CHAPTER 1

VECTOR ANALYSIS

1.1 DEFINITIONS, ELEMENTARY APPROACH

In science and engineering we frequently encounter quantities that have magnitude and magnitude only: mass, time, and temperature. These we label **scalar** quantities, which remain the same no matter what coordinates we use. In contrast, many interesting physical quantities have magnitude and, in addition, an associated direction. This second group includes displacement, velocity, acceleration, force, momentum, and angular momentum. Quantities with magnitude and direction are labeled **vector** quantities. Usually, in elementary treatments, a vector is defined as a quantity having magnitude and direction. To distinguish vectors from scalars, we identify vector quantities with boldface type, that is, **V**.

Our vector may be conveniently represented by an arrow, with length proportional to the magnitude. The direction of the arrow gives the direction of the vector, the positive sense of direction being indicated by the point. In this representation, vector addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.1}$$

consists in placing the rear end of vector **B** at the point of vector **A**. Vector **C** is then represented by an arrow drawn from the rear of **A** to the point of **B**. This procedure, the triangle law of addition, assigns meaning to Eq. (1.1) and is illustrated in Fig. 1.1. By completing the parallelogram, we see that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1.2}$$

as shown in Fig. 1.2. In words, vector addition is **commutative**.

For the sum of three vectors

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C},$$

Fig. 1.3, we may first add **A** and **B**:

$$\mathbf{A} + \mathbf{B} = \mathbf{E}.$$

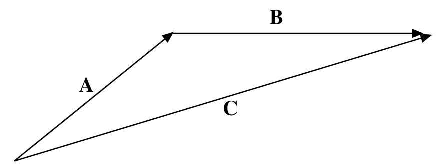

FIGURE 1.1 Triangle law of vector addition.

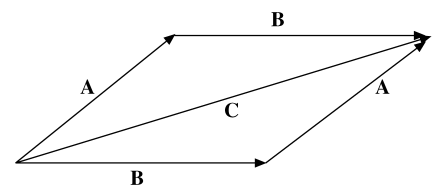

FIGURE 1.2 Parallelogram law of vector addition.

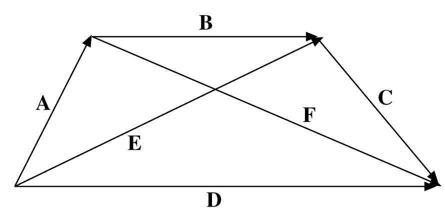

FIGURE 1.3 Vector addition is associative.

Then this sum is added to **C**:

$$\mathbf{D} = \mathbf{E} + \mathbf{C}.$$

Similarly, we may first add **B** and **C**:

$$\mathbf{B} + \mathbf{C} = \mathbf{F}.$$

Then

$$\mathbf{D} = \mathbf{A} + \mathbf{F}.$$

In terms of the original expression,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

Vector addition is **associative**.

A direct physical example of the parallelogram addition law is provided by a weight suspended by two cords. If the junction point (O in Fig. 1.4) is in equilibrium, the vector

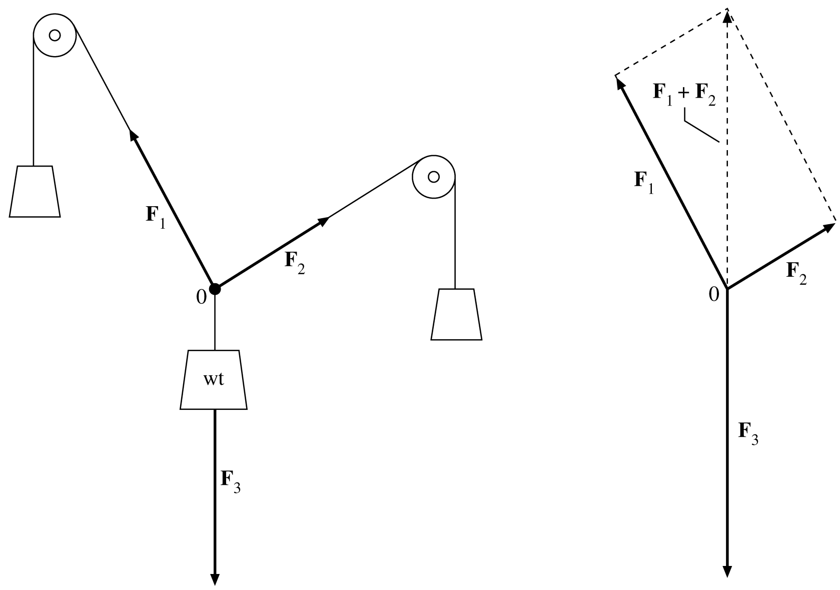

FIGURE 1.4 Equilibrium of forces: $\mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{F}_3$.

sum of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must just cancel the downward force of gravity, $\mathbf{F}_3$. Here the parallelogram addition law is subject to immediate experimental verification.[1]

Subtraction may be handled by defining the negative of a vector as a vector of the same magnitude but with reversed direction. Then

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

In Fig. 1.3,

$$\mathbf{A} = \mathbf{E} - \mathbf{B}.$$

Note that the vectors are treated as geometrical objects that are independent of any coordinate system. This concept of independence of a preferred coordinate system is developed in detail in the next section.

The representation of vector $\mathbf{A}$ by an arrow suggests a second possibility. Arrow $\mathbf{A}$ (Fig. 1.5), starting from the origin,[2] terminates at the point (A_x, A_y, A_z). Thus, if we agree that the vector is to start at the origin, the positive end may be specified by giving the Cartesian coordinates (A_x, A_y, A_z) of the arrowhead.

Although $\mathbf{A}$ could have represented any vector quantity (momentum, electric field, etc.), one particularly important vector quantity, the displacement from the origin to the point

[1] Strictly speaking, the parallelogram addition was introduced as a definition. Experiments show that if we assume that the forces are vector quantities and we combine them by parallelogram addition, the equilibrium condition of zero resultant force is satisfied.

[2] We could start from any point in our Cartesian reference frame; we choose the origin for simplicity. This freedom of shifting the origin of the coordinate system without affecting the geometry is called **translation invariance**.

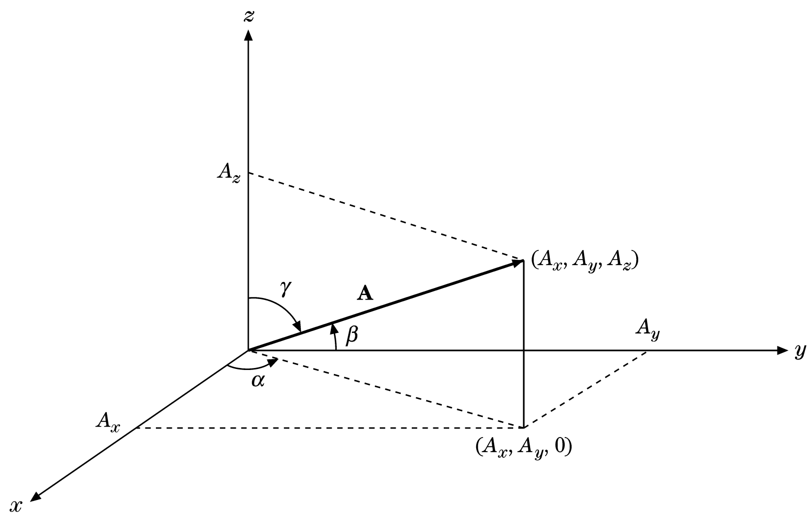

FIGURE 1.5 Cartesian components and direction cosines of **A**.

(x, y, z), is denoted by the special symbol **r**. We then have a choice of referring to the displacement as either the vector **r** or the collection (x, y, z), the coordinates of its endpoint:

$$\mathbf{r} \leftrightarrow (x, y, z). \tag{1.3}$$

Using r for the magnitude of vector **r**, we find that Fig. 1.5 shows that the endpoint coordinates and the magnitude are related by

$$x = r\cos\alpha, \qquad y = r\cos\beta, \qquad z = r\cos\gamma. \tag{1.4}$$

Here $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines**, α being the angle between the given vector and the positive x-axis, and so on. One further bit of vocabulary: The quantities A_x, A_y, and A_z are known as the (Cartesian) **components** of **A** or the **projections** of **A**, with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Thus, any vector **A** may be resolved into its components (or projected onto the coordinate axes) to yield $A_x = A\cos\alpha$, etc., as in Eq. (1.4). We may choose to refer to the vector as a single quantity **A** or to its components (A_x, A_y, A_z). Note that the subscript x in A_x denotes the x component and not a dependence on the variable x. The choice between using **A** or its components (A_x, A_y, A_z) is essentially a choice between a geometric and an algebraic representation. Use either representation at your convenience. The geometric "arrow in space" may aid in visualization. The algebraic set of components is usually more suitable for precise numerical or algebraic calculations.

Vectors enter physics in two distinct forms. (1) Vector **A** may represent a single force acting at a single point. The force of gravity acting at the center of gravity illustrates this form. (2) Vector **A** may be defined over some extended region; that is, **A** and its components may be functions of position: $A_x = A_x(x, y, z)$, and so on. Examples of this sort include the velocity of a fluid varying from point to point over a given volume and electric and magnetic fields. These two cases may be distinguished by referring to the vector defined over a region as a **vector field**. The concept of the vector defined over a region and being a function of position will become extremely important when we differentiate and integrate vectors.

At this stage it is convenient to introduce unit vectors along each of the coordinate axes. Let $\hat{\mathbf{x}}$ be a vector of unit magnitude pointing in the positive x-direction, $\hat{\mathbf{y}}$, a vector of unit magnitude in the positive y-direction, and $\hat{\mathbf{z}}$ a vector of unit magnitude in the positive z-direction. Then $\hat{\mathbf{x}}A_x$ is a vector with magnitude equal to $|A_x|$ and in the x-direction. By vector addition,

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z. \tag{1.5}$$

Note that if $\mathbf{A}$ vanishes, all of its components must vanish individually; that is, if

$$\mathbf{A} = 0, \qquad \text{then } A_x = A_y = A_z = 0.$$

This means that these unit vectors serve as a **basis**, or complete set of vectors, in the three-dimensional Euclidean space in terms of which any vector can be expanded. Thus, Eq. (1.5) is an assertion that the three unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ span our real three-dimensional space: Any vector may be written as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Since $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are linearly independent (no one is a linear combination of the other two), they form a **basis** for the real three-dimensional Euclidean space. Finally, by the Pythagorean theorem, the magnitude of vector $\mathbf{A}$ is

$$\boxed{|\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}.} \tag{1.6}$$

Note that the coordinate unit vectors are not the only complete set, or basis. This resolution of a vector into its components can be carried out in a variety of coordinate systems, as shown in Chapter 2. Here we restrict ourselves to Cartesian coordinates, where the unit vectors have the coordinates $\hat{\mathbf{x}} = (1, 0, 0)$, $\hat{\mathbf{y}} = (0, 1, 0)$ and $\hat{\mathbf{z}} = (0, 0, 1)$ and are all constant in length and direction, properties characteristic of Cartesian coordinates.

As a replacement of the graphical technique, addition and subtraction of vectors may now be carried out in terms of their components. For $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$,

$$\mathbf{A} \pm \mathbf{B} = \hat{\mathbf{x}}(A_x \pm B_x) + \hat{\mathbf{y}}(A_y \pm B_y) + \hat{\mathbf{z}}(A_z \pm B_z). \tag{1.7}$$

It should be emphasized here that the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are used for convenience. They are not essential; we can describe vectors and use them entirely in terms of their components: $\mathbf{A} \leftrightarrow (A_x, A_y, A_z)$. This is the approach of the two more powerful, more sophisticated definitions of vector to be discussed in the next section. However, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ emphasize the **direction**.

So far we have defined the operations of addition and subtraction of vectors. In the next sections, three varieties of multiplication will be defined on the basis of their applicability: a scalar, or inner, product, a vector product peculiar to three-dimensional space, and a direct, or outer, product yielding a second-rank tensor. Division by a vector is not defined.

Exercises

1.1.1 Show how to find $\mathbf{A}$ and $\mathbf{B}$, given $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$.

1.1.2 The vector $\mathbf{A}$ whose magnitude is 1.732 units makes equal angles with the coordinate axes. Find A_x, A_y, and A_z.

1.1.3 Calculate the components of a unit vector that lies in the xy-plane and makes equal angles with the positive directions of the x- and y-axes.

1.1.4 The velocity of sailboat A relative to sailboat B, $\mathbf{v}_{\text{rel}}$, is defined by the equation $\mathbf{v}_{\text{rel}} = \mathbf{v}_A - \mathbf{v}_B$, where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_B$ is the velocity of B. Determine the velocity of A relative to B if

$$\begin{aligned}\mathbf{v}_A &= 30 \text{ km/hr east}\\ \mathbf{v}_B &= 40 \text{ km/hr north.}\end{aligned}$$

ANS. $\mathbf{v}_{\text{rel}} = 50$ km/hr, 53.1° south of east.

1.1.5 A sailboat sails for 1 hr at 4 km/hr (relative to the water) on a steady compass heading of 40° east of north. The sailboat is simultaneously carried along by a current. At the end of the hour the boat is 6.12 km from its starting point. The line from its starting point to its location lies 60° east of north. Find the x (easterly) and y (northerly) components of the water's velocity.

ANS. $v_{\text{east}} = 2.73$ km/hr, $v_{\text{north}} \approx 0$ km/hr.

1.1.6 A vector equation can be reduced to the form $\mathbf{A} = \mathbf{B}$. From this show that the one vector equation is equivalent to **three** scalar equations. Assuming the validity of Newton's second law, $\mathbf{F} = m\mathbf{a}$, as a **vector** equation, this means that a_x depends only on F_x and is independent of F_y and F_z.

1.1.7 The vertices A, B, and C of a triangle are given by the points $(-1, 0, 2)$, $(0, 1, 0)$, and $(1, -1, 0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

ANS. $(0, -2, 2)$ or $(2, 0, -2)$.

1.1.8 A triangle is defined by the vertices of three vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ that extend from the origin. In terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ show that the **vector** sum of the successive sides of the triangle $(AB + BC + CA)$ is zero, where the side AB is from A to B, etc.

1.1.9 A sphere of radius a is centered at a point $\mathbf{r}_1$.

(a) Write out the algebraic equation for the sphere.
(b) Write out a **vector** equation for the sphere.

ANS. (a) $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$.
(b) $\mathbf{r} = \mathbf{r}_1 + \mathbf{a}$, with $\mathbf{r}_1 =$ center.
($\mathbf{a}$ takes on all directions but has a fixed magnitude a.)

1.1.10 A corner reflector is formed by three mutually perpendicular reflecting surfaces. Show that a ray of light incident upon the corner reflector (striking all three surfaces) is reflected back along a line parallel to the line of incidence.
Hint. Consider the effect of a reflection on the components of a vector describing the direction of the light ray.

1.1.11 *Hubble's law*. Hubble found that distant galaxies are receding with a velocity proportional to their distance from where we are on Earth. For the ith galaxy,

$$\mathbf{v}_i = H_0 \mathbf{r}_i,$$

with us at the origin. Show that this recession of the galaxies from us does **not** imply that we are at the center of the universe. Specifically, take the galaxy at $\mathbf{r_1}$ as a new origin and show that Hubble's law is still obeyed.

1.1.12 Find the diagonal vectors of a unit cube with one corner at the origin and its three sides lying along Cartesian coordinates axes. Show that there are four diagonals with length $\sqrt{3}$. Representing these as vectors, what are their components? Show that the diagonals of the cube's faces have length $\sqrt{2}$ and determine their components.

1.2 Rotation of the Coordinate Axes[3]

In the preceding section vectors were defined or represented in two equivalent ways: (1) geometrically by specifying magnitude and direction, as with an arrow, and (2) algebraically by specifying the components relative to Cartesian coordinate axes. The second definition is adequate for the vector analysis of this chapter. In this section two more refined, sophisticated, and powerful definitions are presented. First, the vector field is defined in terms of the behavior of its components under rotation of the coordinate axes. This transformation theory approach leads into the tensor analysis of Chapter 2 and groups of transformations in Chapter 4. Second, the component definition of Section 1.1 is refined and generalized according to the mathematician's concepts of vector and vector space. This approach leads to function spaces, including the Hilbert space.

The definition of vector as a quantity with magnitude and direction is incomplete. On the one hand, we encounter quantities, such as elastic constants and index of refraction in anisotropic crystals, that have magnitude and direction **but** that are not vectors. On the other hand, our naïve approach is awkward to generalize to extend to more complex quantities. We seek a new definition of vector field using our coordinate vector $\mathbf{r}$ as a prototype.

There is a physical basis for our development of a new definition. We describe our physical world by mathematics, but it and any physical predictions we may make must be **independent** of our mathematical conventions.

In our specific case we assume that space is isotropic; that is, there is no preferred direction, or all directions are equivalent. Then the physical system being analyzed or the physical law being enunciated cannot and must not depend on our choice or **orientation** of the coordinate axes. Specifically, if a quantity S does not depend on the orientation of the coordinate axes, it is called a scalar.

[3] This section is optional here. It will be essential for Chapter 2.

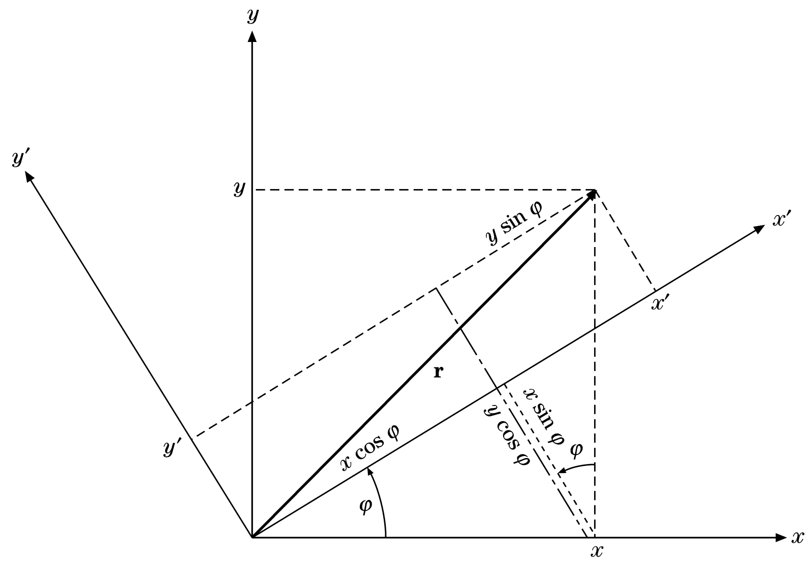

FIGURE 1.6 Rotation of Cartesian coordinate axes about the z-axis.

Now we return to the concept of vector **r** as a geometric object independent of the coordinate system. Let us look at **r** in two different systems, one rotated in relation to the other.

For simplicity we consider first the two-dimensional case. If the x-, y-coordinates are rotated counterclockwise through an angle φ, **keeping r, fixed** (Fig. 1.6), we get the following relations between the components resolved in the original system (unprimed) and those resolved in the new rotated system (primed):

$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \end{aligned} \tag{1.8}$$

We saw in Section 1.1 that a vector could be represented by the coordinates of a point; that is, the coordinates were proportional to the vector components. Hence the components of a vector must transform under rotation as coordinates of a point (such as **r**). Therefore whenever any pair of quantities A_x and A_y in the xy-coordinate system is transformed into (A'_x, A'_y) by this rotation of the coordinate system with

$$\begin{aligned} A'_x &= A_x\cos\varphi + A_y\sin\varphi, \\ A'_y &= -A_x\sin\varphi + A_y\cos\varphi, \end{aligned} \tag{1.9}$$

we **define**[4] A_x and A_y as the components of a vector **A**. Our vector now is defined in terms of the transformation of its components under rotation of the coordinate system. If A_x and A_y transform in the same way as x and y, the components of the general two-dimensional coordinate vector **r**, they are the components of a vector **A**. If A_x and A_y do not show this

[4]A scalar quantity does not depend on the orientation of coordinates; $S' = S$ expresses the fact that it is invariant under rotation of the coordinates.

form invariance (also called **covariance**) when the coordinates are rotated, they do not form a vector.

The vector field components A_x and A_y satisfying the defining equations, Eqs. (1.9), associate a magnitude A and a direction with each point in space. The magnitude is a scalar quantity, invariant to the rotation of the coordinate system. The direction (relative to the unprimed system) is likewise invariant to the rotation of the coordinate system (see Exercise 1.2.1). The result of all this is that the components of a vector may vary according to the rotation of the primed coordinate system. This is what Eqs. (1.9) say. But the variation with the angle is just such that the components in the rotated coordinate system A'_x and A'_y define a vector with the same magnitude and the same direction as the vector defined by the components A_x and A_y relative to the x-, y-coordinate axes. (Compare Exercise 1.2.1.) The components of $\mathbf{A}$ in a particular coordinate system constitute the **representation** of $\mathbf{A}$ in that coordinate system. Equations (1.9), the transformation relations, are a guarantee that the entity $\mathbf{A}$ is independent of the rotation of the coordinate system.

To go on to three and, later, four dimensions, we find it convenient to use a more compact notation. Let

$$\begin{aligned} x &\to x_1 \\ y &\to x_2 \end{aligned} \tag{1.10}$$

$$\begin{aligned} a_{11} &= \cos\varphi, & a_{12} &= \sin\varphi, \\ a_{21} &= -\sin\varphi, & a_{22} &= \cos\varphi. \end{aligned} \tag{1.11}$$

Then Eqs. (1.8) become

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2, \\ x'_2 &= a_{21}x_1 + a_{22}x_2. \end{aligned} \tag{1.12}$$

The coefficient a_{ij} may be interpreted as a direction cosine, the cosine of the angle between x'_i and x_j; that is,

$$\begin{aligned} a_{12} &= \cos(x'_1, x_2) = \sin\varphi, \\ a_{21} &= \cos(x'_2, x_1) = \cos\left(\varphi + \tfrac{\pi}{2}\right) = -\sin\varphi. \end{aligned} \tag{1.13}$$

The advantage of the new notation[5] is that it permits us to use the summation symbol $\sum$ and to rewrite Eqs. (1.12) as

$$x'_i = \sum_{j=1}^{2} a_{ij}x_j, \qquad i = 1, 2. \tag{1.14}$$

Note that i remains as a parameter that gives rise to one equation when it is set equal to 1 and to a second equation when it is set equal to 2. The index j, of course, is a summation index, a dummy index, and, as with a variable of integration, j may be replaced by any other convenient symbol.

[5]You may wonder at the replacement of one parameter φ by four parameters a_{ij}. Clearly, the a_{ij} do not constitute a minimum set of parameters. For two dimensions the four a_{ij} are subject to the three constraints given in Eq. (1.18). The justification for this redundant set of direction cosines is the convenience it provides. Hopefully, this convenience will become more apparent in Chapters 2 and 3. For three-dimensional rotations (9 a_{ij} but only three independent) alternate descriptions are provided by: (1) the Euler angles discussed in Section 3.3, (2) quaternions, and (3) the Cayley–Klein parameters. These alternatives have their respective advantages and disadvantages.

