the planes on which there is no normal stress.

**Answer**

\[ \sigma_I = 65.9 \text{ N/mm}^2, \quad \theta = 21^\circ 38' \]
\[ \sigma_{II} = -50.9 \text{ N/mm}^2, \quad \theta = 111^\circ 38' \]

No normal stress on planes at 70°21’ and −27°5’ to vertical.

**P.1.3.** The following are varying combinations of stresses acting at a point and referred to axes \(x\) and \(y\) in an elastic material. Using Mohr’s circle of stress determine the principal stresses at the point and their directions for each combination.

<table>
<thead>
<tr>
<th>( \sigma_x ) (N/mm²)</th>
<th>( \sigma_y ) (N/mm²)</th>
<th>( \tau_{xy} ) (N/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) +54</td>
<td>+30</td>
<td>+5</td>
</tr>
<tr>
<td>(ii) +30</td>
<td>+54</td>
<td>−5</td>
</tr>
<tr>
<td>(iii) −60</td>
<td>−36</td>
<td>+5</td>
</tr>
<tr>
<td>(iv) +30</td>
<td>−50</td>
<td>+30</td>
</tr>
</tbody>
</table>

**Answer**

(i) \( \sigma_I = +55 \text{ N/mm}^2 \quad \sigma_{II} = +29 \text{ N/mm}^2 \quad \sigma_I \text{ at } 11.5^\circ \text{ to } x \text{ axis.} \)
(ii) \( \sigma_I = +55 \text{ N/mm}^2 \quad \sigma_{II} = +29 \text{ N/mm}^2 \quad \sigma_{II} \text{ at } 11.5^\circ \text{ to } x \text{ axis.} \)
(iii) \( \sigma_I = -34.5 \text{ N/mm}^2 \quad \sigma_{II} = -61 \text{ N/mm}^2 \quad \sigma_I \text{ at } 79.5^\circ \text{ to } x \text{ axis.} \)
(iv) \( \sigma_I = +40 \text{ N/mm}^2 \quad \sigma_{II} = -60 \text{ N/mm}^2 \quad \sigma_I \text{ at } 18.5^\circ \text{ to } x \text{ axis.} \)

**P.1.3 MATLAB** Repeat P.1.3 by creating a script in MATLAB in place of constructing Mohr’s circle. In addition to calculating the principal stresses, also calculate the maximum shear stress at the point for each combination. Do not repeat the calculation of the principal stress directions.
Answer: (i) $\tau_{\text{max}} = 13 \text{ N/mm}^2$, (ii) $\tau_{\text{max}} = 13 \text{ N/mm}^2$
(ii) $\tau_{\text{max}} = 13 \text{ N/mm}^2$, (iv) $\tau_{\text{max}} = 50 \text{ N/mm}^2$

$\sigma_1$ and $\sigma_{\text{II}}$ for each combination are as shown in P.1.3

P.1.4. The state of stress at a point is caused by three separate actions, each of which produces a pure, unidirectional tension of 10 N/mm$^2$ individually but in three different directions, as shown in Fig. P.1.4. By transforming the individual stresses to a common set of axes ($x$, $y$), determine the principal stresses at the point and their directions.

Answer: $\sigma_1 = \sigma_{\text{II}} = 15 \text{ N/mm}^2$. All directions are principal directions.

P.1.5. A shear stress $\tau_{xy}$ acts in a two-dimensional field in which the maximum allowable shear stress is denoted by $\tau_{\text{max}}$ and the major principal stress by $\sigma_1$. Derive, using the geometry of Mohr’s circle of stress, expressions for the maximum values of direct stress which may be applied to the $x$ and $y$ planes in terms of these three parameters.

Answer: 
\[
\sigma_x = \sigma_1 - \tau_{\text{max}} + \sqrt{\tau_{\text{max}}^2 - \tau_{xy}^2}
\]
\[
\sigma_y = \sigma_1 - \tau_{\text{max}} - \sqrt{\tau_{\text{max}}^2 - \tau_{xy}^2}
\]

P.1.6. A solid shaft of circular cross-section supports a torque of 50 kNm and a bending moment of 25 kNm. If the diameter of the shaft is 150 mm, calculate the values of the principal stresses and their directions at a point on the surface of the shaft.

Answer: $\sigma_1 = 121.4 \text{ N/mm}^2$, $\theta = 31^\circ 43'$
$\sigma_{\text{II}} = -46.4 \text{ N/mm}^2$, $\theta = 121^\circ 43'$

P.1.7. An element of an elastic body is subjected to a three-dimensional stress system $\sigma_x$, $\sigma_y$, and $\sigma_z$. Show that, if the direct strains in the directions $x$, $y$, and $z$ are $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$, then

\[
\sigma_x = \lambda \varepsilon_x + 2G\varepsilon_y, \quad \sigma_y = \lambda \varepsilon_y + 2G\varepsilon_x, \quad \sigma_z = \lambda \varepsilon_z + 2G\varepsilon_x
\]
where
\[ \lambda = \frac{V E}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad e = \varepsilon_x + \varepsilon_y + \varepsilon_z \]
the volumetric strain.

**P.1.8.** Show that the compatibility equation for the case of plane strain, namely,
\[ \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} \]
may be expressed in terms of direct stresses \( \sigma_x \) and \( \sigma_y \) in the form
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)(\sigma_x + \sigma_y) = 0 \]

**P.1.9.** A bar of mild steel has a diameter of 75 mm and is placed inside a hollow aluminum cylinder of internal diameter 75 mm and external diameter 100 mm; both bar and cylinder are the same length. The resulting composite bar is subjected to an axial compressive load of 1000 kN. If the bar and cylinder contract by the same amount, calculate the stress in each. The temperature of the compressed composite bar is then reduced by 150°C but no change in length is permitted. Calculate the final stress in the bar and in the cylinder if \( E_{\text{steel}} = 200,000 \text{ N/mm}^2, E_{\text{aluminum}} = 80,000 \text{ N/mm}^2, \alpha_{\text{steel}} = 0.000012/°C \) and \( \alpha_{\text{aluminum}} = 0.000005/°C \).

**Answer:**
Due to load:
- \( \sigma_{\text{steel}} = 172.6 \text{ N/mm}^2 \) (compression)
- \( \sigma_{\text{aluminum}} = 69.1 \text{ N/mm}^2 \) (compression).

Final stress:
- \( \sigma_{\text{steel}} = 187.4 \text{ N/mm}^2 \) (tension)
- \( \sigma_{\text{aluminum}} = 9.1 \text{ N/mm}^2 \) (compression).

**MATLAB** Repeat P.1.9 by creating a script in MATLAB using the Symbolic Math Toolbox for an axial tension load of 1000 kN.

**Answer:**
Due to load:
- \( \sigma_{\text{steel}} = 172.6 \text{ N/mm}^2 \) (tension)
- \( \sigma_{\text{aluminum}} = 69.1 \text{ N/mm}^2 \) (tension)

Final stress:
- \( \sigma_{\text{steel}} = 532.6 \text{ N/mm}^2 \) (tension)
- \( \sigma_{\text{aluminum}} = 129.1 \text{ N/mm}^2 \) (tension)

**P.1.10.** In Fig. P.1.10, the direct strains in the directions \( a, b, c \) are \(-0.002, -0.002, \) and \(+0.002\), respectively. If \( I \) and \( II \) denote principal directions, find \( \varepsilon_I, \varepsilon_{II}, \) and \( \theta.\)

**Answer:** \( \varepsilon_I = +0.00283, \quad \varepsilon_{II} = -0.00283, \quad \theta = -22.5° \) or \(+67.5°\

**P.1.11.** The simply supported rectangular beam shown in Fig. P.1.11 is subjected to two symmetrically placed transverse loads each of magnitude \( Q.\) A rectangular strain gauge rosette located at a point \( P \) on the centroidal axis on one vertical face of the beam gives strain readings as follows: \( \varepsilon_a = -222 \times 10^{-6}, \quad \varepsilon_b = -213 \times 10^{-6}, \) and \( \varepsilon_c = +45 \times 10^{-6}.\) The longitudinal stress \( \sigma_x \) at the point \( P \) due to an external
compressive force is 7 N/mm². Calculate the shear stress $\tau$ at the point $P$ in the vertical plane and hence the transverse load $Q$:

$$(Q = 2bd\tau/3, \text{ where } b = \text{ breadth, } d = \text{ depth of beam})$$

$$E = 31,000 \text{ N/mm}^2, \quad v = 0.2$$

Answer: $\tau_{xy} = 3.16 \text{ N/mm}^2, \quad Q = 94.8 \text{ kN}$

P.1.11. MATLAB Repeat P.1.11 by creating a script in MATLAB using the Symbolic Math Toolbox for the following cross-section dimensions and longitudinal stress combinations:

<table>
<thead>
<tr>
<th>$b$</th>
<th>$d$</th>
<th>$\sigma_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>150 mm</td>
<td>300 mm</td>
<td>7 N/mm²</td>
</tr>
<tr>
<td>100 mm</td>
<td>250 mm</td>
<td>11 N/mm²</td>
</tr>
<tr>
<td>270 mm</td>
<td>270 mm</td>
<td>9 N/mm²</td>
</tr>
</tbody>
</table>

Answer:

(i) $\tau_{xy} = 3.16 \text{ N/mm}^2, \quad Q = 94.8 \text{ kN}$

(ii) $\tau_{xy} = 2.5 \text{ N/mm}^2, \quad Q = 41.7 \text{ kN}$

(iii) $\tau_{xy} = 1.41 \text{ N/mm}^2, \quad Q = 68.5 \text{ kN}$