The generalization to three, four, or N dimensions is now simple. The set of N quantities V_j is said to be the components of an N-dimensional vector $\mathbf{V}$ if and only if their values relative to the rotated coordinate axes are given by

$$V'_i = \sum_{j=1}^{N} a_{ij} V_j, \qquad i = 1, 2, \ldots, N. \tag{1.15}$$

As before, a_{ij} is the cosine of the angle between x'_i and x_j. Often the upper limit N and the corresponding range of i will not be indicated. It is taken for granted that you know how many dimensions your space has.

From the definition of a_{ij} as the cosine of the angle between the positive x'_i direction and the positive x_j direction we may write (Cartesian coordinates)[6]

$$a_{ij} = \frac{\partial x'_i}{\partial x_j}. \tag{1.16a}$$

Using the inverse rotation ($\varphi \to -\varphi$) yields

$$x_j = \sum_{i=1}^{2} a_{ij} x'_i \qquad \text{or} \qquad \frac{\partial x_j}{\partial x'_i} = a_{ij}. \tag{1.16b}$$

Note that these are **partial derivatives**. By use of Eqs. (1.16a) and (1.16b), Eq. (1.15) becomes

$$V'_i = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x'_i} V_j. \tag{1.17}$$

The direction cosines a_{ij} satisfy an **orthogonality condition**

$$\sum_i a_{ij} a_{ik} = \delta_{jk} \tag{1.18}$$

or, equivalently,

$$\sum_i a_{ji} a_{ki} = \delta_{jk}. \tag{1.19}$$

Here, the symbol δ_{jk} is the Kronecker delta, defined by

$$\begin{aligned} \delta_{jk} &= 1 \qquad \text{for} \qquad j = k, \\ \delta_{jk} &= 0 \qquad \text{for} \qquad j \neq k. \end{aligned} \tag{1.20}$$

It is easily verified that Eqs. (1.18) and (1.19) hold in the two-dimensional case by substituting in the specific a_{ij} from Eqs. (1.11). The result is the well-known identity $\sin^2\varphi + \cos^2\varphi = 1$ for the nonvanishing case. To verify Eq. (1.18) in general form, we may use the partial derivative forms of Eqs. (1.16a) and (1.16b) to obtain

$$\sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x'_i}{\partial x_k} = \frac{\partial x_j}{\partial x_k}. \tag{1.21}$$

[6]Differentiate x'_i with respect to x_j. See discussion following Eq. (1.21).

The last step follows by the standard rules for partial differentiation, assuming that x_j is a function of x'_1, x'_2, x'_3, and so on. The final result, $\partial x_j/\partial x_k$, is equal to δ_{jk}, since x_j and x_k as coordinate lines $(j \neq k)$ are assumed to be perpendicular (two or three dimensions) or orthogonal (for any number of dimensions). Equivalently, we may assume that x_j and x_k $(j \neq k)$ are totally independent variables. If $j = k$, the partial derivative is clearly equal to 1.

In redefining a vector in terms of how its components transform under a rotation of the coordinate system, we should emphasize two points:

1. This definition is developed because it is useful and appropriate in describing our physical world. Our vector equations will be independent of any particular coordinate system. (The coordinate system need not even be Cartesian.) The vector equation can always be expressed in some particular coordinate system, and, to obtain numerical results, we must ultimately express the equation in some specific coordinate system.
2. This definition is subject to a generalization that will open up the branch of mathematics known as tensor analysis (Chapter 2).

A qualification is in order. The behavior of the vector components under rotation of the coordinates is used in Section 1.3 to prove that a scalar product is a scalar, in Section 1.4 to prove that a vector product is a vector, and in Section 1.6 to show that the gradient of a scalar ψ, $\nabla\psi$, is a vector. The remainder of this chapter proceeds on the basis of the less restrictive definitions of the vector given in Section 1.1.

Summary: Vectors and Vector Space

It is customary in mathematics to label an ordered triple of real numbers (x_1, x_2, x_3) a **vector** $\mathbf{x}$. The number x_n is called the nth component of vector $\mathbf{x}$. The collection of all such vectors (obeying the properties that follow) form a three-dimensional real **vector space**. We ascribe five properties to our vectors: If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$,

1. Vector equality: $\mathbf{x} = \mathbf{y}$ means $x_i = y_i$, $i = 1, 2, 3$.
2. Vector addition: $\mathbf{x} + \mathbf{y} = \mathbf{z}$ means $x_i + y_i = z_i$, $i = 1, 2, 3$.
3. Scalar multiplication: $a\mathbf{x} \leftrightarrow (ax_1, ax_2, ax_3)$ (with a real).
4. Negative of a vector: $-\mathbf{x} = (-1)\mathbf{x} \leftrightarrow (-x_1, -x_2, -x_3)$.
5. Null vector: There exists a null vector $\mathbf{0} \leftrightarrow (0, 0, 0)$.

Since our vector components are real (or complex) numbers, the following properties also hold:

1. Addition of vectors is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. Addition of vectors is associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
3. Scalar multiplication is distributive:

$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}, \qquad \text{also} \qquad (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}.$$

4. Scalar multiplication is associative: $(ab)\mathbf{x} = a(b\mathbf{x})$.

Further, the null vector **0** is unique, as is the negative of a given vector **x**.

So far as the vectors themselves are concerned this approach merely formalizes the component discussion of Section 1.1. The importance lies in the extensions, which will be considered in later chapters. In Chapter 4, we show that vectors form both an Abelian group under addition and a linear space with the transformations in the linear space described by matrices. Finally, and perhaps most important, for advanced physics the concept of vectors presented here may be generalized to (1) complex quantities,[7] (2) functions, and (3) an infinite number of components. This leads to infinite-dimensional function spaces, the Hilbert spaces, which are important in modern quantum theory. A brief introduction to function expansions and Hilbert space appears in Section 10.4.

Exercises

1.2.1 (a) Show that the magnitude of a vector **A**, $A = (A_x^2 + A_y^2)^{1/2}$, is independent of the orientation of the rotated coordinate system,

$$\left(A_x^2 + A_y^2\right)^{1/2} = \left(A_x'^2 + A_y'^2\right)^{1/2},$$

that is, independent of the rotation angle φ.

This independence of angle is expressed by saying that A is **invariant** under rotations.

(b) At a given point (x, y), **A** defines an angle α relative to the positive x-axis and α' relative to the positive x'-axis. The angle from x to x' is φ. Show that $\mathbf{A} = \mathbf{A}'$ defines the **same** direction in space when expressed in terms of its primed components as in terms of its unprimed components; that is,

$$\alpha' = \alpha - \varphi.$$

1.2.2 Prove the orthogonality condition $\sum_i a_{ji} a_{ki} = \delta_{jk}$. As a special case of this, the direction cosines of Section 1.1 satisfy the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

a result that also follows from Eq. (1.6).

1.3 Scalar or Dot Product

Having defined vectors, we now proceed to combine them. The laws for combining vectors must be mathematically consistent. From the possibilities that are consistent we select two that are both mathematically and physically interesting. A third possibility is introduced in Chapter 2, in which we form tensors.

The projection of a vector **A** onto a coordinate axis, which gives its Cartesian components in Eq. (1.4), defines a special geometrical case of the scalar product of **A** and the coordinate unit vectors:

$$A_x = A\cos\alpha \equiv \mathbf{A}\cdot\hat{\mathbf{x}}, \quad A_y = A\cos\beta \equiv \mathbf{A}\cdot\hat{\mathbf{y}}, \quad A_z = A\cos\gamma \equiv \mathbf{A}\cdot\hat{\mathbf{z}}. \tag{1.22}$$

[7] The n-dimensional vector space of real n-tuples is often labeled $\mathbb{R}^n$ and the n-dimensional vector space of complex n-tuples is labeled $\mathbb{C}^n$.

This special case of a scalar product in conjunction with general properties the scalar product is sufficient to derive the general case of the scalar product.

Just as the projection is linear in **A**, we want the scalar product of two vectors to be linear in **A** and **B**, that is, obey the distributive and associative laws

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \tag{1.23a}$$

$$\mathbf{A}\cdot(y\mathbf{B}) = (y\mathbf{A})\cdot\mathbf{B} = y\mathbf{A}\cdot\mathbf{B}, \tag{1.23b}$$

where y is a number. Now we can use the decomposition of **B** into its Cartesian components according to Eq. (1.5), $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$, to construct the general scalar or dot product of the vectors **A** and **B** as

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \mathbf{A}\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= B_x\mathbf{A}\cdot\hat{\mathbf{x}} + B_y\mathbf{A}\cdot\hat{\mathbf{y}} + B_z\mathbf{A}\cdot\hat{\mathbf{z}} \qquad \text{upon applying Eqs. (1.23a) and (1.23b)}\\ &= B_xA_x + B_yA_y + B_zA_z \qquad \text{upon substituting Eq. (1.22).}\end{aligned}$$

Hence

$$\boxed{\mathbf{A}\cdot\mathbf{B} \equiv \sum_i B_iA_i = \sum_i A_iB_i = \mathbf{B}\cdot\mathbf{A}.} \tag{1.24}$$

If $\mathbf{A} = \mathbf{B}$ in Eq. (1.24), we recover the magnitude $A = (\sum A_i^2)^{1/2}$ of **A** in Eq. (1.6) from Eq. (1.24).

It is obvious from Eq. (1.24) that the scalar product treats **A** and **B** alike, or is symmetric in **A** and **B**, and is commutative. Thus, alternatively and equivalently, we can first generalize Eqs. (1.22) to the projection A_B of **A** onto the direction of a vector $\mathbf{B} \neq 0$ as $A_B = A\cos\theta \equiv \mathbf{A}\cdot\hat{\mathbf{B}}$, where $\hat{\mathbf{B}} = \mathbf{B}/B$ is the unit vector in the direction of **B** and θ is the angle between **A** and **B**, as shown in Fig. 1.7. Similarly, we project **B** onto **A** as $B_A = B\cos\theta \equiv \mathbf{B}\cdot\hat{\mathbf{A}}$. Second, we make these projections symmetric in **A** and **B**, which leads to the definition

$$\boxed{\mathbf{A}\cdot\mathbf{B} \equiv A_BB = AB_A = AB\cos\theta.} \tag{1.25}$$

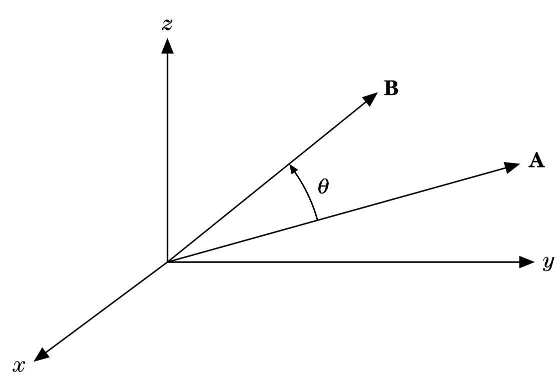

FIGURE 1.7 Scalar product $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$.

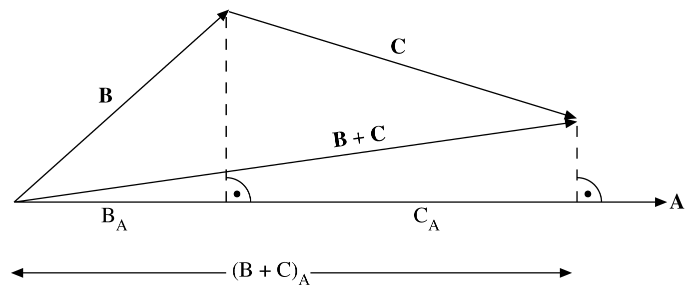

FIGURE 1.8 The distributive law
$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = AB_A + AC_A = A(\mathbf{B} + \mathbf{C})_A$, Eq. (1.23a).

The distributive law in Eq. (1.23a) is illustrated in Fig. 1.8, which shows that the sum of the projections of **B** and **C** onto **A**, $B_A + C_A$ is equal to the projection of $\mathbf{B} + \mathbf{C}$ onto **A**, $(\mathbf{B} + \mathbf{C})_A$.

It follows from Eqs. (1.22), (1.24), and (1.25) that the coordinate unit vectors satisfy the relations

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \tag{1.26a}$$

whereas

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0. \tag{1.26b}$$

If the component definition, Eq. (1.24), is labeled an algebraic definition, then Eq. (1.25) is a geometric definition. One of the most common applications of the scalar product in physics is in the calculation of **work** = **force**·**displacement**· $\cos\theta$, which is interpreted as displacement times the projection of the force along the displacement direction, i.e., the scalar product of force and displacement, $W = \mathbf{F} \cdot \mathbf{S}$.

If $\mathbf{A} \cdot \mathbf{B} = 0$ and we know that $\mathbf{A} \neq 0$ and $\mathbf{B} \neq 0$, then, from Eq. (1.25), $\cos\theta = 0$, or $\theta = 90^\circ, 270^\circ$, and so on. The vectors **A** and **B** must be perpendicular. Alternately, we may say **A** and **B** are orthogonal. The unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually orthogonal. To develop this notion of orthogonality one more step, suppose that **n** is a unit vector and **r** is a nonzero vector in the xy-plane; that is, $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ (Fig. 1.9). If

$$\mathbf{n} \cdot \mathbf{r} = 0$$

for **all** choices of **r**, then **n** must be perpendicular (orthogonal) to the xy-plane.

Often it is convenient to replace $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ by subscripted unit vectors $\mathbf{e}_m$, $m = 1, 2, 3$, with $\hat{\mathbf{x}} = \mathbf{e}_1$, and so on. Then Eqs. (1.26a) and (1.26b) become

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}. \tag{1.26c}$$

For $m \neq n$ the unit vectors $\mathbf{e}_m$ and $\mathbf{e}_n$ are orthogonal. For $m = n$ each vector is normalized to unity, that is, has unit magnitude. The set $\mathbf{e}_m$ is said to be **orthonormal**. A major advantage of Eq. (1.26c) over Eqs. (1.26a) and (1.26b) is that Eq. (1.26c) may readily be generalized to N-dimensional space: $m, n = 1, 2, \ldots, N$. Finally, we are picking sets of unit vectors $\mathbf{e}_m$ that are orthonormal for convenience – a very great convenience.

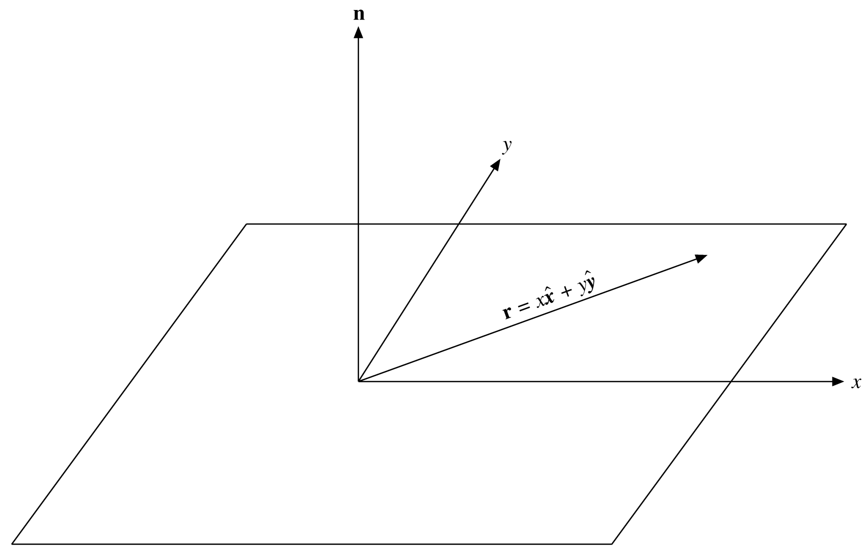

FIGURE 1.9 A normal vector.

Invariance of the Scalar Product Under Rotations

We have not yet shown that the word *scalar* is justified or that the scalar product is indeed a scalar quantity. To do this, we investigate the behavior of $\mathbf{A} \cdot \mathbf{B}$ under a rotation of the coordinate system. By use of Eq. (1.15),

$$A'_x B'_x + A'_y B'_y + A'_z B'_z = \sum_i a_{xi} A_i \sum_j a_{xj} B_j + \sum_i a_{yi} A_i \sum_j a_{yj} B_j + \sum_i a_{zi} A_i \sum_j a_{zj} B_j. \tag{1.27}$$

Using the indices k and l to sum over x, y, and z, we obtain

$$\sum_k A'_k B'_k = \sum_l \sum_i \sum_j a_{li} A_i a_{lj} B_j, \tag{1.28}$$

and, by rearranging the terms on the right-hand side, we have

$$\sum_k A'_k B'_k = \sum_l \sum_i \sum_j (a_{li} a_{lj}) A_i B_j = \sum_i \sum_j \delta_{ij} A_i B_j = \sum_i A_i B_i. \tag{1.29}$$

The last two steps follow by using Eq. (1.18), the orthogonality condition of the direction cosines, and Eqs. (1.20), which define the Kronecker delta. The effect of the Kronecker delta is to cancel all terms in a summation over either index except the term for which the indices are equal. In Eq. (1.29) its effect is to set $j = i$ and to eliminate the summation over j. Of course, we could equally well set $i = j$ and eliminate the summation over i.

Equation (1.29) gives us

$$\boxed{\sum_k A'_k B'_k = \sum_i A_i B_i,} \tag{1.30}$$

which is just our definition of a scalar quantity, one that remains invariant under the rotation of the coordinate system.

In a similar approach that exploits this concept of invariance, we take $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and dot it into itself:

$$\begin{aligned} \mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B}. \end{aligned} \tag{1.31}$$

Since

$$\mathbf{C} \cdot \mathbf{C} = C^2, \tag{1.32}$$

the square of the magnitude of vector $\mathbf{C}$ and thus an invariant quantity, we see that

$$\mathbf{A} \cdot \mathbf{B} = \frac{1}{2}\left(C^2 - A^2 - B^2\right), \qquad \text{invariant}. \tag{1.33}$$

Since the right-hand side of Eq. (1.33) is invariant — that is, a scalar quantity — the left-hand side, $\mathbf{A} \cdot \mathbf{B}$, must also be invariant under rotation of the coordinate system. Hence $\mathbf{A} \cdot \mathbf{B}$ is a scalar.

Equation (1.31) is really another form of the law of cosines, which is

$$C^2 = A^2 + B^2 + 2AB\cos\theta. \tag{1.34}$$

Comparing Eqs. (1.31) and (1.34), we have another verification of Eq. (1.25), or, if preferred, a vector derivation of the law of cosines (Fig. 1.10).

The dot product, given by Eq. (1.24), may be generalized in two ways. The space need not be restricted to three dimensions. In n-dimensional space, Eq. (1.24) applies with the sum running from 1 to n. Moreover, n may be infinity, with the sum then a convergent infinite series (Section 5.2). The other generalization extends the concept of vector to embrace functions. The function analog of a dot, or inner, product appears in Section 10.4.

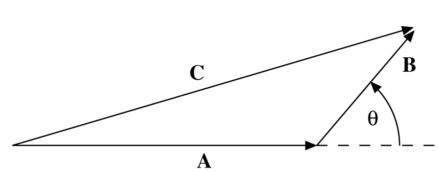

FIGURE 1.10 The law of cosines.

Exercises

1.3.1 Two unit magnitude vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ are required to be either parallel or perpendicular to each other. Show that $\mathbf{e}_i \cdot \mathbf{e}_j$ provides an interpretation of Eq. (1.18), the direction cosine orthogonality relation.

1.3.2 Given that (1) the dot product of a unit vector with itself is unity and (2) this relation is valid in all (rotated) coordinate systems, show that $\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}}' = 1$ (with the primed system rotated 45° about the z-axis relative to the unprimed) implies that $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$.

1.3.3 The vector $\mathbf{r}$, starting at the origin, terminates at and specifies the point in space (x, y, z). Find the surface swept out by the tip of $\mathbf{r}$ if

(a) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{a} = 0$. Characterize $\mathbf{a}$ geometrically.
(b) $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{r} = 0$. Describe the geometric role of $\mathbf{a}$.
The vector $\mathbf{a}$ is constant (in magnitude and direction).

1.3.4 The interaction energy between two dipoles of moments $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ may be written in the vector form

$$V = -\frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3} + \frac{3(\boldsymbol{\mu}_1 \cdot \mathbf{r})(\boldsymbol{\mu}_2 \cdot \mathbf{r})}{r^5}$$

and in the scalar form

$$V = \frac{\mu_1 \mu_2}{r^3}(2\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\varphi).$$

Here θ_1 and θ_2 are the angles of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ relative to $\mathbf{r}$, while φ is the azimuth of $\boldsymbol{\mu}_2$ relative to the $\boldsymbol{\mu}_1$–$\mathbf{r}$ plane (Fig. 1.11). Show that these two forms are equivalent.
Hint: Equation (12.178) will be helpful.

1.3.5 A pipe comes diagonally down the south wall of a building, making an angle of 45° with the horizontal. Coming into a corner, the pipe turns and continues diagonally down a west-facing wall, still making an angle of 45° with the horizontal. What is the angle between the south-wall and west-wall sections of the pipe?

ANS. 120°.

1.3.6 Find the shortest distance of an observer at the point $(2, 1, 3)$ from a rocket in free flight with velocity $(1, 2, 3)$ m/s. The rocket was launched at time $t = 0$ from $(1, 1, 1)$. Lengths are in kilometers.

1.3.7 Prove the law of cosines from the triangle with corners at the point of $\mathbf{C}$ and $\mathbf{A}$ in Fig. 1.10 and the projection of vector $\mathbf{B}$ onto vector $\mathbf{A}$.

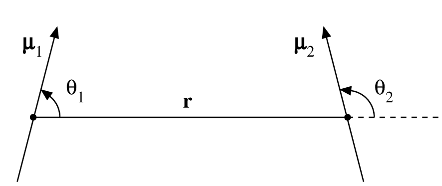

FIGURE 1.11 Two dipole moments.

1.4 VECTOR OR CROSS PRODUCT

A second form of vector multiplication employs the sine of the included angle instead of the cosine. For instance, the angular momentum of a body shown at the point of the distance vector in Fig. 1.12 is defined as

$$\begin{aligned}\text{angular momentum} &= \text{radius arm} \times \text{linear momentum}\\ &= \text{distance} \times \text{linear momentum} \times \sin\theta.\end{aligned}$$

For convenience in treating problems relating to quantities such as angular momentum, torque, and angular velocity, we define the vector product, or cross product, as

$$\boxed{\mathbf{C} = \mathbf{A} \times \mathbf{B}, \qquad \text{with } C = AB\sin\theta.} \tag{1.35}$$

Unlike the preceding case of the scalar product, $\mathbf{C}$ is now a vector, and we assign it a direction perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ form a right-handed system. With this choice of direction we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \qquad \text{anticommutation.} \tag{1.36a}$$

From this definition of cross product we have

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}, \tag{1.36b}$$

whereas

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}},\\ \hat{\mathbf{y}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}, \qquad \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}.\end{aligned} \tag{1.36c}$$

Among the examples of the cross product in mathematical physics are the relation between linear momentum $\mathbf{p}$ and angular momentum $\mathbf{L}$, with $\mathbf{L}$ defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

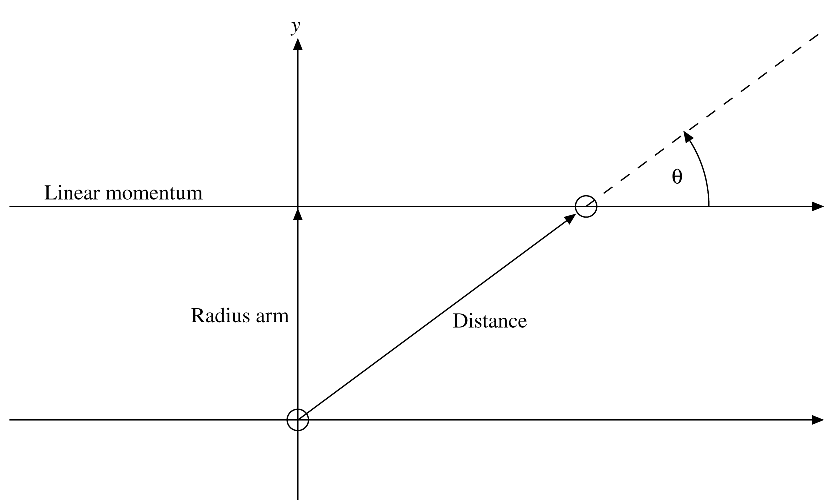

FIGURE 1.12 Angular momentum.

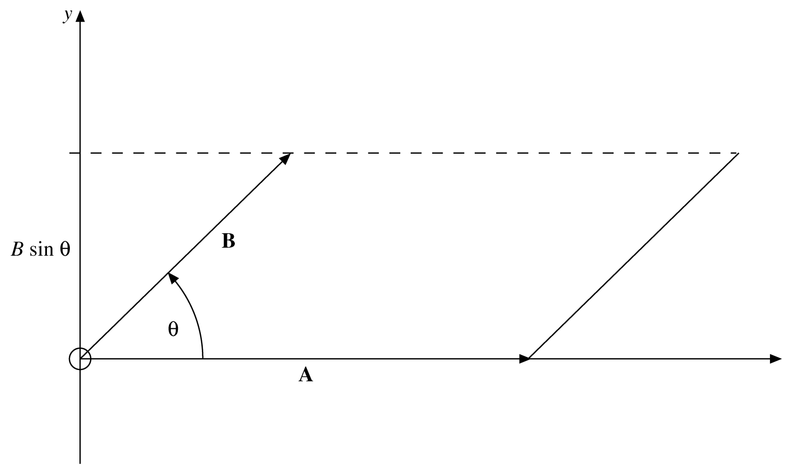

FIGURE 1.13 Parallelogram representation of the vector product.

and the relation between linear velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Vectors $\mathbf{v}$ and $\mathbf{p}$ describe properties of the particle or physical system. However, the position vector $\mathbf{r}$ is determined by the choice of the origin of the coordinates. This means that $\boldsymbol{\omega}$ and $\mathbf{L}$ depend on the choice of the origin.

The familiar magnetic induction $\mathbf{B}$ is usually defined by the vector product force equation[8]

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B} \text{ (mks units).}$$

Here $\mathbf{v}$ is the velocity of the electric charge q and $\mathbf{F}_M$ is the resulting force on the moving charge.

The cross product has an important geometrical interpretation, which we shall use in subsequent sections. In the parallelogram defined by $\mathbf{A}$ and $\mathbf{B}$ (Fig. 1.13), $B\sin\theta$ is the height if A is taken as the length of the base. Then $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$ is the **area** of the parallelogram. As a vector, $\mathbf{A} \times \mathbf{B}$ is the area of the parallelogram defined by $\mathbf{A}$ and $\mathbf{B}$, with the area vector normal to the plane of the parallelogram. This suggests that area (with its orientation in space) may be treated as a vector quantity.

An alternate definition of the vector product can be derived from the special case of the coordinate unit vectors in Eqs. (1.36c) in conjunction with the linearity of the cross product in both vector arguments, in analogy with Eqs. (1.23) for the dot product,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \tag{1.37a}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}, \tag{1.37b}$$

$$\mathbf{A} \times (y\mathbf{B}) = y\mathbf{A} \times \mathbf{B} = (y\mathbf{A}) \times \mathbf{B}, \tag{1.37c}$$

[8]The electric field $\mathbf{E}$ is assumed here to be zero.

where y is a number again. Using the decomposition of $\mathbf{A}$ and $\mathbf{B}$ into their Cartesian components according to Eq. (1.5), we find

$$\boxed{\begin{aligned}\mathbf{A}\times\mathbf{B} \equiv \mathbf{C} = (C_x, C_y, C_z) &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= (A_xB_y - A_yB_x)\hat{\mathbf{x}}\times\hat{\mathbf{y}} + (A_xB_z - A_zB_x)\hat{\mathbf{x}}\times\hat{\mathbf{z}}\\ &\quad + (A_yB_z - A_zB_y)\hat{\mathbf{y}}\times\hat{\mathbf{z}}\end{aligned}}$$

upon applying Eqs. (1.37a) and (1.37b) and substituting Eqs. (1.36a), (1.36b), and (1.36c) so that the Cartesian components of $\mathbf{A}\times\mathbf{B}$ become

$$C_x = A_yB_z - A_zB_y, \qquad C_y = A_zB_x - A_xB_z, \qquad C_z = A_xB_y - A_yB_x, \tag{1.38}$$

or

$$C_i = A_jB_k - A_kB_j, \qquad i, j, k \text{ all different}, \tag{1.39}$$

and with cyclic permutation of the indices i, j, and k corresponding to x, y, and z, respectively. The vector product $\mathbf{C}$ may be mnemonically represented by a determinant,[9]

$$\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \equiv \hat{\mathbf{x}}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\mathbf{y}}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{\mathbf{z}}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}, \tag{1.40}$$

which is meant to be expanded across the top row to reproduce the three components of $\mathbf{C}$ listed in Eqs. (1.38).

Equation (1.35) might be called a geometric definition of the vector product. Then Eqs. (1.38) would be an algebraic definition.

To show the equivalence of Eq. (1.35) and the component definition, Eqs. (1.38), let us form $\mathbf{A}\cdot\mathbf{C}$ and $\mathbf{B}\cdot\mathbf{C}$, using Eqs. (1.38). We have

$$\begin{aligned}\mathbf{A}\cdot\mathbf{C} &= \mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})\\ &= A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x)\\ &= 0.\end{aligned} \tag{1.41}$$

Similarly,

$$\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0. \tag{1.42}$$

Equations (1.41) and (1.42) show that $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ ($\cos\theta = 0, \theta = \pm 90^\circ$) and therefore perpendicular to the plane they determine. The positive direction is determined by considering special cases, such as the unit vectors $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ ($C_z = +A_xB_y$).

The magnitude is obtained from

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) &= A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2\\ &= A^2B^2 - A^2B^2\cos^2\theta\\ &= A^2B^2\sin^2\theta.\end{aligned} \tag{1.43}$$

[9] See Section 3.1 for a brief summary of determinants.

Hence

$$C = AB\sin\theta. \tag{1.44}$$

The first step in Eq. (1.43) may be verified by expanding out in component form, using Eqs. (1.38) for $\mathbf{A} \times \mathbf{B}$ and Eq. (1.24) for the dot product. From Eqs. (1.41), (1.42), and (1.44) we see the equivalence of Eqs. (1.35) and (1.38), the two definitions of vector product.

There still remains the problem of verifying that $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is indeed a vector, that is, that it obeys Eq. (1.15), the vector transformation law. Starting in a rotated (primed system),

$$\begin{aligned}
C'_i &= A'_j B'_k - A'_k B'_j, \qquad i, j, \text{ and } k \text{ in cyclic order,} \\
&= \sum_l a_{jl} A_l \sum_m a_{km} B_m - \sum_l a_{kl} A_l \sum_m a_{jm} B_m \\
&= \sum_{l,m} (a_{jl} a_{km} - a_{kl} a_{jm}) A_l B_m.
\end{aligned} \tag{1.45}$$

The combination of direction cosines in parentheses vanishes for $m = l$. We therefore have j and k taking on fixed values, dependent on the choice of i, and six combinations of l and m. If $i = 3$, then $j = 1, k = 2$ (cyclic order), and we have the following direction cosine combinations:[10]

$$\begin{aligned}
a_{11}a_{22} - a_{21}a_{12} &= a_{33}, \\
a_{13}a_{21} - a_{23}a_{11} &= a_{32}, \\
a_{12}a_{23} - a_{22}a_{13} &= a_{31}
\end{aligned} \tag{1.46}$$

and their negatives. Equations (1.46) are identities satisfied by the direction cosines. They may be verified with the use of determinants and matrices (see Exercise 3.3.3). Substituting back into Eq. (1.45),

$$\begin{aligned}
C'_3 &= a_{33} A_1 B_2 + a_{32} A_3 B_1 + a_{31} A_2 B_3 - a_{33} A_2 B_1 - a_{32} A_1 B_3 - a_{31} A_3 B_2 \\
&= a_{31} C_1 + a_{32} C_2 + a_{33} C_3 \\
&= \sum_n a_{3n} C_n.
\end{aligned} \tag{1.47}$$

By permuting indices to pick up C'_1 and C'_2, we see that Eq. (1.15) is satisfied and $\mathbf{C}$ is indeed a vector. It should be mentioned here that this **vector nature** of the **cross product** is an accident associated with the **three-dimensional** nature of ordinary space.[11] It will be seen in Chapter 2 that the cross product may also be treated as a second-rank antisymmetric tensor.

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[10] Equations (1.46) hold for rotations because they preserve volumes. For a more general orthogonal transformation, the r.h.s. of Eqs. (1.46) is multiplied by the determinant of the transformation matrix (see Chapter 3 for matrices and determinants).

[11] Specifically Eqs. (1.46) hold only for three-dimensional space. See D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Dordrecht: Reidel, 1984) for a far-reaching generalization of the cross product.

If we define a vector as an ordered triplet of numbers (or functions), as in the latter part of Section 1.2, then there is no problem identifying the cross product as a vector. The cross-product operation maps the two triples $\mathbf{A}$ and $\mathbf{B}$ into a third triple, $\mathbf{C}$, which by definition is a vector.

We now have two ways of multiplying vectors; a third form appears in Chapter 2. But what about division by a vector? It turns out that the ratio $\mathbf{B}/\mathbf{A}$ is not uniquely specified (Exercise 3.2.21) unless $\mathbf{A}$ and $\mathbf{B}$ are also required to be parallel. Hence division of one vector by another is not defined.

Exercises

1.4.1 Show that the medians of a triangle intersect in the center, which is 2/3 of the median's length from each corner. Construct a numerical example and plot it.

1.4.2 Prove the law of cosines starting from $\mathbf{A}^2 = (\mathbf{B} - \mathbf{C})^2$.

1.4.3 Starting with $\mathbf{C} = \mathbf{A} + \mathbf{B}$, show that $\mathbf{C} \times \mathbf{C} = 0$ leads to

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

1.4.4 Show that

(a) $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = A^2 - B^2$,
(b) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B}) = 2\mathbf{A} \times \mathbf{B}$.

The distributive laws needed here,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C},$$

and

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

may easily be verified (if desired) by expansion in Cartesian components.

1.4.5 Given the three vectors,

$$\mathbf{P} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}},$$
$$\mathbf{Q} = -6\hat{\mathbf{x}} - 4\hat{\mathbf{y}} + 2\hat{\mathbf{z}},$$
$$\mathbf{R} = \hat{\mathbf{x}} - 2\hat{\mathbf{y}} - \hat{\mathbf{z}},$$

find two that are perpendicular and two that are parallel or antiparallel.

1.4.6 If $\mathbf{P} = \hat{\mathbf{x}} P_x + \hat{\mathbf{y}} P_y$ and $\mathbf{Q} = \hat{\mathbf{x}} Q_x + \hat{\mathbf{y}} Q_y$ are any two nonparallel (also nonantiparallel) vectors in the xy-plane, show that $\mathbf{P} \times \mathbf{Q}$ is in the z-direction.

1.4.7 Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (AB)^2 - (\mathbf{A} \cdot \mathbf{B})^2$.

1.4.8 Using the vectors

$$\mathbf{P} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta,$$
$$\mathbf{Q} = \hat{\mathbf{x}}\cos\varphi - \hat{\mathbf{y}}\sin\varphi,$$
$$\mathbf{R} = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi,$$

prove the familiar trigonometric identities

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$
$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

1.4.9 (a) Find a vector $\mathbf{A}$ that is perpendicular to

$$\mathbf{U} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}},$$
$$\mathbf{V} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}.$$

(b) What is $\mathbf{A}$ if, in addition to this requirement, we demand that it have unit magnitude?

1.4.10 If four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ all lie in the same plane, show that

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0.$$

Hint. Consider the directions of the cross-product vectors.

1.4.11 The coordinates of the three vertices of a triangle are $(2, 1, 5)$, $(5, 2, 8)$, and $(4, 8, 2)$. Compute its area by vector methods, its center and medians. Lengths are in centimeters. *Hint*. See Exercise 1.4.1.

1.4.12 The vertices of parallelogram $ABCD$ are $(1, 0, 0)$, $(2, -1, 0)$, $(0, -1, 1)$, and $(-1, 0, 1)$ in order. Calculate the vector areas of triangle ABD and of triangle BCD. Are the two vector areas equal?

ANS. $\text{Area}_{ABD} = -\frac{1}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}})$.

1.4.13 The origin and the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ (all of which start at the origin) define a tetrahedron. Taking the outward direction as positive, calculate the total vector area of the four tetrahedral surfaces.
Note. In Section 1.11 this result is generalized to any closed surface.

1.4.14 Find the sides and angles of the spherical triangle ABC defined by the three vectors

$$\mathbf{A} = (1, 0, 0),$$
$$\mathbf{B} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$
$$\mathbf{C} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Each vector starts from the origin (Fig. 1.14).

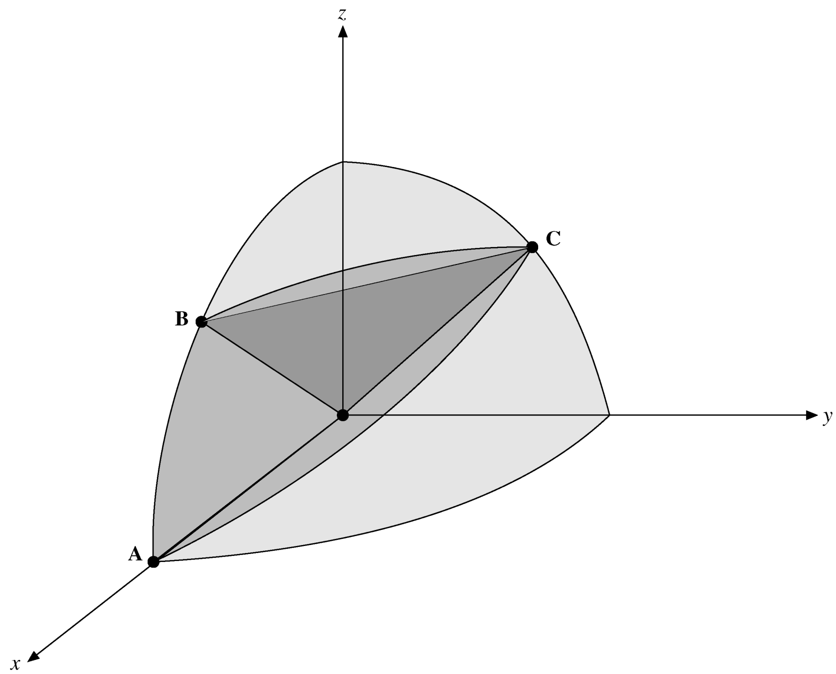

FIGURE 1.14 Spherical triangle.

1.4.15 Derive the law of sines (Fig. 1.15):

$$\frac{\sin\alpha}{|\mathbf{A}|} = \frac{\sin\beta}{|\mathbf{B}|} = \frac{\sin\gamma}{|\mathbf{C}|}.$$

1.4.16 The magnetic induction **B** is **defined** by the Lorentz force equation,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}).$$

Carrying out three experiments, we find that if

$$\mathbf{v} = \hat{\mathbf{x}}, \qquad \frac{\mathbf{F}}{q} = 2\hat{\mathbf{z}} - 4\hat{\mathbf{y}},$$
$$\mathbf{v} = \hat{\mathbf{y}}, \qquad \frac{\mathbf{F}}{q} = 4\hat{\mathbf{x}} - \hat{\mathbf{z}},$$
$$\mathbf{v} = \hat{\mathbf{z}}, \qquad \frac{\mathbf{F}}{q} = \hat{\mathbf{y}} - 2\hat{\mathbf{x}}.$$

From the results of these three separate experiments calculate the magnetic induction **B**.

1.4.17 Define a cross product of two vectors in two-dimensional space and give a geometrical interpretation of your construction.

1.4.18 Find the shortest distance between the paths of two rockets in free flight. Take the first rocket path to be $\mathbf{r} = \mathbf{r}_1 + t_1\mathbf{v}_1$ with launch at $\mathbf{r}_1 = (1, 1, 1)$ and velocity $\mathbf{v}_1 = (1, 2, 3)$

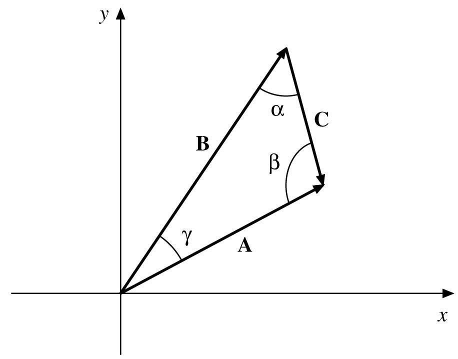

FIGURE 1.15 Law of sines.

and the second rocket path as $\mathbf{r} = \mathbf{r}_2 + t_2\mathbf{v}_2$ with $\mathbf{r}_2 = (5, 2, 1)$ and $\mathbf{v}_2 = (-1, -1, 1)$. Lengths are in kilometers, velocities in kilometers per hour.

1.5 TRIPLE SCALAR PRODUCT, TRIPLE VECTOR PRODUCT

Triple Scalar Product

Sections 1.3 and 1.4 cover the two types of multiplication of interest here. However, there are combinations of three vectors, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, that occur with sufficient frequency to deserve further attention. The combination

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

is known as the **triple scalar product**. $\mathbf{B}\times\mathbf{C}$ yields a vector that, dotted into $\mathbf{A}$, gives a scalar. We note that $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$ represents a scalar crossed into a vector, an operation that is not defined. Hence, if we agree to exclude this undefined interpretation, the parentheses may be omitted and the triple scalar product written $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

Using Eqs. (1.38) for the cross product and Eq. (1.24) for the dot product, we obtain

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)\\ &= \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}\\ &= -\mathbf{A}\cdot\mathbf{C}\times\mathbf{B} = -\mathbf{C}\cdot\mathbf{B}\times\mathbf{A} = -\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}, \text{ and so on.}\end{aligned} \tag{1.48}$$

There is a high degree of symmetry in the component expansion. Every term contains the factors A_i, B_j, and C_k. If i, j, and k are in cyclic order (x, y, z), the sign is positive. If the order is anticyclic, the sign is negative. Further, the dot and the cross may be interchanged,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}. \tag{1.49}$$

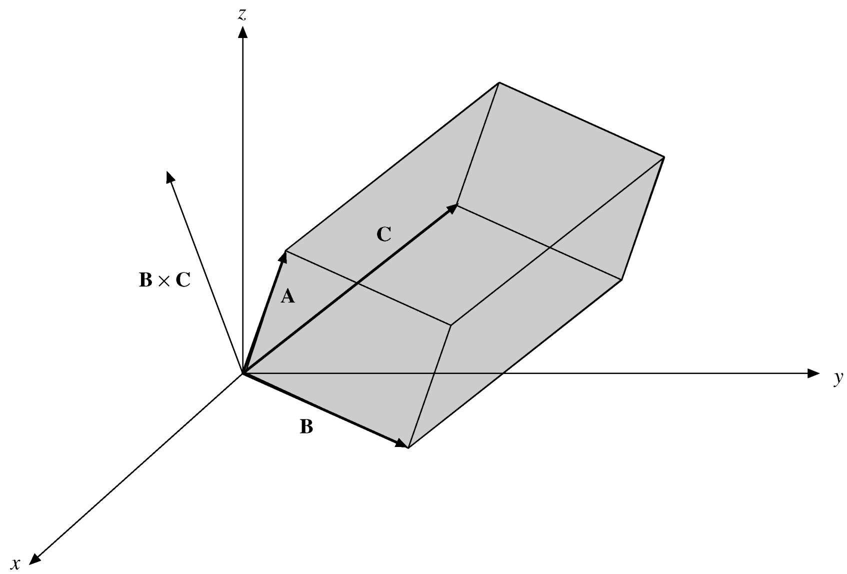

FIGURE 1.16 Parallelepiped representation of triple scalar product.

A convenient representation of the component expansion of Eq. (1.48) is provided by the determinant

$$\boxed{\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.} \tag{1.50}$$

The rules for interchanging rows and columns of a determinant[12] provide an immediate verification of the permutations listed in Eq. (1.48), whereas the symmetry of **A**, **B**, and **C** in the determinant form suggests the relation given in Eq. (1.49). The triple products encountered in Section 1.4, which showed that $\mathbf{A}\times\mathbf{B}$ was perpendicular to both **A** and **B**, were special cases of the general result (Eq. (1.48)).

The triple scalar product has a direct geometrical interpretation. The three vectors **A**, **B**, and **C** may be interpreted as defining a parallelepiped (Fig. 1.16):

$$\begin{aligned} |\mathbf{B}\times\mathbf{C}| &= BC\sin\theta \\ &= \text{area of parallelogram base.} \end{aligned} \tag{1.51}$$

The direction, of course, is normal to the base. Dotting **A** into this means multiplying the base area by the projection of **A** onto the normal, or base times height. Therefore

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \text{volume of parallelepiped defined by } \mathbf{A}, \mathbf{B}, \text{ and } \mathbf{C}.$$

The triple scalar product finds an interesting and important application in the construction of a reciprocal crystal lattice. Let **a**, **b**, and **c** (not necessarily mutually perpendicular)

[12]See Section 3.1 for a summary of the properties of determinants.

represent the vectors that define a crystal lattice. The displacement from one lattice point to another may then be written

$$\mathbf{r} = n_a\mathbf{a} + n_b\mathbf{b} + n_c\mathbf{c}, \tag{1.52}$$

with n_a, n_b, and n_c taking on integral values. With these vectors we may form

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}. \tag{1.53a}$$

We see that $\mathbf{a}'$ is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, and we can readily show that

$$\mathbf{a}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{b} = \mathbf{c}'\cdot\mathbf{c} = 1, \tag{1.53b}$$

whereas

$$\mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0. \tag{1.53c}$$

It is from Eqs. (1.53b) and (1.53c) that the name **reciprocal lattice** is associated with the points $\mathbf{r}' = n'_a\mathbf{a}' + n'_b\mathbf{b}' + n'_c\mathbf{c}'$. The mathematical space in which this reciprocal lattice exists is sometimes called a *Fourier space*, on the basis of relations to the Fourier analysis of Chapters 14 and 15. This reciprocal lattice is useful in problems involving the scattering of waves from the various planes in a crystal. Further details may be found in R. B. Leighton's *Principles of Modern Physics*, pp. 440–448 [New York: McGraw-Hill (1959)].

Triple Vector Product

The second triple product of interest is $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, which is a vector. Here the parentheses must be retained, as may be seen from a special case $(\hat{\mathbf{x}}\times\hat{\mathbf{x}})\times\hat{\mathbf{y}} = 0$, while $\hat{\mathbf{x}}\times(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = \hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}$.

Example 1.5.1 A Triple Vector Product

For the vectors

$$\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}} = (1, 2, -1), \quad \mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}} = (0, 1, 1), \quad \mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{y}} = (0, 1, 1),$$

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}},$$

and

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{\mathbf{x}} - \hat{\mathbf{z}} = -(\hat{\mathbf{y}} + \hat{\mathbf{z}}) - (\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

$$= -\mathbf{B} - \mathbf{C}.$$ ■

By rewriting the result in the last line of Example 1.5.1 as a linear combination of $\mathbf{B}$ and $\mathbf{C}$, we notice that, taking a geometric approach, the triple vector product is perpendicular

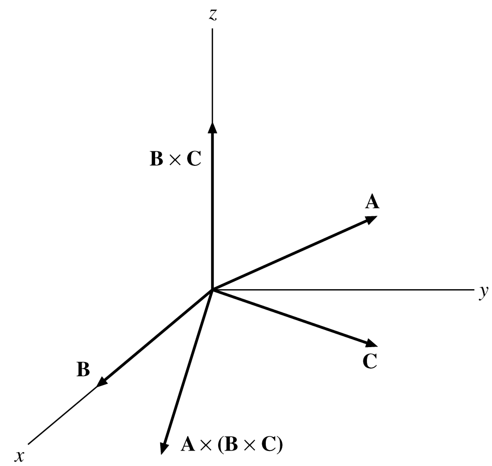

FIGURE 1.17 **B** and **C** are in the xy-plane. $\mathbf{B}\times\mathbf{C}$ is perpendicular to the xy-plane and is shown here along the z-axis. Then $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is perpendicular to the z-axis and therefore is back in the xy-plane.

to **A** and to $\mathbf{B}\times\mathbf{C}$. The plane defined by **B** and **C** is perpendicular to $\mathbf{B}\times\mathbf{C}$, and so the triple product lies in this plane (see Fig. 1.17):

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = u\mathbf{B} + v\mathbf{C}. \tag{1.54}$$

Taking the scalar product of Eq. (1.54) with **A** gives zero for the left-hand side, so $u\mathbf{A}\cdot\mathbf{B} + v\mathbf{A}\cdot\mathbf{C} = 0$. Hence $u = w\mathbf{A}\cdot\mathbf{C}$ and $v = -w\mathbf{A}\cdot\mathbf{B}$ for a suitable w. Substituting these values into Eq. (1.54) gives

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = w\big[\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\big];} \tag{1.55}$$

we want to show that

$$w = 1$$

in Eq. (1.55), an important relation sometimes known as the **BAC–CAB** rule. Since Eq. (1.55) is linear in A, B, and C, w is independent of these magnitudes. That is, we only need to show that $w = 1$ for unit vectors $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$. Let us denote $\hat{\mathbf{B}}\cdot\hat{\mathbf{C}} = \cos\alpha$, $\hat{\mathbf{C}}\cdot\hat{\mathbf{A}} = \cos\beta$, $\hat{\mathbf{A}}\cdot\hat{\mathbf{B}} = \cos\gamma$, and square Eq. (1.55) to obtain

$$\begin{aligned}
\big[\hat{\mathbf{A}}\times(\hat{\mathbf{B}}\times\hat{\mathbf{C}})\big]^2 &= \hat{\mathbf{A}}^2(\hat{\mathbf{B}}\times\hat{\mathbf{C}})^2 - \big[\hat{\mathbf{A}}\cdot(\hat{\mathbf{B}}\times\hat{\mathbf{C}})\big]^2 \\
&= 1 - \cos^2\alpha - \big[\hat{\mathbf{A}}\cdot(\hat{\mathbf{B}}\times\hat{\mathbf{C}})\big]^2 \\
&= w^2\big[(\hat{\mathbf{A}}\cdot\hat{\mathbf{C}})^2 + (\hat{\mathbf{A}}\cdot\hat{\mathbf{B}})^2 - 2(\hat{\mathbf{A}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{A}}\cdot\hat{\mathbf{C}})(\hat{\mathbf{B}}\cdot\hat{\mathbf{C}})\big] \\
&= w^2\big(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma\big),
\end{aligned} \tag{1.56}$$

using $(\hat{\mathbf{A}} \times \hat{\mathbf{B}})^2 = \hat{\mathbf{A}}^2\hat{\mathbf{B}}^2 - (\hat{\mathbf{A}} \cdot \hat{\mathbf{B}})^2$ repeatedly (see Eq. (1.43) for a proof). Consequently, the (squared) volume spanned by $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ that occurs in Eq. (1.56) can be written as

$$\left[\hat{\mathbf{A}} \cdot (\hat{\mathbf{B}} \times \hat{\mathbf{C}})\right]^2 = 1 - \cos^2\alpha - w^2\left(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma\right).$$

Here $w^2 = 1$, since this volume is symmetric in α, β, γ. That is, $w = \pm 1$ and is independent of $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$. Using again the special case $\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \hat{\mathbf{y}}) = -\hat{\mathbf{y}}$ in Eq. (1.55) finally gives $w = 1$. (An alternate derivation using the Levi-Civita symbol ε_{ijk} of Chapter 2 is the topic of Exercise 2.9.8.)

It might be noted here that just as vectors are independent of the coordinates, so a vector equation is independent of the particular coordinate system. The coordinate system only determines the components. If the vector equation can be established in Cartesian coordinates, it is established and valid in any of the coordinate systems to be introduced in Chapter 2. Thus, Eq. (1.55) may be verified by a direct though not very elegant method of expanding into Cartesian components (see Exercise 1.5.2).

Exercises

1.5.1 One vertex of a glass parallelepiped is at the origin (Fig. 1.18). The three adjacent vertices are at $(3, 0, 0)$, $(0, 0, 2)$, and $(0, 3, 1)$. All lengths are in centimeters. Calculate the number of cubic centimeters of glass in the parallelepiped using the triple scalar product.

1.5.2 Verify the expansion of the triple vector product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

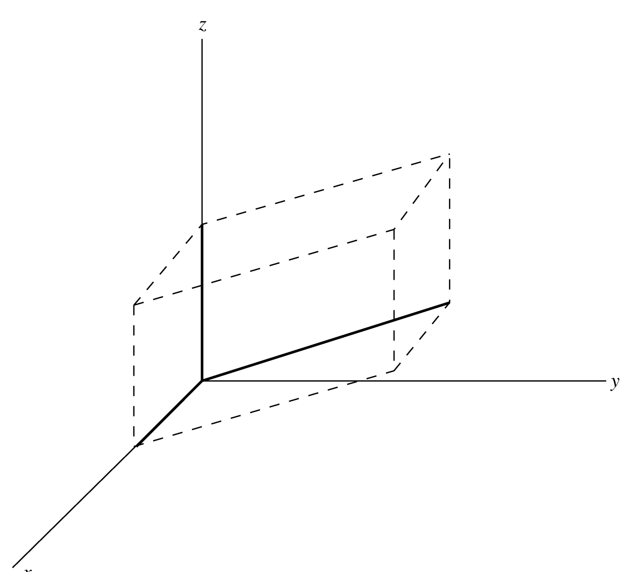

FIGURE 1.18 Parallelepiped: triple scalar product.

by direct expansion in Cartesian coordinates.

1.5.3 Show that the first step in Eq. (1.43), which is

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2,$$

is consistent with the *BAC–CAB* rule for a triple vector product.

1.5.4 You are given the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$,

$$\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}},$$
$$\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}},$$
$$\mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{z}}.$$

(a) Compute the triple scalar product, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$. Noting that $\mathbf{A} = \mathbf{B}+\mathbf{C}$, give a geometric interpretation of your result for the triple scalar product.
(b) Compute $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

1.5.5 The orbital angular momentum $\mathbf{L}$ of a particle is given by $\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times\mathbf{v}$, where $\mathbf{p}$ is the linear momentum. With linear and angular velocity related by $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, show that

$$\mathbf{L} = mr^2\big[\boldsymbol{\omega} - \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\boldsymbol{\omega})\big].$$

Here $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$-direction. For $\mathbf{r}\cdot\boldsymbol{\omega} = 0$ this reduces to $\mathbf{L} = I\boldsymbol{\omega}$, with the moment of inertia I given by mr^2. In Section 3.5 this result is generalized to form an inertia tensor.

1.5.6 The kinetic energy of a single particle is given by $T = \frac{1}{2}mv^2$. For rotational motion this becomes $\frac{1}{2}m(\boldsymbol{\omega}\times\mathbf{r})^2$. Show that

$$T = \frac{1}{2}m\big[r^2\omega^2 - (\mathbf{r}\cdot\boldsymbol{\omega})^2\big].$$

For $\mathbf{r}\cdot\boldsymbol{\omega} = 0$ this reduces to $T = \frac{1}{2}I\omega^2$, with the moment of inertia I given by mr^2.

1.5.7 Show that[13]

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0.$$

1.5.8 A vector $\mathbf{A}$ is decomposed into a radial vector $\mathbf{A}_r$ and a tangential vector $\mathbf{A}_t$. If $\hat{\mathbf{r}}$ is a unit vector in the radial direction, show that

(a) $\mathbf{A}_r = \hat{\mathbf{r}}(\mathbf{A}\cdot\hat{\mathbf{r}})$ and
(b) $\mathbf{A}_t = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{A})$.

1.5.9 Prove that a necessary and sufficient condition for the three (nonvanishing) vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ to be coplanar is the vanishing of the triple scalar product

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0.$$

[13]This is Jacobi's identity for vector products; for commutators it is important in the context of Lie algebras (see Eq. (4.16) in Section 4.2).

1.5.10 Three vectors **A**, **B**, and **C** are given by

$$\mathbf{A} = 3\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 2\hat{\mathbf{z}},$$
$$\mathbf{B} = 6\hat{\mathbf{x}} + 4\hat{\mathbf{y}} - 2\hat{\mathbf{z}},$$
$$\mathbf{C} = -3\hat{\mathbf{x}} - 2\hat{\mathbf{y}} - 4\hat{\mathbf{z}}.$$

Compute the values of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, $\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$ and $\mathbf{B} \times (\mathbf{C} \times \mathbf{A})$.

1.5.11 Vector **D** is a linear combination of three noncoplanar (and nonorthogonal) vectors:

$$\mathbf{D} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}.$$

Show that the coefficients are given by a ratio of triple scalar products,

$$a = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{C}}{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}, \qquad \text{and so on.}$$

1.5.12 Show that

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$

1.5.13 Show that

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{D})\mathbf{C} - (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})\mathbf{D}.$$

1.5.14 For a **spherical** triangle such as pictured in Fig. 1.14 show that

$$\frac{\sin A}{\sin \overline{BC}} = \frac{\sin B}{\sin \overline{CA}} = \frac{\sin C}{\sin \overline{AB}}.$$

Here $\sin A$ is the sine of the included angle at A, while $\overline{BC}$ is the side opposite (in radians).

1.5.15 Given

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}},$$

and $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$, show that

(a) $\mathbf{x} \cdot \mathbf{y}' = \delta_{xy}$, $(\mathbf{x}, \mathbf{y} = \mathbf{a}, \mathbf{b}, \mathbf{c})$,
(b) $\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}' = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})^{-1}$,
(c) $\mathbf{a} = \dfrac{\mathbf{b}' \times \mathbf{c}'}{\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}'}$.

1.5.16 If $\mathbf{x} \cdot \mathbf{y}' = \delta_{xy}$, $(\mathbf{x}, \mathbf{y} = \mathbf{a}, \mathbf{b}, \mathbf{c})$, prove that

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}.$$

(This is the converse of Problem 1.5.15.)

1.5.17 Show that any vector **V** may be expressed in terms of the reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ (of Problem 1.5.15) by

$$\mathbf{V} = (\mathbf{V} \cdot \mathbf{a})\mathbf{a}' + (\mathbf{V} \cdot \mathbf{b})\mathbf{b}' + (\mathbf{V} \cdot \mathbf{c})\mathbf{c}'.$$

1.5.18 An electric charge q_1 moving with velocity $\mathbf{v}_1$ produces a magnetic induction $\mathbf{B}$ given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} q_1 \frac{\mathbf{v}_1 \times \hat{\mathbf{r}}}{r^2} \qquad \text{(mks units)},$$

where $\hat{\mathbf{r}}$ points from q_1 to the point at which $\mathbf{B}$ is measured (Biot and Savart law).

(a) Show that the magnetic force on a second charge q_2, velocity $\mathbf{v}_2$, is given by the triple vector product

$$\mathbf{F}_2 = \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r^2} \mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{r}}).$$

(b) Write out the corresponding magnetic force $\mathbf{F}_1$ that q_2 exerts on q_1. Define your unit radial vector. How do $\mathbf{F}_1$ and $\mathbf{F}_2$ compare?

(c) Calculate $\mathbf{F}_1$ and $\mathbf{F}_2$ for the case of q_1 and q_2 moving along parallel trajectories side by side.

ANS.

(b) $\mathbf{F}_1 = -\frac{\mu_0}{4\pi} \frac{q_1 q_2}{r^2} \mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{r}}).$
In general, there is no simple relation between $\mathbf{F}_1$ and $\mathbf{F}_2$. Specifically, Newton's third law, $\mathbf{F}_1 = -\mathbf{F}_2$, does not hold.

(c) $\mathbf{F}_1 = \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r^2} v^2 \hat{\mathbf{r}} = -\mathbf{F}_2.$
Mutual attraction.

1.6 Gradient, ∇

To provide a motivation for the vector nature of partial derivatives, we now introduce the **total variation of a function** $F(x, y)$,

$$\boxed{dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.}$$

It consists of independent variations in the x- and y-directions. We write dF as a sum of two increments, one purely in the x- and the other in the y-direction,

$$\begin{aligned} dF(x, y) &\equiv F(x + dx, y + dy) - F(x, y) \\ &= \big[F(x + dx, y + dy) - F(x, y + dy)\big] + \big[F(x, y + dy) - F(x, y)\big] \\ &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy, \end{aligned}$$

by adding and subtracting $F(x, y + dy)$. The mean value theorem (that is, continuity of F) tells us that here $\partial F/\partial x$, $\partial F/\partial y$ are evaluated at some point ξ, η between x and $x + dx$, y

and $y + dy$, respectively. As $dx \to 0$ and $dy \to 0$, $\xi \to x$ and $\eta \to y$. This result generalizes to three and higher dimensions. For example, for a function φ of three variables,

$$\begin{aligned} d\varphi(x, y, z) &\equiv \left[\varphi(x + dx, y + dy, z + dz) - \varphi(x, y + dy, z + dz)\right] \\ &\quad + \left[\varphi(x, y + dy, z + dz) - \varphi(x, y, z + dz)\right] \\ &\quad + \left[\varphi(x, y, z + dz) - \varphi(x, y, z)\right] \\ &= \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz. \end{aligned} \tag{1.57}$$

Algebraically, $d\varphi$ in the total variation is a scalar product of the change in position $d\mathbf{r}$ and the **directional** change of φ. And now we are ready to recognize the three-dimensional partial derivative as a vector, which leads us to the concept of gradient.

Suppose that $\varphi(x, y, z)$ is a scalar point function, that is, a function whose value depends on the values of the coordinates (x, y, z). As a scalar, it must have the same value at a given fixed point in space, independent of the rotation of our coordinate system, or

$$\varphi'(x_1', x_2', x_3') = \varphi(x_1, x_2, x_3). \tag{1.58}$$

By differentiating with respect to x_i' we obtain

$$\frac{\partial \varphi'(x_1', x_2', x_3')}{\partial x_i'} = \frac{\partial \varphi(x_1, x_2, x_3)}{\partial x_i'} = \sum_j \frac{\partial \varphi}{\partial x_j} \frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij} \frac{\partial \varphi}{\partial x_j} \tag{1.59}$$

by the rules of partial differentiation and Eqs. (1.16a) and (1.16b). But comparison with Eq. (1.17), the vector transformation law, now shows that we have **constructed** a vector with components $\partial\varphi/\partial x_j$. This vector we label the gradient of φ.

A convenient symbolism is

$$\nabla\varphi = \hat{\mathbf{x}} \frac{\partial \varphi}{\partial x} + \hat{\mathbf{y}} \frac{\partial \varphi}{\partial y} + \hat{\mathbf{z}} \frac{\partial \varphi}{\partial z} \tag{1.60}$$

or

$$\boxed{\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}.} \tag{1.61}$$

$\nabla\varphi$ (or del φ) is our gradient of the scalar φ, whereas ∇ (del) itself is a vector differential operator (available to operate on or to differentiate a scalar φ). All the relationships for ∇ (del) can be derived from the hybrid nature of del in terms of both the partial derivatives and its vector nature.

The gradient of a scalar is extremely important in physics and engineering in expressing the relation between a force field and a potential field,

$$\text{force } \mathbf{F} = -\nabla(\text{potential } V), \tag{1.62}$$

which holds for both gravitational and electrostatic fields, among others. Note that the minus sign in Eq. (1.62) results in water flowing downhill rather than uphill! If a force can be described, as in Eq. (1.62), by a single function $V(\mathbf{r})$ everywhere, we call the scalar function V its **potential**. Because the force is the directional derivative of the potential, we can find the potential, if it exists, by integrating the force along a suitable path. Because the total variation $dV = \nabla V \cdot d\mathbf{r} = -\mathbf{F} \cdot d\mathbf{r}$ is the work done against the force along the path $d\mathbf{r}$, we recognize the physical meaning of the potential (difference) as work and energy. Moreover, in a sum of path increments the intermediate points cancel,

$$\left[V(\mathbf{r}+d\mathbf{r}_1+d\mathbf{r}_2)-V(\mathbf{r}+d\mathbf{r}_1)\right]+\left[V(\mathbf{r}+d\mathbf{r}_1)-V(\mathbf{r})\right]=V(\mathbf{r}+d\mathbf{r}_2+d\mathbf{r}_1)-V(\mathbf{r}),$$

so the integrated work along some path from an initial point $\mathbf{r}_i$ to a final point $\mathbf{r}$ is given by the potential difference $V(\mathbf{r}) - V(\mathbf{r}_i)$ at the endpoints of the path. Therefore, such forces are especially simple and well behaved: They are called **conservative**. When there is loss of energy due to friction along the path or some other dissipation, the work will depend on the path, and such forces cannot be conservative: No potential exists. We discuss conservative forces in more detail in Section 1.13.

Example 1.6.1 The Gradient of a Potential $V(r)$

Let us calculate the gradient of $V(r) = V(\sqrt{x^2+y^2+z^2})$, so

$$\nabla V(r) = \hat{\mathbf{x}}\frac{\partial V(r)}{\partial x} + \hat{\mathbf{y}}\frac{\partial V(r)}{\partial y} + \hat{\mathbf{z}}\frac{\partial V(r)}{\partial z}.$$

Now, $V(r)$ depends on x through the dependence of r on x. Therefore[14]

$$\frac{\partial V(r)}{\partial x} = \frac{dV(r)}{dr} \cdot \frac{\partial r}{\partial x}.$$

From r as a function of x, y, z,

$$\frac{\partial r}{\partial x} = \frac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}.$$

Therefore

$$\frac{\partial V(r)}{\partial x} = \frac{dV(r)}{dr} \cdot \frac{x}{r}.$$

Permuting coordinates $(x \to y, y \to z, z \to x)$ to obtain the y and z derivatives, we get

$$\boxed{\begin{aligned} \nabla V(r) &= (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)\frac{1}{r}\frac{dV}{dr} \\ &= \frac{\mathbf{r}}{r}\frac{dV}{dr} = \hat{\mathbf{r}}\frac{dV}{dr}. \end{aligned}}$$

Here $\hat{\mathbf{r}}$ is a unit vector $(\mathbf{r}/r)$ in the **positive** radial direction. The gradient of a function of r is a vector in the (positive or negative) radial direction. In Section 2.5, $\hat{\mathbf{r}}$ is seen as one of the three orthonormal unit vectors of spherical polar coordinates and $\hat{\mathbf{r}}\partial/\partial r$ as the radial component of ∇. ■

[14]This is a special case of the **chain rule** of partial differentiation:

$$\frac{\partial V(r,\theta,\varphi)}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial x},$$

where $\partial V/\partial\theta = \partial V/\partial\varphi = 0$, $\partial V/\partial r \to dV/dr$.

A Geometrical Interpretation

One immediate application of $\nabla\varphi$ is to dot it into an increment of length

$$d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz.$$

Thus we obtain

$$\nabla\varphi \cdot d\mathbf{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi,$$

the change in the scalar function φ corresponding to a change in position $d\mathbf{r}$. Now consider P and Q to be two points on a surface $\varphi(x, y, z) = C$, a constant. These points are chosen so that Q is a distance $d\mathbf{r}$ from P. Then, moving from P to Q, the change in $\varphi(x, y, z) = C$ is given by

$$d\varphi = (\nabla\varphi) \cdot d\mathbf{r} = 0 \tag{1.63}$$

since we stay on the surface $\varphi(x, y, z) = C$. This shows that $\nabla\varphi$ is perpendicular to $d\mathbf{r}$. Since $d\mathbf{r}$ may have any direction from P **as long as it stays in the surface** of constant φ, point Q being restricted to the surface but having arbitrary direction, $\nabla\varphi$ is seen as normal to the surface $\varphi = \text{constant}$ (Fig. 1.19).

If we now permit $d\mathbf{r}$ to take us from one surface $\varphi = C_1$ to an adjacent surface $\varphi = C_2$ (Fig. 1.20),

$$d\varphi = C_1 - C_2 = \Delta C = (\nabla\varphi) \cdot d\mathbf{r}. \tag{1.64}$$

For a given $d\varphi$, $|d\mathbf{r}|$ is a minimum when it is chosen parallel to $\nabla\varphi$ ($\cos\theta = 1$); or, for a given $|d\mathbf{r}|$, the change in the scalar function φ is maximized by choosing $d\mathbf{r}$ parallel to

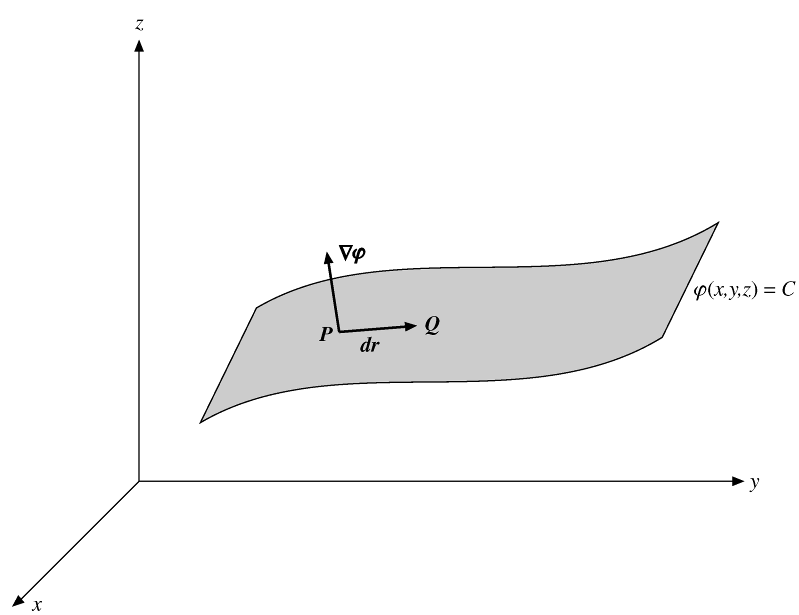

FIGURE 1.19 The length increment $d\mathbf{r}$ has to stay on the surface $\varphi = C$.

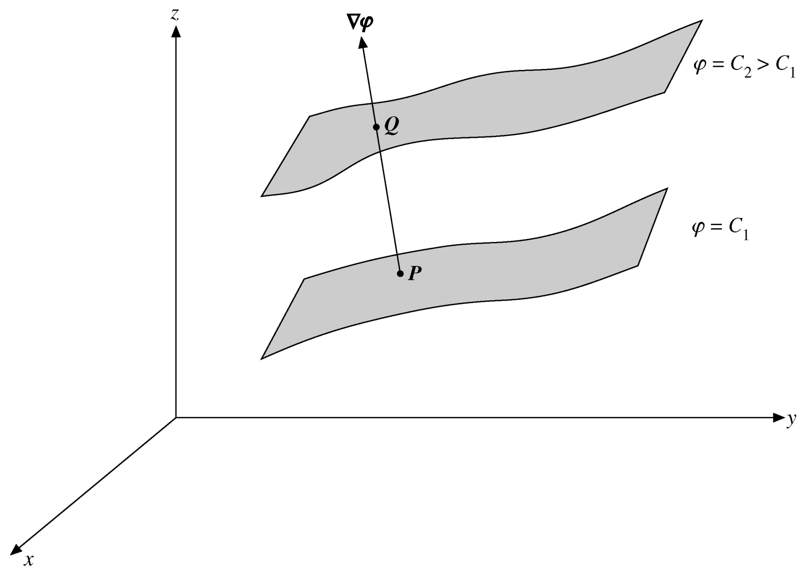

FIGURE 1.20 Gradient.

$\nabla\varphi$. **This identifies** $\nabla\varphi$ **as a vector having the direction of the maximum space rate of change of** φ, an identification that will be useful in Chapter 2 when we consider non-Cartesian coordinate systems. This identification of $\nabla\varphi$ may also be developed by using the calculus of variations subject to a constraint, Exercise 17.6.9.

Example 1.6.2 FORCE AS GRADIENT OF A POTENTIAL

As a specific example of the foregoing, and as an extension of Example 1.6.1, we consider the surfaces consisting of concentric spherical shells, Fig. 1.21. We have

$$\varphi(x, y, z) = \left(x^2 + y^2 + z^2\right)^{1/2} = r = C,$$

where r is the radius, equal to C, our constant. $\Delta C = \Delta\varphi = \Delta r$, the distance between two shells. From Example 1.6.1

$$\nabla\varphi(r) = \hat{\mathbf{r}}\frac{d\varphi(r)}{dr} = \hat{\mathbf{r}}.$$

The gradient is in the radial direction and is normal to the spherical surface $\varphi = C$. ■

Example 1.6.3 INTEGRATION BY PARTS OF GRADIENT

Let us prove the formula $\int \mathbf{A}(\mathbf{r}) \cdot \nabla f(\mathbf{r})\, d^3r = -\int f(\mathbf{r})\nabla \cdot \mathbf{A}(\mathbf{r})\, d^3r$, where $\mathbf{A}$ or f or both vanish at infinity so that the integrated parts vanish. This condition is satisfied if, for example, $\mathbf{A}$ is the electromagnetic vector potential and f is a bound-state wave function $\psi(\mathbf{r})$.

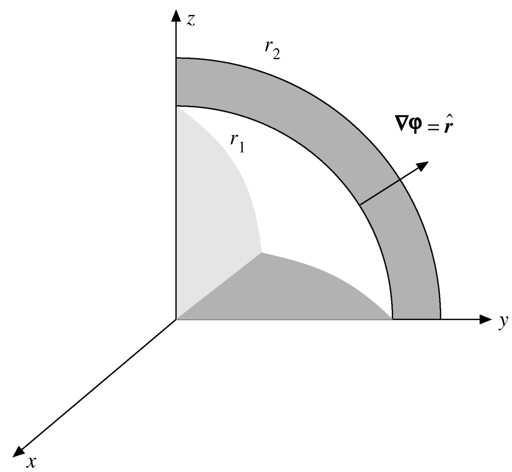

FIGURE 1.21 Gradient for $\varphi(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$, spherical shells: $(x_2^2 + y_2^2 + z_2^2)^{1/2} = r_2 = C_2$, $(x_1^2 + y_1^2 + z_1^2)^{1/2} = r_1 = C_1$.

Writing the inner product in Cartesian coordinates, integrating each one-dimensional integral by parts, and dropping the integrated terms, we obtain

$$\begin{aligned}\int \mathbf{A}(\mathbf{r}) \cdot \nabla f(\mathbf{r})\, d^3r &= \iint \left[A_x f|_{x=-\infty}^{\infty} - \int f \frac{\partial A_x}{\partial x} dx \right] dy\, dz + \cdots \\ &= -\iiint f \frac{\partial A_x}{\partial x} dx\, dy\, dz - \iiint f \frac{\partial A_y}{\partial y} dy\, dx\, dz - \iiint f \frac{\partial A_z}{\partial z} dz\, dx\, dy \\ &= -\int f(\mathbf{r}) \nabla \cdot \mathbf{A}(\mathbf{r})\, d^3r.\end{aligned}$$

If $\mathbf{A} = e^{ikz}\hat{\mathbf{e}}$ describes an outgoing photon in the direction of the constant polarization unit vector $\hat{\mathbf{e}}$ and $f = \psi(\mathbf{r})$ is an exponentially decaying bound-state wave function, then

$$\int e^{ikz}\hat{\mathbf{e}} \cdot \nabla\psi(\mathbf{r})\, d^3r = -e_z \int \psi(\mathbf{r}) \frac{de^{ikz}}{dz} d^3r = -ike_z \int \psi(\mathbf{r}) e^{ikz}\, d^3r,$$

because only the z-component of the gradient contributes. ■

Exercises

1.6.1 If $S(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}$, find

(a) ∇S at the point $(1, 2, 3)$;
(b) the magnitude of the gradient of S, $|\nabla S|$ at $(1, 2, 3)$; and
(c) the direction cosines of ∇S at $(1, 2, 3)$.

1.6.2 (a) Find a unit vector perpendicular to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point (1, 1, 1). Lengths are in centimeters.

(b) Derive the equation of the plane tangent to the surface at (1, 1, 1).

ANS. (a) $(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})/\sqrt{3}$, (b) $x + y + z = 3$.

1.6.3 Given a vector $\mathbf{r}_{12} = \hat{\mathbf{x}}(x_1 - x_2) + \hat{\mathbf{y}}(y_1 - y_2) + \hat{\mathbf{z}}(z_1 - z_2)$, show that $\boldsymbol{\nabla}_1 r_{12}$ (gradient with respect to x_1, y_1, and z_1 of the magnitude r_{12}) is a unit vector in the direction of $\mathbf{r}_{12}$.

1.6.4 If a vector function $\mathbf{F}$ depends on both space coordinates (x, y, z) and time t, show that

$$d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} dt.$$

1.6.5 Show that $\boldsymbol{\nabla}(uv) = v\boldsymbol{\nabla}u + u\boldsymbol{\nabla}v$, where u and v are differentiable scalar functions of x, y, and z.

(a) Show that a necessary and sufficient condition that $u(x, y, z)$ and $v(x, y, z)$ are related by some function $f(u, v) = 0$ is that $(\boldsymbol{\nabla}u) \times (\boldsymbol{\nabla}v) = 0$.

(b) If $u = u(x, y)$ and $v = v(x, y)$, show that the condition $(\boldsymbol{\nabla}u) \times (\boldsymbol{\nabla}v) = 0$ leads to the two-dimensional Jacobian

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

The functions u and v are assumed differentiable.

1.7 DIVERGENCE, ∇

Differentiating a vector function is a simple extension of differentiating scalar quantities. Suppose $\mathbf{r}(t)$ describes the position of a satellite at some time t. Then, for differentiation with respect to time,

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \mathbf{v}, \text{ linear velocity.}$$

Graphically, we again have the slope of a curve, orbit, or trajectory, as shown in Fig. 1.22.

If we resolve $\mathbf{r}(t)$ into its Cartesian components, $d\mathbf{r}/dt$ always reduces directly to a vector sum of not more than three (for three-dimensional space) scalar derivatives. In other coordinate systems (Chapter 2) the situation is more complicated, for the unit vectors are no longer constant in direction. Differentiation with respect to the space coordinates is handled in the same way as differentiation with respect to time, as seen in the following paragraphs.

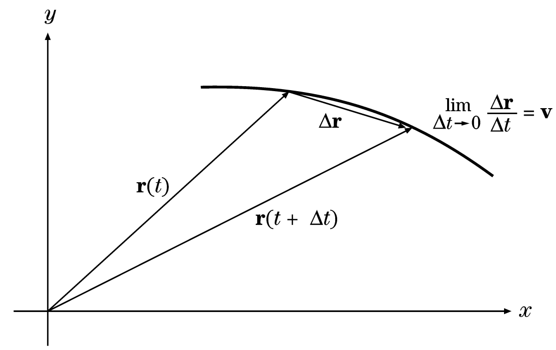

FIGURE 1.22 Differentiation of a vector.

In Section 1.6, ∇ was defined as a vector operator. Now, paying attention to both its vector and its differential properties, we let it operate on a vector. First, as a vector we dot it into a second vector to obtain

$$\boxed{\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},} \tag{1.65a}$$

known as the divergence of $\mathbf{V}$. This is a scalar, as discussed in Section 1.3.

Example 1.7.1 DIVERGENCE OF COORDINATE VECTOR

Calculate $\nabla \cdot \mathbf{r}$:

$$\begin{aligned}\nabla \cdot \mathbf{r} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)\\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},\end{aligned}$$

or $\nabla \cdot \mathbf{r} = 3$. ■

Example 1.7.2 DIVERGENCE OF CENTRAL FORCE FIELD

Generalizing Example 1.7.1,

$$\begin{aligned}\nabla \cdot \big(\mathbf{r} f(r)\big) &= \frac{\partial}{\partial x}\big[x\, f(r)\big] + \frac{\partial}{\partial y}\big[y\, f(r)\big] + \frac{\partial}{\partial z}\big[z\, f(r)\big]\\ &= 3 f(r) + \frac{x^2}{r}\frac{df}{dr} + \frac{y^2}{r}\frac{df}{dr} + \frac{z^2}{r}\frac{df}{dr}\\ &= 3 f(r) + r\frac{df}{dr}.\end{aligned}$$

The manipulation of the partial derivatives leading to the second equation in Example 1.7.2 is discussed in Example 1.6.1. In particular, if $f(r) = r^{n-1}$,

$$\begin{aligned}\nabla \cdot \left(\mathbf{r}r^{n-1}\right) &= \nabla \cdot \hat{\mathbf{r}}r^{n} \\ &= 3r^{n-1} + (n-1)r^{n-1} \\ &= (n+2)r^{n-1}. \end{aligned} \tag{1.65b}$$

This divergence vanishes for $n = -2$, except at $r = 0$, an important fact in Section 1.14. ■

Example 1.7.3 Integration by Parts of Divergence

Let us prove the formula $\int f(\mathbf{r})\nabla \cdot \mathbf{A}(\mathbf{r})\, d^3r = -\int \mathbf{A} \cdot \nabla f\, d^3r$, where $\mathbf{A}$ or f or both vanish at infinity.

To show this, we proceed, as in Example 1.6.3, by integration by parts after writing the inner product in Cartesian coordinates. Because the integrated terms are evaluated at infinity, where they vanish, we obtain

$$\begin{aligned}\int f(\mathbf{r})\nabla \cdot \mathbf{A}(\mathbf{r})\, d^3r &= \int f\left(\frac{\partial A_x}{\partial x}dx\,dy\,dz + \frac{\partial A_y}{\partial y}dy\,dx\,dz + \frac{\partial A_z}{\partial z}dz\,dx\,dy\right) \\ &= -\int \left(A_x\frac{\partial f}{\partial x}dx\,dy\,dz + A_y\frac{\partial f}{\partial y}dy\,dx\,dz + A_z\frac{\partial f}{\partial z}dz\,dx\,dy\right) \\ &= -\int \mathbf{A} \cdot \nabla f\, d^3r. \end{aligned}$$

■

A Physical Interpretation

To develop a feeling for the physical significance of the divergence, consider $\nabla \cdot (\rho\mathbf{v})$ with $\mathbf{v}(x, y, z)$, the velocity of a compressible fluid, and $\rho(x, y, z)$, its density at point (x, y, z). If we consider a small volume $dx\,dy\,dz$ (Fig. 1.23) at $x = y = z = 0$, the fluid flowing into this volume per unit time (positive x-direction) through the face $EFGH$ is (rate of flow in)$_{EFGH} = \rho v_x|_{x=0} = dy\,dz$. The components of the flow ρv_y and ρv_z tangential to this face contribute nothing to the flow through this face. The rate of flow out (still positive x-direction) through face $ABCD$ is $\rho v_x|_{x=dx}\, dy\,dz$. To compare these flows and to find the net flow out, we expand this last result, like the total variation in Section 1.6.[15] This yields

$$\begin{aligned}(\text{rate of flow out})_{ABCD} &= \rho v_x|_{x=dx}\, dy\,dz \\ &= \left[\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\, dx\right]_{x=0} dy\,dz.\end{aligned}$$

Here the derivative term is a first correction term, allowing for the possibility of nonuniform density or velocity or both.[16] The zero-order term $\rho v_x|_{x=0}$ (corresponding to uniform flow)

[15] Here we have the increment dx and we show a partial derivative with respect to x since ρv_x may also depend on y and z.

[16] Strictly speaking, ρv_x is averaged over face $EFGH$ and the expression $\rho v_x + (\partial/\partial x)(\rho v_x)\, dx$ is similarly averaged over face $ABCD$. Using an arbitrarily small differential volume, we find that the averages reduce to the values employed here.

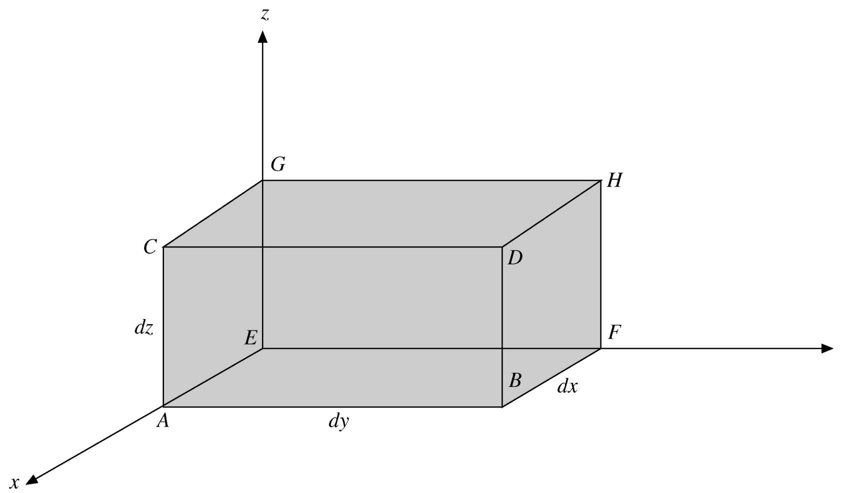

FIGURE 1.23 Differential rectangular parallelepiped (in first octant).

cancels out:

$$\text{Net rate of flow out}|_x = \frac{\partial}{\partial x}(\rho v_x)\, dx\, dy\, dz.$$

Equivalently, we can arrive at this result by

$$\lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x, 0, 0) - \rho v_x(0, 0, 0)}{\Delta x} \equiv \left.\frac{\partial[\rho v_x(x, y, z)]}{\partial x}\right|_{0,0,0}.$$

Now, the x-axis is not entitled to any preferred treatment. The preceding result for the two faces perpendicular to the x-axis must hold for the two faces perpendicular to the y-axis, with x replaced by y and the corresponding changes for y and z: $y \to z$, $z \to x$. This is a cyclic permutation of the coordinates. A further cyclic permutation yields the result for the remaining two faces of our parallelepiped. Adding the net rate of flow out for all three pairs of surfaces of our volume element, we have

$$\begin{aligned} \begin{matrix}\text{net flow out} \\ \text{(per unit time)}\end{matrix} &= \left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dx\, dy\, dz \\ &= \nabla \cdot (\rho \mathbf{v})\, dx\, dy\, dz. \end{aligned} \tag{1.66}$$

Therefore the net flow of our compressible fluid out of the volume element $dx\,dy\,dz$ per unit volume per unit time is $\nabla \cdot (\rho \mathbf{v})$. Hence the name **divergence**. A direct application is in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.67a}$$

which states that a net flow out of the volume results in a decreased density inside the volume. Note that in Eq. (1.67a), ρ is considered to be a possible function of time as well as of space: $\rho(x, y, z, t)$. The divergence appears in a wide variety of physical problems, ranging from a probability current density in quantum mechanics to neutron leakage in a nuclear reactor.

The combination $\nabla \cdot (f\mathbf{V})$, in which f is a scalar function and $\mathbf{V}$ is a vector function, may be written

$$\begin{aligned}\nabla \cdot (f\mathbf{V}) &= \frac{\partial}{\partial x}(fV_x) + \frac{\partial}{\partial y}(fV_y) + \frac{\partial}{\partial z}(fV_z)\\ &= \frac{\partial f}{\partial x}V_x + f\frac{\partial V_x}{\partial x} + \frac{\partial f}{\partial y}V_y + f\frac{\partial V_y}{\partial y} + \frac{\partial f}{\partial z}V_z + f\frac{\partial V_z}{\partial z}\\ &= (\nabla f)\cdot \mathbf{V} + f\nabla \cdot \mathbf{V}, \qquad (1.67\text{b})\end{aligned}$$

which is just what we would expect for the derivative of a product. Notice that ∇ as a differential operator differentiates both f and $\mathbf{V}$; as a vector it is dotted into $\mathbf{V}$ (in each term).

If we have the special case of the divergence of a vector vanishing,

$$\nabla \cdot \mathbf{B} = 0, \qquad (1.68)$$

the vector $\mathbf{B}$ is said to be **solenoidal**, the term coming from the example in which $\mathbf{B}$ is the magnetic induction and Eq. (1.68) appears as one of Maxwell's equations. When a vector is solenoidal, it may be written as the curl of another vector known as the vector potential. (In Section 1.13 we shall calculate such a vector potential.)

Exercises

1.7.1 For a particle moving in a circular orbit $\mathbf{r} = \hat{\mathbf{x}}r\cos\omega t + \hat{\mathbf{y}}r\sin\omega t$,

(a) evaluate $\mathbf{r}\times\dot{\mathbf{r}}$, with $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \mathbf{v}$.
(b) Show that $\ddot{\mathbf{r}} + \omega^2\mathbf{r} = 0$ with $\ddot{\mathbf{r}} = \frac{d\mathbf{v}}{dt}$.
The radius r and the angular velocity ω are constant.

ANS. (a) $\hat{\mathbf{z}}\omega r^2$.

1.7.2 Vector $\mathbf{A}$ satisfies the vector transformation law, Eq. (1.15). Show directly that its time derivative $d\mathbf{A}/dt$ also satisfies Eq. (1.15) and is therefore a vector.

1.7.3 Show, by differentiating components, that

(a) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$,
(b) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$,
just like the derivative of the product of two algebraic functions.

1.7.4 In Chapter 2 it will be seen that the unit vectors in non-Cartesian coordinate systems are usually functions of the coordinate variables, $\mathbf{e}_i = \mathbf{e}_i(q_1, q_2, q_3)$ but $|\mathbf{e}_i| = 1$. Show that either $\partial\mathbf{e}_i/\partial q_j = 0$ or $\partial\mathbf{e}_i/\partial q_j$ is orthogonal to $\mathbf{e}_i$.
Hint. $\partial\mathbf{e}_i^2/\partial q_j = 0$.

1.7.5 Prove $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$.
Hint. Treat as a triple scalar product.

1.7.6 The electrostatic field of a point charge q is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\hat{\mathbf{r}}}{r^2}.$$

Calculate the divergence of $\mathbf{E}$. What happens at the origin?

1.8 Curl, $\nabla\times$

Another possible operation with the vector operator ∇ is to cross it into a vector. We obtain

$$\begin{aligned}\nabla \times \mathbf{V} &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)\\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \end{aligned} \tag{1.69}$$

which is called the **curl** of $\mathbf{V}$. In expanding this determinant we must consider the derivative nature of ∇. Specifically, $\mathbf{V} \times \nabla$ is defined only as an operator, another vector differential operator. It is certainly not equal, in general, to $-\nabla \times \mathbf{V}$.[17] In the case of Eq. (1.69) the determinant must be expanded **from the top down** so that we get the derivatives as shown in the middle portion of Eq. (1.69). If ∇ is crossed into the product of a scalar and a vector, we can show

$$\begin{aligned}\nabla \times (f\mathbf{V})|_x &= \left[\frac{\partial}{\partial y}(fV_z) - \frac{\partial}{\partial z}(fV_y)\right]\\ &= \left(f\frac{\partial V_z}{\partial y} + \frac{\partial f}{\partial y}V_z - f\frac{\partial V_y}{\partial z} - \frac{\partial f}{\partial z}V_y\right)\\ &= f\nabla \times \mathbf{V}|_x + (\nabla f) \times \mathbf{V}|_x. \end{aligned} \tag{1.70}$$

If we permute the coordinates $x \to y$, $y \to z$, $z \to x$ to pick up the y-component and then permute them a second time to pick up the z-component, then

$$\nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + (\nabla f) \times \mathbf{V}, \tag{1.71}$$

which is the vector product analog of Eq. (1.67b). Again, as a differential operator ∇ differentiates both f and $\mathbf{V}$. As a vector it is crossed into $\mathbf{V}$ (in each term).

[17] In this same spirit, if $\mathbf{A}$ is a differential operator, it is not necessarily true that $\mathbf{A} \times \mathbf{A} = 0$. Specifically, for the quantum mechanical angular momentum **operator** $\mathbf{L} = -i(\mathbf{r} \times \nabla)$, we find that $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$. See Sections 4.3 and 4.4 for more details.

Example 1.8.1 VECTOR POTENTIAL OF A CONSTANT **B** FIELD

From electrodynamics we know that $\nabla \cdot \mathbf{B} = 0$, which has the general solution $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}(\mathbf{r})$ is called the vector potential (of the magnetic induction), because $\nabla \cdot (\nabla \times \mathbf{A}) = (\nabla \times \nabla) \cdot \mathbf{A} \equiv 0$, as a triple scalar product with two identical vectors. This last identity will not change if we add the gradient of some scalar function to the vector potential, which, therefore, is not unique.

In our case, we want to show that a vector potential is $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$.

Using the *BAC–BAC* rule in conjunction with Example 1.7.1, we find that

$$2\nabla \times \mathbf{A} = \nabla \times (\mathbf{B} \times \mathbf{r}) = (\nabla \cdot \mathbf{r})\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{r} = 3\mathbf{B} - \mathbf{B} = 2\mathbf{B},$$

where we indicate by the ordering of the scalar product of the second term that the gradient still acts on the coordinate vector. ■

Example 1.8.2 CURL OF A CENTRAL FORCE FIELD

Calculate $\nabla \times (\mathbf{r} f(r))$.

By Eq. (1.71),

$$\nabla \times \big(\mathbf{r} f(r)\big) = f(r)\nabla \times \mathbf{r} + \big[\nabla f(r)\big] \times \mathbf{r}. \tag{1.72}$$

First,

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0. \tag{1.73}$$

Second, using $\nabla f(r) = \hat{\mathbf{r}}(df/dr)$ (Example 1.6.1), we obtain

$$\nabla \times \mathbf{r} f(r) = \frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{r} = 0. \tag{1.74}$$

This vector product vanishes, since $\mathbf{r} = \hat{\mathbf{r}}r$ and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$. ■

To develop a better feeling for the physical significance of the curl, we consider the circulation of fluid around a differential loop in the xy-plane, Fig. 1.24.

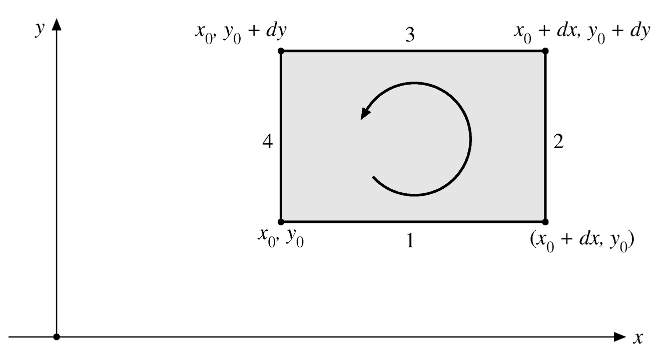

FIGURE 1.24 Circulation around a differential loop.

Although the circulation is technically given by a vector line integral $\int \mathbf{V}\cdot d\boldsymbol{\lambda}$ (Section 1.10), we can set up the equivalent scalar integrals here. Let us take the circulation to be

$$\begin{aligned}\text{circulation}_{1234} &= \int_1 V_x(x,y)\,d\lambda_x + \int_2 V_y(x,y)\,d\lambda_y \\ &\quad + \int_3 V_x(x,y)\,d\lambda_x + \int_4 V_y(x,y)\,d\lambda_y. \end{aligned} \tag{1.75}$$

The numbers 1, 2, 3, and 4 refer to the numbered line segments in Fig. 1.24. In the first integral, $d\lambda_x = +dx$; but in the third integral, $d\lambda_x = -dx$ because the third line segment is traversed in the negative x-direction. Similarly, $d\lambda_y = +dy$ for the second integral, $-dy$ for the fourth. Next, the integrands are referred to the point (x_0, y_0) with a Taylor expansion[18] taking into account the displacement of line segment 3 from 1 and that of 2 from 4. For our differential line segments this leads to

$$\begin{aligned}\text{circulation}_{1234} &= V_x(x_0,y_0)\,dx + \left[V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx\right]dy \\ &\quad + \left[V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy\right](-dx) + V_y(x_0,y_0)(-dy) \\ &= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy. \end{aligned} \tag{1.76}$$

Dividing by $dx\,dy$, we have

$$\text{circulation per unit area} = \nabla\times\mathbf{V}|_z. \tag{1.77}$$

The circulation[19] about our differential area in the xy-plane is given by the z-component of $\nabla\times\mathbf{V}$. In principle, the curl $\nabla\times\mathbf{V}$ at (x_0, y_0) could be determined by inserting a (differential) paddle wheel into the moving fluid at point (x_0, y_0). The rotation of the little paddle wheel would be a measure of the curl, and its axis would be along the direction of $\nabla\times\mathbf{V}$, which is perpendicular to the plane of circulation.

We shall use the result, Eq. (1.76), in Section 1.12 to derive Stokes' theorem. Whenever the curl of a vector $\mathbf{V}$ vanishes,

$$\nabla\times\mathbf{V} = 0, \tag{1.78}$$

$\mathbf{V}$ is labeled **irrotational**. The most important physical examples of irrotational vectors are the gravitational and electrostatic forces. In each case

$$\mathbf{V} = C\frac{\hat{\mathbf{r}}}{r^2} = C\frac{\mathbf{r}}{r^3}, \tag{1.79}$$

where C is a constant and $\hat{\mathbf{r}}$ is the unit vector in the outward radial direction. For the gravitational case we have $C = -Gm_1m_2$, given by Newton's law of universal gravitation. If $C = q_1q_2/4\pi\varepsilon_0$, we have Coulomb's law of electrostatics (mks units). The force $\mathbf{V}$

[18]Here, $V_y(x_0+dx, y_0) = V_y(x_0, y_0) + (\frac{\partial V_y}{\partial x})_{x_0y_0}\,dx + \cdots$. The higher-order terms will drop out in the limit as $dx\to 0$. A correction term for the variation of V_y with y is canceled by the corresponding term in the fourth integral.

[19]In fluid dynamics $\nabla\times\mathbf{V}$ is called the "vorticity."

given in Eq. (1.79) may be shown to be irrotational by direct expansion into Cartesian components, as we did in Example 1.8.1. Another approach is developed in Chapter 2, in which we express $\nabla\times$, the curl, in terms of spherical polar coordinates. In Section 1.13 we shall see that whenever a vector is irrotational, the vector may be written as the (negative) gradient of a scalar potential. In Section 1.16 we shall prove that a vector field may be resolved into an irrotational part and a solenoidal part (subject to conditions at infinity). In terms of the electromagnetic field this corresponds to the resolution into an irrotational electric field and a solenoidal magnetic field.

For waves in an elastic medium, if the displacement $\mathbf{u}$ is irrotational, $\nabla \times \mathbf{u} = 0$, plane waves (or spherical waves at large distances) become longitudinal. If $\mathbf{u}$ is solenoidal, $\nabla \cdot \mathbf{u} = 0$, then the waves become transverse. A seismic disturbance will produce a displacement that may be resolved into a solenoidal part and an irrotational part (compare Section 1.16). The irrotational part yields the longitudinal P (primary) earthquake waves. The solenoidal part gives rise to the slower transverse S (secondary) waves.

Using the gradient, divergence, and curl, and of course the *BAC–CAB* rule, we may construct or verify a large number of useful vector identities. For verification, complete expansion into Cartesian components is always a possibility. Sometimes if we use insight instead of routine shuffling of Cartesian components, the verification process can be shortened drastically.

Remember that ∇ is a vector operator, a hybrid creature satisfying two sets of rules:

1. vector rules, and
2. partial differentiation rules — including differentiation of a product.

Example 1.8.3 Gradient of a Dot Product

Verify that

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}). \tag{1.80}$$

This particular example hinges on the recognition that $\nabla(\mathbf{A}\cdot\mathbf{B})$ is the type of term that appears in the *BAC–CAB* expansion of a triple vector product, Eq. (1.55). For instance,

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B},$$

with the ∇ differentiating only $\mathbf{B}$, not $\mathbf{A}$. From the commutativity of factors in a scalar product we may interchange $\mathbf{A}$ and $\mathbf{B}$ and write

$$\mathbf{B}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{B}\cdot\nabla)\mathbf{A},$$

now with ∇ differentiating only $\mathbf{A}$, not $\mathbf{B}$. Adding these two equations, we obtain ∇ differentiating the product $\mathbf{A}\cdot\mathbf{B}$ and the identity, Eq. (1.80). This identity is used frequently in electromagnetic theory. Exercise 1.8.13 is a simple illustration. ■

Example 1.8.4 Integration by Parts of Curl

Let us prove the formula $\int \mathbf{C}(\mathbf{r}) \cdot (\nabla \times \mathbf{A}(\mathbf{r}))\, d^3r = \int \mathbf{A}(\mathbf{r}) \cdot (\nabla \times \mathbf{C}(\mathbf{r}))\, d^3r$, where $\mathbf{A}$ or $\mathbf{C}$ or both vanish at infinity.

To show this, we proceed, as in Examples 1.6.3 and 1.7.3, by integration by parts after writing the inner product and the curl in Cartesian coordinates. Because the integrated terms vanish at infinity we obtain

$$\begin{aligned}
&\int \mathbf{C}(\mathbf{r}) \cdot \big(\nabla \times \mathbf{A}(\mathbf{r})\big)\, d^3r \\
&\quad = \int \left[C_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + C_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + C_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] d^3r \\
&\quad = \int \left[A_x \left(\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z} \right) + A_y \left(\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x} \right) + A_z \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} \right) \right] d^3r \\
&\quad = \int \mathbf{A}(\mathbf{r}) \cdot \big(\nabla \times \mathbf{C}(\mathbf{r})\big)\, d^3r,
\end{aligned}$$

just rearranging appropriately the terms after integration by parts. ■

Exercises

1.8.1 Show, by rotating the coordinates, that the components of the curl of a vector transform as a vector.
Hint. The direction cosine identities of Eq. (1.46) are available as needed.

1.8.2 Show that $\mathbf{u} \times \mathbf{v}$ is solenoidal if $\mathbf{u}$ and $\mathbf{v}$ are each irrotational.

1.8.3 If $\mathbf{A}$ is irrotational, show that $\mathbf{A} \times \mathbf{r}$ is solenoidal.

1.8.4 A rigid body is rotating with constant angular velocity $\boldsymbol{\omega}$. Show that the linear velocity $\mathbf{v}$ is solenoidal.

1.8.5 If a vector function $\mathbf{f}(x, y, z)$ is not irrotational but the product of f and a scalar function $g(x, y, z)$ is irrotational, show that then

$$\mathbf{f} \cdot \nabla \times \mathbf{f} = 0.$$

1.8.6 If (a) $\mathbf{V} = \hat{\mathbf{x}} V_x(x, y) + \hat{\mathbf{y}} V_y(x, y)$ and (b) $\nabla \times \mathbf{V} \neq 0$, prove that $\nabla \times \mathbf{V}$ is perpendicular to $\mathbf{V}$.

1.8.7 Classically, orbital angular momentum is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the linear momentum. To go from classical mechanics to quantum mechanics, replace $\mathbf{p}$ by the operator $-i\nabla$ (Section 15.6). Show that the quantum mechanical angular momentum operator has Cartesian components (in units of $\hbar$)

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

1.8.8 Using the angular momentum operators previously given, show that they satisfy commutation relations of the form

$$[L_x, L_y] \equiv L_x L_y - L_y L_x = i L_z$$

and hence

$$\mathbf{L} \times \mathbf{L} = i\mathbf{L}.$$

These commutation relations will be taken later as the defining relations of an angular momentum operator—Exercise 3.2.15 and the following one and Chapter 4.

1.8.9 With the commutator bracket notation $[L_x, L_y] = L_x L_y - L_y L_x$, the angular momentum vector $\mathbf{L}$ satisfies $[L_x, L_y] = i L_z$, etc., or $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$.
If two other vectors $\mathbf{a}$ and $\mathbf{b}$ commute with each other and with $\mathbf{L}$, that is, $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{L}] = [\mathbf{b}, \mathbf{L}] = 0$, show that

$$[\mathbf{a} \cdot \mathbf{L}, \mathbf{b} \cdot \mathbf{L}] = i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{L}.$$

1.8.10 For $\mathbf{A} = \hat{\mathbf{x}} A_x(x, y, z)$ and $\mathbf{B} = \hat{\mathbf{x}} B_x(x, y, z)$ evaluate each term in the vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$$

and verify that the identity is satisfied.

1.8.11 Verify the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B}).$$

1.8.12 As an alternative to the vector identity of Example 1.8.3 show that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \times \nabla) \times \mathbf{B} + (\mathbf{B} \times \nabla) \times \mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) + \mathbf{B}(\nabla \cdot \mathbf{A}).$$

1.8.13 Verify the identity

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla\left(A^2\right) - (\mathbf{A} \cdot \nabla)\mathbf{A}.$$

1.8.14 If $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that

$$\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \mathbf{A} \times \mathbf{B}.$$

1.8.15 A distribution of electric currents creates a constant magnetic moment $\mathbf{m} = \text{const}$. The force on $\mathbf{m}$ in an external magnetic induction $\mathbf{B}$ is given by

$$\mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m}).$$

Show that

$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B}.$$

Note. Assuming no time dependence of the fields, Maxwell's equations yield $\nabla \times \mathbf{B} = 0$. Also, $\nabla \cdot \mathbf{B} = 0$.

1.8.16 An electric dipole of moment $\mathbf{p}$ is located at the origin. The dipole creates an electric potential at $\mathbf{r}$ given by

$$\psi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\varepsilon_0 r^3}.$$

Find the electric field, $\mathbf{E} = -\nabla\psi$ at $\mathbf{r}$.

1.8.17 The vector potential $\mathbf{A}$ of a magnetic dipole, dipole moment $\mathbf{m}$, is given by $\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)(\mathbf{m} \times \mathbf{r}/r^3)$. Show that the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}.$$

Note. The limiting process leading to point dipoles is discussed in Section 12.1 for electric dipoles, in Section 12.5 for magnetic dipoles.

1.8.18 The velocity of a two-dimensional flow of liquid is given by

$$\mathbf{V} = \hat{\mathbf{x}}u(x, y) - \hat{\mathbf{y}}v(x, y).$$

If the liquid is incompressible and the flow is irrotational, show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the Cauchy–Riemann conditions of Section 6.2.

1.8.19 The evaluation in this section of the four integrals for the circulation omitted Taylor series terms such as $\partial V_x/\partial x$, $\partial V_y/\partial y$ and all second derivatives. Show that $\partial V_x/\partial x$, $\partial V_y/\partial y$ cancel out when the four integrals are added and that the second derivative terms drop out in the limit as $dx \to 0$, $dy \to 0$.
Hint. Calculate the circulation per unit area and then take the limit $dx \to 0$, $dy \to 0$.

1.9 Successive Applications of ∇

We have now defined gradient, divergence, and curl to obtain vector, scalar, and vector quantities, respectively. Letting ∇ operate on each of these quantities, we obtain

(a) $\nabla \cdot \nabla\varphi$ (b) $\nabla \times \nabla\varphi$ (c) $\nabla\nabla \cdot \mathbf{V}$
(d) $\nabla \cdot \nabla \times \mathbf{V}$ (e) $\nabla \times (\nabla \times \mathbf{V})$

all five expressions involving second derivatives and all five appearing in the second-order differential equations of mathematical physics, particularly in electromagnetic theory.

The first expression, $\nabla \cdot \nabla\varphi$, the divergence of the gradient, is named the Laplacian of φ. We have

$$\nabla \cdot \nabla\varphi = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{x}}\frac{\partial\varphi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\varphi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}. \tag{1.81a}$$

When φ is the electrostatic potential, we have

$$\nabla \cdot \nabla\varphi = 0 \tag{1.81b}$$

at points where the charge density vanishes, which is Laplace's equation of electrostatics. Often the combination $\nabla \cdot \nabla$ is written ∇^2, or Δ in the European literature.

Example 1.9.1 LAPLACIAN OF A POTENTIAL

Calculate $\nabla \cdot \nabla V(r)$.

Referring to Examples 1.6.1 and 1.7.2,

$$\nabla \cdot \nabla V(r) = \nabla \cdot \hat{\mathbf{r}}\frac{dV}{dr} = \frac{2}{r}\frac{dV}{dr} + \frac{d^2V}{dr^2},$$

replacing $f(r)$ in Example 1.7.2 by $1/r \cdot dV/dr$. If $V(r) = r^n$, this reduces to

$$\nabla \cdot \nabla r^n = n(n+1)r^{n-2}.$$

This vanishes for $n = 0$ [$V(r) = \text{constant}$] and for $n = -1$; that is, $V(r) = 1/r$ is a solution of Laplace's equation, $\nabla^2 V(r) = 0$. This is for $r \neq 0$. At $r = 0$, a Dirac delta function is involved (see Eq. (1.169) and Section 9.7). ■

Expression (b) may be written

$$\nabla \times \nabla\varphi = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}.$$

By expanding the determinant, we obtain

$$\nabla \times \nabla\varphi = \hat{\mathbf{x}}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2\varphi}{\partial z\,\partial x} - \frac{\partial^2\varphi}{\partial x\,\partial z}\right)$$

$$+ \hat{\mathbf{z}}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right) = 0, \tag{1.82}$$

assuming that the order of partial differentiation may be interchanged. This is true as long as these second partial derivatives of φ are continuous functions. Then, from Eq. (1.82), the curl of a gradient is identically zero. All gradients, therefore, are irrotational. Note that the zero in Eq. (1.82) comes as a mathematical identity, independent of any physics. The zero in Eq. (1.81b) is a consequence of physics.

Expression (d) is a triple scalar product that may be written

$$\nabla \cdot \nabla \times \mathbf{V} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{1.83}$$

Again, assuming continuity so that the order of differentiation is immaterial, we obtain

$$\nabla \cdot \nabla \times \mathbf{V} = 0. \tag{1.84}$$

The divergence of a curl vanishes or all curls are solenoidal. In Section 1.16 we shall see that vectors may be resolved into solenoidal and irrotational parts by Helmholtz's theorem.

The two remaining expressions satisfy a relation

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V}, \tag{1.85}$$

valid in Cartesian coordinates (but not in curved coordinates). This follows immediately from Eq. (1.55), the *BAC–CAB* rule, which we rewrite so that **C** appears at the extreme right of each term. The term $\nabla \cdot \nabla\mathbf{V}$ was not included in our list, but it may be **defined** by Eq. (1.85).

Example 1.9.2 ELECTROMAGNETIC WAVE EQUATION

One important application of this vector relation (Eq. (1.85)) is in the derivation of the electromagnetic wave equation. In vacuum Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0, \tag{1.86a}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{1.86b}$$

$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.86c}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.86d}$$

Here **E** is the electric field, **B** is the magnetic induction, ε_0 is the electric permittivity, and μ_0 is the magnetic permeability (SI units), so $\varepsilon_0\mu_0 = 1/c^2$, c being the velocity of light. The relation has important consequences. Because ε_0, μ_0 can be measured in any frame, the velocity of light is the same in any frame.

Suppose we eliminate **B** from Eqs. (1.86c) and (1.86d). We may do this by taking the curl of both sides of Eq. (1.86d) and the time derivative of both sides of Eq. (1.86c). Since the space and time derivatives commute,

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \nabla \times \frac{\partial \mathbf{B}}{\partial t},$$

and we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\varepsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Application of Eqs. (1.85) and (1.86b) yields

$$\nabla \cdot \nabla \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{1.87}$$

the electromagnetic vector wave equation. Again, if $\mathbf{E}$ is expressed in Cartesian coordinates, Eq. (1.87) separates into three scalar wave equations, each involving the scalar Laplacian.

When external electric charge and current densities are kept as driving terms in Maxwell's equations, similar wave equations are valid for the electric potential and the vector potential. To show this, we solve Eq. (1.86a) by writing $\mathbf{B} = \nabla \times \mathbf{A}$ as a curl of the vector potential. This expression is substituted into Faraday's induction law in differential form, Eq. (1.86d), to yield $\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0$. The vanishing curl implies that $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ is a gradient and, therefore, can be written as $-\nabla \varphi$, where $\varphi(\mathbf{r}, t)$ is defined as the (nonstatic) electric potential. These results for the $\mathbf{B}$ and $\mathbf{E}$ fields,

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t},} \tag{1.88}$$

solve the homogeneous Maxwell's equations.

We now show that the inhomogeneous Maxwell's equations,

$$\text{Gauss' law: } \boxed{\nabla \cdot \mathbf{E} = \rho/\varepsilon_0,} \qquad \text{Oersted's law: } \boxed{\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}} \tag{1.89}$$

in differential form lead to wave equations for the potentials φ and $\mathbf{A}$, provided that $\nabla \cdot \mathbf{A}$ is determined by the constraint $\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0$. This choice of fixing the divergence of the vector potential, called the **Lorentz gauge**, serves to uncouple the differential equations of both potentials. This gauge constraint is not a restriction; it has no physical effect.

Substituting our electric field solution into Gauss' law yields

$$\boxed{\frac{\rho}{\varepsilon_0} = \nabla \cdot \mathbf{E} = -\nabla^2 \varphi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\nabla^2 \varphi + \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2},} \tag{1.90}$$

the wave equation for the electric potential. In the last step we have used the Lorentz gauge to replace the divergence of the vector potential by the time derivative of the electric potential and thus decouple φ from $\mathbf{A}$.

Finally, we substitute $\mathbf{B} = \nabla \times \mathbf{A}$ into Oersted's law and use Eq. (1.85), which expands ∇^2 in terms of a longitudinal (the gradient term) and a transverse component (the curl term). This yields

$$\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \frac{1}{c^2}\left(\nabla \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right),$$

where we have used the electric field solution (Eq. (1.88)) in the last step. Now we see that the Lorentz gauge condition eliminates the gradient terms, so the wave equation

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}} \tag{1.91}$$

for the vector potential remains.

Finally, looking back at Oersted's law, taking the divergence of Eq. (1.89), dropping $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, and substituting Gauss' law for $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, we find $\mu_0 \nabla \cdot \mathbf{J} = -\frac{1}{\epsilon_0 c^2}\frac{\partial \rho}{\partial t}$, where $\epsilon_0 \mu_0 = 1/c^2$, that is, the continuity equation for the current density. This step justifies the inclusion of Maxwell's displacement current in the generalization of Oersted's law to nonstationary situations. ■

Exercises

1.9.1 Verify Eq. (1.85),

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V},$$

by direct expansion in Cartesian coordinates.

1.9.2 Show that the identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V}$$

follows from the *BAC–CAB* rule for a triple vector product. Justify any alteration of the order of factors in the *BAC* and *CAB* terms.

1.9.3 Prove that $\nabla \times (\varphi \nabla \varphi) = 0$.

1.9.4 You are given that the curl of **F** equals the curl of **G**. Show that **F** and **G** may differ by (a) a constant and (b) a gradient of a scalar function.

1.9.5 The Navier–Stokes equation of hydrodynamics contains a nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$. Show that the curl of this term may be written as $-\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$.

1.9.6 From the Navier–Stokes equation for the steady flow of an incompressible viscous fluid we have the term

$$\nabla \times \left[\mathbf{v} \times (\nabla \times \mathbf{v})\right],$$

where **v** is the fluid velocity. Show that this term vanishes for the special case

$$\mathbf{v} = \hat{\mathbf{x}} v(y, z).$$

1.9.7 Prove that $(\nabla u) \times (\nabla v)$ is solenoidal, where u and v are differentiable scalar functions.

1.9.8 φ is a scalar satisfying Laplace's equation, $\nabla^2 \varphi = 0$. Show that $\nabla \varphi$ is **both** solenoidal and irrotational.

1.9.9 With ψ a scalar (wave) function, show that

$$(\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla)\psi = r^2 \nabla^2 \psi - r^2 \frac{\partial^2 \psi}{\partial r^2} - 2r \frac{\partial \psi}{\partial r}.$$

(This can actually be shown more easily in spherical polar coordinates, Section 2.5.)

1.9.10 In a (nonrotating) isolated mass such as a star, the condition for equilibrium is

$$\nabla P + \rho \nabla \varphi = 0.$$

Here P is the total pressure, ρ is the density, and φ is the gravitational potential. Show that at any given point the normals to the surfaces of constant pressure and constant gravitational potential are parallel.

1.9.11 In the Pauli theory of the electron, one encounters the expression

$$(\mathbf{p} - e\mathbf{A}) \times (\mathbf{p} - e\mathbf{A})\psi,$$

where ψ is a scalar (wave) function. $\mathbf{A}$ is the magnetic vector potential related to the magnetic induction $\mathbf{B}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. Given that $\mathbf{p} = -i\nabla$, show that this expression reduces to $ie\mathbf{B}\psi$. Show that this leads to the orbital g-factor $g_L = 1$ upon writing the magnetic moment as $\boldsymbol{\mu} = g_L \mathbf{L}$ in units of Bohr magnetons and $\mathbf{L} = -i\mathbf{r} \times \nabla$. See also Exercise 1.13.7.

1.9.12 Show that any solution of the equation

$$\nabla \times (\nabla \times \mathbf{A}) - k^2 \mathbf{A} = 0$$

automatically satisfies the vector Helmholtz equation

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = 0$$

and the solenoidal condition

$$\nabla \cdot \mathbf{A} = 0.$$

Hint. Let $\nabla \cdot$ operate on the first equation.

1.9.13 The theory of heat conduction leads to an equation

$$\nabla^2 \Psi = k|\nabla \Phi|^2,$$

where Φ is a potential satisfying Laplace's equation: $\nabla^2 \Phi = 0$. Show that a solution of this equation is

$$\Psi = \frac{1}{2} k \Phi^2.$$

1.10 Vector Integration

The next step after differentiating vectors is to integrate them. Let us start with line integrals and then proceed to surface and volume integrals. In each case the method of attack will be to reduce the vector integral to scalar integrals with which the reader is assumed familiar.

Line Integrals

Using an increment of length $d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$, we may encounter the line integrals

$$\int_C \varphi\, d\mathbf{r}, \tag{1.92a}$$

$$\int_C \mathbf{V}\cdot d\mathbf{r}, \tag{1.92b}$$

$$\int_C \mathbf{V}\times d\mathbf{r}, \tag{1.92c}$$

in each of which the integral is over some contour C that may be open (with starting point and ending point separated) or closed (forming a loop). Because of its physical interpretation that follows, the second form, Eq. (1.92b) is by far the most important of the three.

With φ, a scalar, the first integral reduces immediately to

$$\int_C \varphi\, d\mathbf{r} = \hat{\mathbf{x}}\int_C \varphi(x,y,z)\,dx + \hat{\mathbf{y}}\int_C \varphi(x,y,z)\,dy + \hat{\mathbf{z}}\int_C \varphi(x,y,z)\,dz. \tag{1.93}$$

This separation has employed the relation

$$\int \hat{\mathbf{x}}\varphi\, dx = \hat{\mathbf{x}}\int \varphi\, dx, \tag{1.94}$$

which is permissible because the Cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are constant in both magnitude and direction. Perhaps this relation is obvious here, but it will not be true in the non-Cartesian systems encountered in Chapter 2.

The three integrals on the right side of Eq. (1.93) are ordinary scalar integrals and, to avoid complications, we assume that they are Riemann integrals. Note, however, that the integral with respect to x cannot be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unless the integrand has special properties so that the integral depends only on the value of the end points, the value will depend on the particular choice of contour C. For instance, if we choose the very special case $\varphi = 1$, Eq. (1.92a) is just the vector distance from the start of contour C to the endpoint, in this case independent of the choice of path connecting fixed endpoints. With $d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$, the second and third forms also reduce to scalar integrals and, like Eq. (1.92a), are dependent, in general, on the choice of path. The form (Eq. (1.92b)) is exactly the same as that encountered when we calculate the work done by a force that varies along the path,

$$W = \int \mathbf{F}\cdot d\mathbf{r} = \int F_x(x,y,z)\,dx + \int F_y(x,y,z)\,dy + \int F_z(x,y,z)\,dz. \tag{1.95a}$$

In this expression $\mathbf{F}$ is the force exerted on a particle.

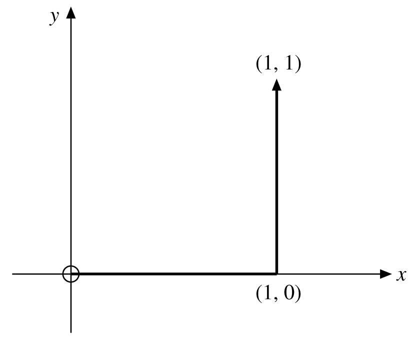

FIGURE 1.25 A path of integration.

Example 1.10.1 PATH-DEPENDENT WORK

The force exerted on a body is $\mathbf{F} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$. The problem is to calculate the work done going from the origin to the point $(1, 1)$:

$$W = \int_{0,0}^{1,1} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{1,1} (-y\,dx + x\,dy). \tag{1.95b}$$

Separating the two integrals, we obtain

$$W = -\int_0^1 y\,dx + \int_0^1 x\,dy. \tag{1.95c}$$

The first integral cannot be evaluated until we specify the values of y as x ranges from 0 to 1. Likewise, the second integral requires x as a function of y. Consider first the path shown in Fig. 1.25. Then

$$W = -\int_0^1 0\,dx + \int_0^1 1\,dy = 1, \tag{1.95d}$$

since $y = 0$ along the first segment of the path and $x = 1$ along the second. If we select the path $[x = 0, 0 \leqslant y \leqslant 1]$ and $[0 \leqslant x \leqslant 1, y = 1]$, then Eq. (1.95c) gives $W = -1$. For this force the work done depends on the choice of path. ■

Surface Integrals

Surface integrals appear in the same forms as line integrals, the element of area also being a vector, $d\boldsymbol{\sigma}$.[20] Often this area element is written $\mathbf{n}\,dA$, in which $\mathbf{n}$ is a unit (normal) vector to indicate the positive direction.[21] There are two conventions for choosing the positive direction. First, if the surface is a closed surface, we agree to take the outward normal as positive. Second, if the surface is an open surface, the positive normal depends on the direction in which the perimeter of the open surface is traversed. If the right-hand fingers

[20] Recall that in Section 1.4 the area (of a parallelogram) is represented by a cross-product **vector**.
[21] Although $\mathbf{n}$ always has unit length, its direction may well be a function of position.

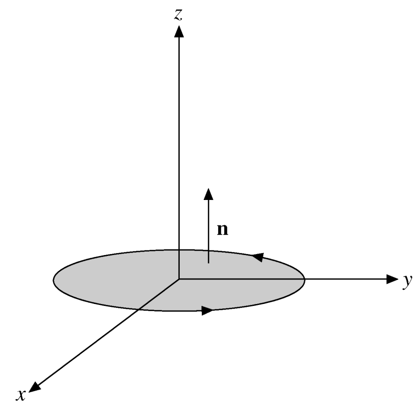

FIGURE 1.26 Right-hand rule for the positive normal.

are placed in the direction of travel around the perimeter, the positive normal is indicated by the thumb of the right hand. As an illustration, a circle in the xy-plane (Fig. 1.26) mapped out from x to y to $-x$ to $-y$ and back to x will have its positive normal parallel to the positive z-axis (for the right-handed coordinate system).

Analogous to the line integrals, Eqs. (1.92a) to (1.92c), surface integrals may appear in the forms

$$\int \varphi\, d\boldsymbol{\sigma}, \qquad \int \mathbf{V}\cdot d\boldsymbol{\sigma}, \qquad \int \mathbf{V}\times d\boldsymbol{\sigma}.$$

Again, the dot product is by far the most commonly encountered form. The surface integral $\int \mathbf{V}\cdot d\boldsymbol{\sigma}$ may be interpreted as a flow or flux through the given surface. This is really what we did in Section 1.7 to obtain the significance of the term divergence. This identification reappears in Section 1.11 as Gauss' theorem. Note that both physically and from the dot product the tangential components of the velocity contribute nothing to the flow through the surface.

Volume Integrals

Volume integrals are somewhat simpler, for the volume element $d\tau$ is a scalar quantity.[22] We have

$$\int_V \mathbf{V}\, d\tau = \hat{\mathbf{x}}\int_V V_x\, d\tau + \hat{\mathbf{y}}\int_V V_y\, d\tau + \hat{\mathbf{z}}\int_V V_z\, d\tau, \tag{1.96}$$

again reducing the vector integral to a vector sum of scalar integrals.

[22] Frequently the symbols d^3r and d^3x are used to denote a volume element in coordinate (xyz or $x_1x_2x_3$) space.

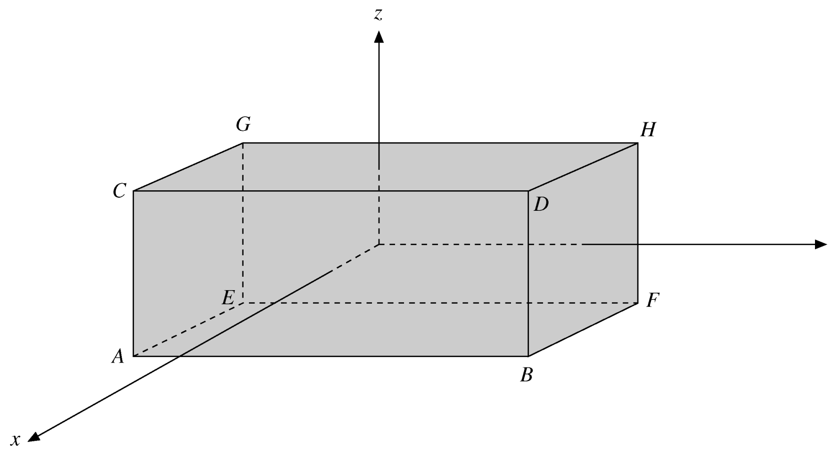

FIGURE 1.27 Differential rectangular parallelepiped (origin at center).

Integral Definitions of Gradient, Divergence, and Curl

One interesting and significant application of our surface and volume integrals is their use in developing alternate definitions of our differential relations. We find

$$\nabla\varphi = \lim_{\int d\tau\to 0} \frac{\int \varphi\, d\boldsymbol{\sigma}}{\int d\tau}, \tag{1.97}$$

$$\nabla\cdot\mathbf{V} = \lim_{\int d\tau\to 0} \frac{\int \mathbf{V}\cdot d\boldsymbol{\sigma}}{\int d\tau}, \tag{1.98}$$

$$\nabla\times\mathbf{V} = \lim_{\int d\tau\to 0} \frac{\int d\boldsymbol{\sigma}\times\mathbf{V}}{\int d\tau}. \tag{1.99}$$

In these three equations $\int d\tau$ is the volume of a small region of space and $d\boldsymbol{\sigma}$ is the vector area element of this volume. The identification of Eq. (1.98) as the divergence of $\mathbf{V}$ was carried out in Section 1.7. Here we show that Eq. (1.97) is consistent with our earlier definition of $\nabla\varphi$ (Eq. (1.60)). For simplicity we choose $d\tau$ to be the differential volume $dx\,dy\,dz$ (Fig. 1.27). This time we place the origin at the geometric center of our volume element. The area integral leads to six integrals, one for each of the six faces. Remembering that $d\boldsymbol{\sigma}$ is outward, $d\boldsymbol{\sigma}\cdot\hat{\mathbf{x}} = -|d\boldsymbol{\sigma}|$ for surface $EFHG$, and $+|d\boldsymbol{\sigma}|$ for surface $ABDC$, we have

$$\begin{aligned}\int \varphi\, d\boldsymbol{\sigma} = &-\hat{\mathbf{x}}\int_{EFHG}\left(\varphi - \frac{\partial\varphi}{\partial x}\frac{dx}{2}\right)dy\,dz + \hat{\mathbf{x}}\int_{ABDC}\left(\varphi + \frac{\partial\varphi}{\partial x}\frac{dx}{2}\right)dy\,dz\\ &-\hat{\mathbf{y}}\int_{AEGC}\left(\varphi - \frac{\partial\varphi}{\partial y}\frac{dy}{2}\right)dx\,dz + \hat{\mathbf{y}}\int_{BFHD}\left(\varphi + \frac{\partial\varphi}{\partial y}\frac{dy}{2}\right)dx\,dz\\ &-\hat{\mathbf{z}}\int_{ABFE}\left(\varphi - \frac{\partial\varphi}{\partial z}\frac{dz}{2}\right)dx\,dy + \hat{\mathbf{z}}\int_{CDHG}\left(\varphi + \frac{\partial\varphi}{\partial z}\frac{dz}{2}\right)dx\,dy.\end{aligned}$$

Using the total variations, we evaluate each integrand at the origin with a correction included to correct for the displacement ($\pm dx/2$, etc.) of the center of the face from the origin. Having chosen the total volume to be of differential size ($\int d\tau = dx\,dy\,dz$), we drop the integral signs on the right and obtain

$$\int \varphi\, d\boldsymbol{\sigma} = \left(\hat{\mathbf{x}}\frac{\partial \varphi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \varphi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \varphi}{\partial z}\right) dx\,dy\,dz. \tag{1.100}$$

Dividing by

$$\int d\tau = dx\,dy\,dz,$$

we verify Eq. (1.97).

This verification has been oversimplified in ignoring other correction terms beyond the first derivatives. These additional terms, which are introduced in Section 5.6 when the Taylor expansion is developed, vanish in the limit

$$\int d\tau \to 0\ (dx \to 0, dy \to 0, dz \to 0).$$

This, of course, is the reason for specifying in Eqs. (1.97), (1.98), and (1.99) that this limit be taken. Verification of Eq. (1.99) follows these same lines exactly, using a differential volume $dx\,dy\,dz$.

Exercises

1.10.1 The force field acting on a two-dimensional linear oscillator may be described by

$$\mathbf{F} = -\hat{\mathbf{x}}kx - \hat{\mathbf{y}}ky.$$

Compare the work done moving against this force field when going from (1, 1) to (4, 4) by the following straight-line paths:

(a) $(1, 1) \to (4, 1) \to (4, 4)$
(b) $(1, 1) \to (1, 4) \to (4, 4)$
(c) $(1, 1) \to (4, 4)$ along $x = y$.

This means evaluating

$$-\int_{(1,1)}^{(4,4)} \mathbf{F} \cdot d\mathbf{r}$$

along each path.

1.10.2 Find the work done going around a unit circle in the xy-plane:

(a) counterclockwise from 0 to π,
(b) clockwise from 0 to $-\pi$, doing work **against** a force field given by

$$\mathbf{F} = \frac{-\hat{\mathbf{x}}y}{x^2 + y^2} + \frac{\hat{\mathbf{y}}x}{x^2 + y^2}.$$

Note that the work done depends on the path.

1.10.3 Calculate the work you do in going from point (1, 1) to point (3, 3). The force **you exert** is given by

$$\mathbf{F} = \hat{\mathbf{x}}(x - y) + \hat{\mathbf{y}}(x + y).$$

Specify clearly the path you choose. Note that this force field is nonconservative.

1.10.4 Evaluate $\oint \mathbf{r} \cdot d\mathbf{r}$.
Note. The symbol $\oint$ means that the path of integration is a closed loop.

1.10.5 Evaluate

$$\frac{1}{3}\int_s \mathbf{r} \cdot d\boldsymbol{\sigma}$$

over the unit cube defined by the point (0, 0, 0) and the unit intercepts on the positive x-, y-, and z-axes. Note that (a) $\mathbf{r} \cdot d\boldsymbol{\sigma}$ is zero for three of the surfaces and (b) each of the three remaining surfaces contributes the same amount to the integral.

1.10.6 Show, by expansion of the surface integral, that

$$\lim_{\int d\tau \to 0} \frac{\int_s d\boldsymbol{\sigma} \times \mathbf{V}}{\int d\tau} = \nabla \times \mathbf{V}.$$

Hint. Choose the volume $\int d\tau$ to be a differential volume $dx\,dy\,dz$.

1.11 GAUSS' THEOREM

Here we derive a useful relation between a surface integral of a vector and the volume integral of the divergence of that vector. Let us assume that the vector $\mathbf{V}$ and its first derivatives are continuous over the simply connected region (that does not have any holes, such as a donut) of interest. Then Gauss' theorem states that

$$\boxed{\oiint_{\partial V} \mathbf{V} \cdot d\boldsymbol{\sigma} = \iiint_V \nabla \cdot \mathbf{V}\, d\tau.} \tag{1.101a}$$

In words, the surface integral of a vector over a closed surface equals the volume integral of the divergence of that vector integrated over the volume enclosed by the surface.

Imagine that volume V is subdivided into an arbitrarily large number of tiny (differential) parallelepipeds. For each parallelepiped

$$\sum_{\text{six surfaces}} \mathbf{V} \cdot d\boldsymbol{\sigma} = \nabla \cdot \mathbf{V}\, d\tau \tag{1.101b}$$

from the analysis of Section 1.7, Eq. (1.66), with $\rho\mathbf{v}$ replaced by $\mathbf{V}$. The summation is over the six faces of the parallelepiped. Summing over all parallelepipeds, we find that the $\mathbf{V} \cdot d\boldsymbol{\sigma}$ terms cancel (pairwise) for all **interior** faces; only the contributions of the **exterior** surfaces survive (Fig. 1.28). Analogous to the definition of a Riemann integral as the limit

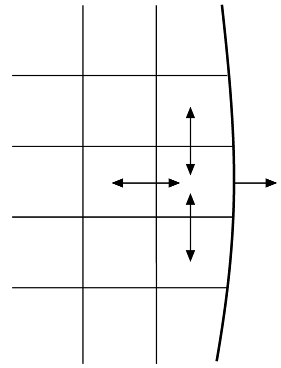

FIGURE 1.28 Exact cancellation of $d\boldsymbol{\sigma}$'s on **interior** surfaces. No cancellation on the **exterior** surface.

of a sum, we take the limit as the number of parallelepipeds approaches infinity ($\to \infty$) and the dimensions of each approach zero ($\to 0$):

$$\begin{array}{ccc} \displaystyle\sum_{\text{exterior surfaces}} \mathbf{V} \cdot d\boldsymbol{\sigma} & = & \displaystyle\sum_{\text{volumes}} \nabla \cdot \mathbf{V}\, d\tau \\ \downarrow & & \downarrow \\ \int_S \mathbf{V} \cdot d\boldsymbol{\sigma} & = & \int_V \nabla \cdot \mathbf{V}\, d\tau. \end{array}$$

The result is Eq. (1.101a), Gauss' theorem.

From a physical point of view Eq. (1.66) has established $\nabla \cdot \mathbf{V}$ as the net outflow of fluid per unit volume. The volume integral then gives the total net outflow. But the surface integral $\int \mathbf{V} \cdot d\boldsymbol{\sigma}$ is just another way of expressing this same quantity, which is the equality, Gauss' theorem.

Green's Theorem

A frequently useful corollary of Gauss' theorem is a relation known as Green's theorem. If u and v are two scalar functions, we have the identities

$$\nabla \cdot (u\,\nabla v) = u\nabla \cdot \nabla v + (\nabla u) \cdot (\nabla v), \tag{1.102}$$

$$\nabla \cdot (v\,\nabla u) = v\nabla \cdot \nabla u + (\nabla v) \cdot (\nabla u). \tag{1.103}$$

Subtracting Eq. (1.103) from Eq. (1.102), integrating over a volume (u, v, and their derivatives, assumed continuous), and applying Eq. (1.101a) (Gauss' theorem), we obtain

$$\boxed{\iiint_V (u\nabla \cdot \nabla v - v\nabla \cdot \nabla u)\, d\tau = \oiint_{\partial V} (u\nabla v - v\nabla u) \cdot d\boldsymbol{\sigma}.} \tag{1.104}$$

This is Green's theorem. We use it for developing Green's functions in Chapter 9. An alternate form of Green's theorem, derived from Eq. (1.102) alone, is

$$\oint\!\!\!\oint_{\partial V} u\nabla v \cdot d\boldsymbol{\sigma} = \iiint_V u\nabla \cdot \nabla v\, d\tau + \iiint_V \nabla u \cdot \nabla v\, d\tau. \tag{1.105}$$

This is the form of Green's theorem used in Section 1.16.

Alternate Forms of Gauss' Theorem

Although Eq. (1.101a) involving the divergence is by far the most important form of Gauss' theorem, volume integrals involving the gradient and the curl may also appear. Suppose

$$\mathbf{V}(x, y, z) = V(x, y, z)\mathbf{a}, \tag{1.106}$$

in which **a** is a vector with constant magnitude and constant but arbitrary direction. (You pick the direction, but once you have chosen it, hold it fixed.) Equation (1.101a) becomes

$$\mathbf{a} \cdot \oint\!\!\!\oint_{\partial V} V\, d\boldsymbol{\sigma} = \iiint_V \nabla \cdot \mathbf{a} V\, d\tau = \mathbf{a} \cdot \iiint_V \nabla V\, d\tau \tag{1.107}$$

by Eq. (1.67b). This may be rewritten

$$\mathbf{a} \cdot \left[\oint\!\!\!\oint_{\partial V} V\, d\boldsymbol{\sigma} - \iiint_V \nabla V\, d\tau\right] = 0. \tag{1.108}$$

Since $|\mathbf{a}| \neq 0$ and its direction is arbitrary, meaning that the cosine of the included angle cannot **always** vanish, the terms in brackets must be zero.[23] The result is

$$\oint\!\!\!\oint_{\partial V} V\, d\boldsymbol{\sigma} = \iiint_V \nabla V\, d\tau. \tag{1.109}$$

In a similar manner, using $\mathbf{V} = \mathbf{a} \times \mathbf{P}$ in which **a** is a constant vector, we may show

$$\oint\!\!\!\oint_{\partial V} d\boldsymbol{\sigma} \times \mathbf{P} = \iiint_V \nabla \times \mathbf{P}\, d\tau. \tag{1.110}$$

These last two forms of Gauss' theorem are used in the vector form of Kirchoff diffraction theory. They may also be used to verify Eqs. (1.97) and (1.99). Gauss' theorem may also be extended to tensors (see Section 2.11).

Exercises

1.11.1 Using Gauss' theorem, prove that

$$\oint\!\!\!\oint_S d\boldsymbol{\sigma} = 0$$

if $S = \partial V$ is a closed surface.

[23] This exploitation of the **arbitrary** nature of a part of a problem is a valuable and widely used technique. The arbitrary vector is used again in Sections 1.12 and 1.13. Other examples appear in Section 1.14 (integrands equated) and in Section 2.8, quotient rule.

1.11.2 Show that

$$\frac{1}{3}\oiint_S \mathbf{r}\cdot d\boldsymbol{\sigma} = V,$$

where V is the volume enclosed by the closed surface $S = \partial V$.
Note. This is a generalization of Exercise 1.10.5.

1.11.3 If $\mathbf{B} = \nabla \times \mathbf{A}$, show that

$$\oiint_S \mathbf{B}\cdot d\boldsymbol{\sigma} = \mathbf{0}$$

for any closed surface S.

1.11.4 Over some volume V let ψ be a solution of Laplace's equation (with the derivatives appearing there continuous). Prove that the integral over any closed surface in V of the normal derivative of ψ ($\partial\psi/\partial n$, or $\nabla\psi\cdot\mathbf{n}$) will be zero.

1.11.5 In analogy to the integral definition of gradient, divergence, and curl of Section 1.10, show that

$$\nabla^2\varphi = \lim_{\int d\tau\to 0}\frac{\int \nabla\varphi\cdot d\boldsymbol{\sigma}}{\int d\tau}.$$

1.11.6 The electric displacement vector $\mathbf{D}$ satisfies the Maxwell equation $\nabla\cdot\mathbf{D} = \rho$, where ρ is the charge density (per unit volume). At the boundary between two media there is a surface charge density σ (per unit area). Show that a boundary condition for $\mathbf{D}$ is

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = \sigma.$$

$\mathbf{n}$ is a unit vector normal to the surface and out of medium 1.
Hint. Consider a thin pillbox as shown in Fig. 1.29.

1.11.7 From Eq. (1.67b), with $\mathbf{V}$ the electric field $\mathbf{E}$ and f the electrostatic potential φ, show that, for integration over all space,

$$\int \rho\varphi\, d\tau = \varepsilon_0 \int E^2\, d\tau.$$

This corresponds to a three-dimensional integration by parts.
Hint. $\mathbf{E} = -\nabla\varphi$, $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$. You may assume that φ vanishes at large r at least as fast as r^{-1}.

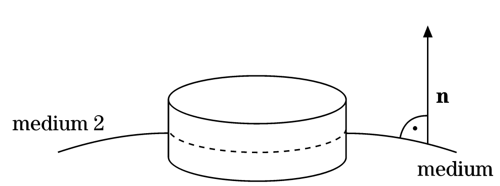

FIGURE 1.29 Pillbox.

1.11.8 A particular steady-state electric current distribution is localized in space. Choosing a bounding surface far enough out so that the current density $\mathbf{J}$ is zero everywhere on the surface, show that

$$\iiint \mathbf{J}\, d\tau = 0.$$

Hint. Take one component of $\mathbf{J}$ at a time. With $\nabla \cdot \mathbf{J} = 0$, show that $\mathbf{J}_i = \nabla \cdot (x_i \mathbf{J})$ and apply Gauss' theorem.

1.11.9 The creation of a **localized** system of steady electric currents (current density $\mathbf{J}$) and magnetic fields may be shown to require an amount of work

$$W = \frac{1}{2} \iiint \mathbf{H} \cdot \mathbf{B}\, d\tau.$$

Transform this into

$$W = \frac{1}{2} \iiint \mathbf{J} \cdot \mathbf{A}\, d\tau.$$

Here $\mathbf{A}$ is the magnetic vector potential: $\nabla \times \mathbf{A} = \mathbf{B}$.
Hint. In Maxwell's equations take the displacement current term $\partial \mathbf{D}/\partial t = 0$. If the fields and currents are localized, a bounding surface may be taken far enough out so that the integrals of the fields and currents over the surface yield zero.

1.11.10 Prove the generalization of Green's theorem:

$$\iiint_V (v\mathcal{L}u - u\mathcal{L}v)\, d\tau = \oint\!\!\oint_{\partial V} p(v\nabla u - u\nabla v) \cdot d\boldsymbol{\sigma}.$$

Here $\mathcal{L}$ is the self-adjoint operator (Section 10.1),

$$\mathcal{L} = \nabla \cdot \left[p(\mathbf{r})\nabla \right] + q(\mathbf{r})$$

and p, q, u, and v are functions of position, p and q having continuous first derivatives and u and v having continuous second derivatives.
Note. This generalized Green's theorem appears in Section 9.7.

1.12 STOKES' THEOREM

Gauss' theorem relates the volume integral of a derivative of a function to an integral of the function over the closed surface bounding the volume. Here we consider an analogous relation between the surface integral of a derivative of a function and the line integral of the function, the path of integration being the perimeter bounding the surface.

Let us take the surface and subdivide it into a network of arbitrarily small rectangles. In Section 1.8 we showed that the circulation about such a differential rectangle (in the xy-plane) is $\nabla \times \mathbf{V}|_z\, dx\, dy$. From Eq. (1.76) applied to **one** differential rectangle,

$$\sum_{\text{four sides}} \mathbf{V} \cdot d\boldsymbol{\lambda} = \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma}. \tag{1.111}$$

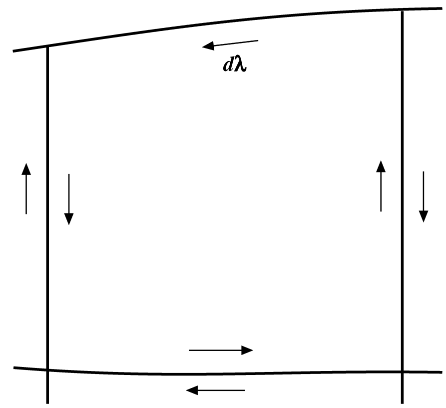

FIGURE 1.30 Exact cancellation on **interior** paths. No cancellation on the **exterior** path.

We sum over all the little rectangles, as in the definition of a Riemann integral. The surface contributions (right-hand side of Eq. (1.111)) are added together. The line integrals (left-hand side of Eq. (1.111)) of all **interior** line segments cancel identically. Only the line integral around the perimeter survives (Fig. 1.30). Taking the usual limit as the number of rectangles approaches infinity while $dx \to 0$, $dy \to 0$, we have

$$\begin{array}{ccc} \displaystyle\sum_{\substack{\text{exterior line}\\ \text{segments}}} \mathbf{V}\cdot d\boldsymbol{\lambda} & = & \displaystyle\sum_{\text{rectangles}} \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma} \\ \downarrow & & \downarrow \\ \multicolumn{3}{c}{\boxed{\oint \mathbf{V}\cdot d\boldsymbol{\lambda} = \int_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}.}} \end{array} \tag{1.112}$$

This is Stokes' theorem. The surface integral on the right is over the surface bounded by the perimeter or contour, for the line integral on the left. The direction of the vector representing the area is out of the paper plane toward the reader if the direction of traversal around the contour for the line integral is in the positive mathematical sense, as shown in Fig. 1.30.

This demonstration of Stokes' theorem is limited by the fact that we used a Maclaurin expansion of $\mathbf{V}(x, y, z)$ in establishing Eq. (1.76) in Section 1.8. Actually we need only demand that the curl of $\mathbf{V}(x, y, z)$ exist and that it be integrable over the surface. A proof of the Cauchy integral theorem analogous to the development of Stokes' theorem here but using these less restrictive conditions appears in Section 6.3.

Stokes' theorem obviously applies to an open surface. It is possible to consider a closed surface as a limiting case of an open surface, with the opening (and therefore the perimeter) shrinking to zero. This is the point of Exercise 1.12.7.

Alternate Forms of Stokes' Theorem

As with Gauss' theorem, other relations between surface and line integrals are possible. We find

$$\int_S d\boldsymbol{\sigma} \times \nabla\varphi = \oint_{\partial S} \varphi\, d\boldsymbol{\lambda} \tag{1.113}$$

and

$$\int_S (d\boldsymbol{\sigma} \times \nabla) \times \mathbf{P} = \oint_{\partial S} d\boldsymbol{\lambda} \times \mathbf{P}. \tag{1.114}$$

Equation (1.113) may readily be verified by the substitution $\mathbf{V} = \mathbf{a}\varphi$, in which $\mathbf{a}$ is a vector of constant magnitude and of constant direction, as in Section 1.11. Substituting into Stokes' theorem, Eq. (1.112),

$$\begin{aligned}\int_S (\nabla \times \mathbf{a}\varphi) \cdot d\boldsymbol{\sigma} &= -\int_S \mathbf{a} \times \nabla\varphi \cdot d\boldsymbol{\sigma} \\ &= -\mathbf{a} \cdot \int_S \nabla\varphi \times d\boldsymbol{\sigma}.\end{aligned} \tag{1.115}$$

For the line integral,

$$\oint_{\partial S} \mathbf{a}\varphi \cdot d\boldsymbol{\lambda} = \mathbf{a} \cdot \oint_{\partial S} \varphi\, d\boldsymbol{\lambda}, \tag{1.116}$$

and we obtain

$$\mathbf{a} \cdot \left(\oint_{\partial S} \varphi\, d\boldsymbol{\lambda} + \int_S \nabla\varphi \times d\boldsymbol{\sigma} \right) = 0. \tag{1.117}$$

Since the choice of direction of $\mathbf{a}$ is arbitrary, the expression in parentheses must vanish, thus verifying Eq. (1.113). Equation (1.114) may be derived similarly by using $\mathbf{V} = \mathbf{a} \times \mathbf{P}$, in which $\mathbf{a}$ is again a constant vector.

We can use Stokes' theorem to derive Oersted's and Faraday's laws from two of Maxwell's equations, and vice versa, thus recognizing that the former are an integrated form of the latter.

Example 1.12.1 Oersted's and Faraday's Laws

Consider the magnetic field generated by a long wire that carries a stationary current I. Starting from Maxwell's differential law $\nabla \times \mathbf{H} = \mathbf{J}$, Eq. (1.89) (with Maxwell's displacement current $\partial\mathbf{D}/\partial t = 0$ for a stationary current case by Ohm's law), we integrate over a closed area S perpendicular to and surrounding the wire and apply Stokes' theorem to get

$$I = \int_S \mathbf{J} \cdot d\boldsymbol{\sigma} = \int_S (\nabla \times \mathbf{H}) \cdot d\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{H} \cdot d\mathbf{r},$$

which is Oersted's law. Here the line integral is along ∂S, the closed curve surrounding the cross-sectional area S.

Similarly, we can integrate Maxwell's equation for $\nabla \times \mathbf{E}$, Eq. (1.86d), to yield Faraday's induction law. Imagine moving a closed loop (∂S) of wire (of area S) across a magnetic induction field $\mathbf{B}$. We integrate Maxwell's equation and use Stokes' theorem, yielding

$$\int_{\partial S} \mathbf{E} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{E}) \cdot d\boldsymbol{\sigma} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\boldsymbol{\sigma} = -\frac{d\Phi}{dt},$$

which is Faraday's law. The line integral on the left-hand side represents the voltage induced in the wire loop, while the right-hand side is the change with time of the magnetic flux Φ through the moving surface S of the wire. ■

Both Stokes' and Gauss' theorems are of tremendous importance in a wide variety of problems involving vector calculus. Some idea of their power and versatility may be obtained from the exercises of Sections 1.11 and 1.12 and the development of potential theory in Sections 1.13 and 1.14.

Exercises

1.12.1 Given a vector $\mathbf{t} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$, show, with the help of Stokes' theorem, that the integral around a continuous closed curve in the xy-plane

$$\frac{1}{2}\oint \mathbf{t} \cdot d\boldsymbol{\lambda} = \frac{1}{2}\oint (x\,dy - y\,dx) = A,$$

the area enclosed by the curve.

1.12.2 The calculation of the magnetic moment of a current loop leads to the line integral

$$\oint \mathbf{r} \times d\mathbf{r}.$$

(a) Integrate around the perimeter of a current loop (in the xy-plane) and show that the scalar magnitude of this line integral is twice the area of the enclosed surface.
(b) The perimeter of an ellipse is described by $\mathbf{r} = \hat{\mathbf{x}}a\cos\theta + \hat{\mathbf{y}}b\sin\theta$. From part (a) show that the area of the ellipse is πab.

1.12.3 Evaluate $\oint \mathbf{r} \times d\mathbf{r}$ by using the alternate form of Stokes' theorem given by Eq. (1.114):

$$\int_S (d\boldsymbol{\sigma} \times \nabla) \times \mathbf{P} = \oint d\boldsymbol{\lambda} \times \mathbf{P}.$$

Take the loop to be entirely in the xy-plane.

1.12.4 In steady state the magnetic field $\mathbf{H}$ satisfies the Maxwell equation $\nabla \times \mathbf{H} = \mathbf{J}$, where $\mathbf{J}$ is the current density (per square meter). At the boundary between two media there is a surface current density $\mathbf{K}$. Show that a boundary condition on $\mathbf{H}$ is

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}.$$

$\mathbf{n}$ is a unit vector normal to the surface and out of medium 1.
Hint. Consider a narrow loop perpendicular to the interface as shown in Fig. 1.31.

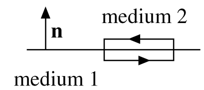

FIGURE 1.31
Integration path at the boundary of two media.

1.12.5 From Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$, with $\mathbf{J}$ here the current density and $\mathbf{E} = 0$. Show from this that

$$\oint \mathbf{H} \cdot d\mathbf{r} = I,$$

where I is the net electric current enclosed by the loop integral. These are the differential and integral forms of Ampère's law of magnetism.

1.12.6 A magnetic induction $\mathbf{B}$ is generated by electric current in a ring of radius R. Show that the **magnitude** of the vector potential $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$) at the ring can be

$$|\mathbf{A}| = \frac{\varphi}{2\pi R},$$

where φ is the total magnetic flux passing through the ring.
Note. $\mathbf{A}$ is tangential to the ring and may be changed by adding the gradient of a scalar function.

1.12.7 Prove that

$$\int_S \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} = 0,$$

if S is a closed surface.

1.12.8 Evaluate $\oint \mathbf{r} \cdot d\mathbf{r}$ (Exercise 1.10.4) by Stokes' theorem.

1.12.9 Prove that

$$\oint u\nabla v \cdot d\boldsymbol{\lambda} = -\oint v\nabla u \cdot d\boldsymbol{\lambda}.$$

1.12.10 Prove that

$$\oint u\nabla v \cdot d\boldsymbol{\lambda} = \int_S (\nabla u) \times (\nabla v) \cdot d\boldsymbol{\sigma}.$$

1.13 POTENTIAL THEORY

Scalar Potential

If a force over a given simply connected region of space S (which means that it has no holes) can be expressed as the negative gradient of a scalar function φ,

$$\mathbf{F} = -\nabla\varphi, \tag{1.118}$$

we call φ a scalar potential that describes the force by one function instead of three. A scalar potential is only determined up to an additive constant, which can be used to adjust its value at infinity (usually zero) or at some other point. The force $\mathbf{F}$ appearing as the negative gradient of a single-valued scalar potential is labeled a **conservative** force. We want to know when a scalar potential function exists. To answer this question we establish two other relations as equivalent to Eq. (1.118). These are

$$\nabla \times \mathbf{F} = 0 \tag{1.119}$$

and

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0, \tag{1.120}$$

for every closed path in our simply connected region S. We proceed to show that each of these three equations implies the other two. Let us start with

$$\mathbf{F} = -\nabla\varphi. \tag{1.121}$$

Then

$$\nabla \times \mathbf{F} = -\nabla \times \nabla\varphi = 0 \tag{1.122}$$

by Eq. (1.82) or Eq. (1.118) implies Eq. (1.119). Turning to the line integral, we have

$$\oint \mathbf{F} \cdot d\mathbf{r} = -\oint \nabla\varphi \cdot d\mathbf{r} = -\oint d\varphi, \tag{1.123}$$

using Eq. (1.118). Now, $d\varphi$ integrates to give φ. Since we have specified a closed loop, the end points coincide and we get zero for every closed path in our region S for which Eq. (1.118) holds. It is important to note the restriction here that the potential be single-valued and that Eq. (1.118) hold for **all** points in S. This problem may arise in using a scalar magnetic potential, a perfectly valid procedure as long as no net current is encircled. As soon as we choose a path in space that encircles a net current, the scalar magnetic potential ceases to be single-valued and our analysis no longer applies.

Continuing this demonstration of equivalence, let us assume that Eq. (1.120) holds. If $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all paths in S, we see that the value of the integral joining two distinct points A and B is independent of the path (Fig. 1.32). Our premise is that

$$\oint_{ACBDA} \mathbf{F} \cdot d\mathbf{r} = 0. \tag{1.124}$$

Therefore

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} = -\int_{BDA} \mathbf{F} \cdot d\mathbf{r} = \int_{ADB} \mathbf{F} \cdot d\mathbf{r}, \tag{1.125}$$

reversing the sign by reversing the direction of integration. Physically, this means that the work done in going from A to B is independent of the path and that the work done in going around a closed path is zero. This is the reason for labeling such a force conservative: Energy is conserved.

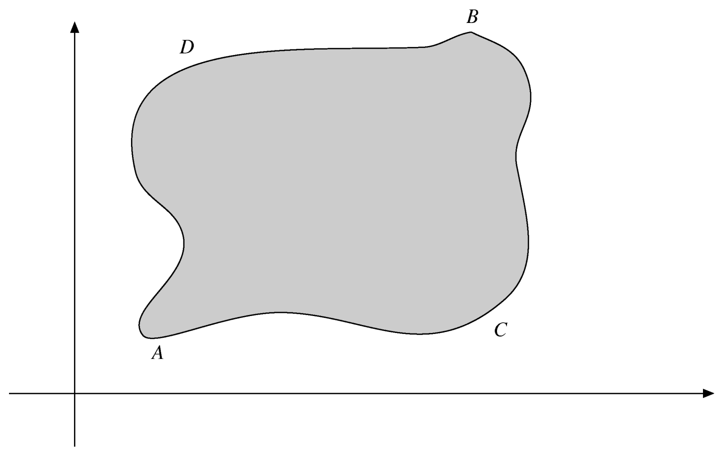

FIGURE 1.32 Possible paths for doing work.

With the result shown in Eq. (1.125), we have the work done dependent only on the endpoints A and B. That is,

$$\text{work done by force} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \varphi(A) - \varphi(B). \tag{1.126}$$

Equation (1.126) defines a scalar potential (strictly speaking, the difference in potential between points A and B) and provides a means of calculating the potential. If point B is taken as a variable, say, (x, y, z), then differentiation with respect to x, y, and z will recover Eq. (1.118).

The choice of sign on the right-hand side is arbitrary. The choice here is made to achieve agreement with Eq. (1.118) and to ensure that water will run downhill rather than uphill. For points A and B separated by a length $d\mathbf{r}$, Eq. (1.126) becomes

$$\mathbf{F} \cdot d\mathbf{r} = -d\varphi = -\nabla\varphi \cdot d\mathbf{r}. \tag{1.127}$$

This may be rewritten

$$(\mathbf{F} + \nabla\varphi) \cdot d\mathbf{r} = 0, \tag{1.128}$$

and since $d\mathbf{r}$ is arbitrary, Eq. (1.118) must follow. If

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0, \tag{1.129}$$

we may obtain Eq. (1.119) by using Stokes' theorem (Eq. (1.112)):

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma}. \tag{1.130}$$

If we take the path of integration to be the perimeter of an arbitrary differential area $d\boldsymbol{\sigma}$, the integrand in the surface integral must vanish. Hence Eq. (1.120) implies Eq. (1.119).

Finally, if $\nabla \times \mathbf{F} = 0$, we need only reverse our statement of Stokes' theorem (Eq. (1.130)) to derive Eq. (1.120). Then, by Eqs. (1.126) to (1.128), the initial statement

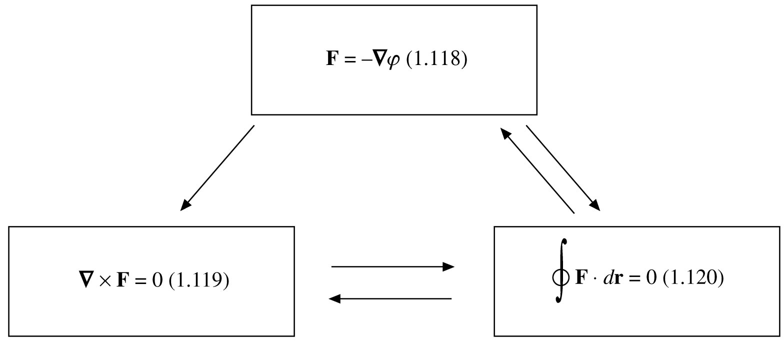

FIGURE 1.33 Equivalent formulations of a conservative force.

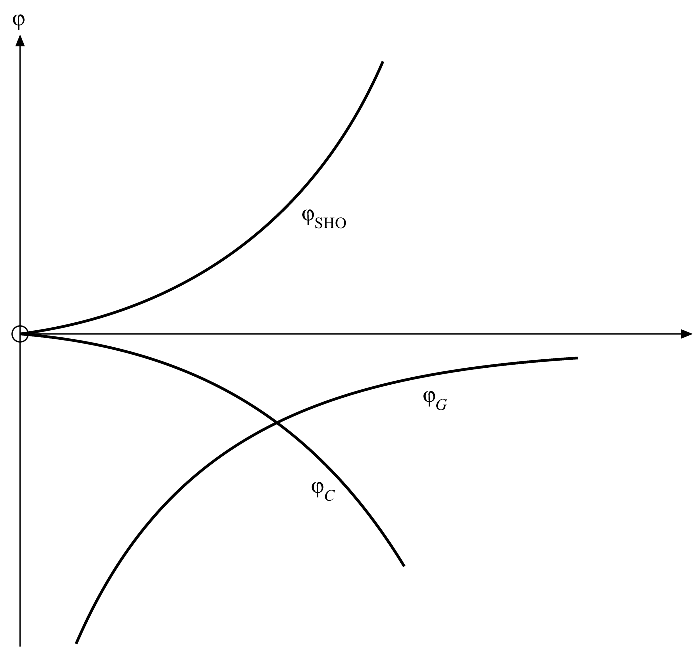

FIGURE 1.34 Potential energy versus distance (gravitational, centrifugal, and simple harmonic oscillator).

$\mathbf{F} = -\nabla\varphi$ is derived. The triple equivalence is demonstrated (Fig. 1.33). To summarize, a single-valued scalar potential function φ exists if and only if $\mathbf{F}$ is irrotational or the work done around every closed loop is zero. The gravitational and electrostatic force fields given by Eq. (1.79) are irrotational and therefore are conservative. Gravitational and electrostatic scalar potentials exist. Now, by calculating the work done (Eq. (1.126)), we proceed to determine three potentials (Fig. 1.34).

Example 1.13.1 GRAVITATIONAL POTENTIAL

Find the scalar potential for the gravitational force on a unit mass m_1,

$$\mathbf{F}_G = -\frac{Gm_1m_2\hat{\mathbf{r}}}{r^2} = -\frac{k\hat{\mathbf{r}}}{r^2}, \tag{1.131}$$

radially **inward**. By integrating Eq. (1.118) from infinity in to position $\mathbf{r}$, we obtain

$$\varphi_G(r) - \varphi_G(\infty) = -\int_\infty^{\mathbf{r}} \mathbf{F}_G \cdot d\mathbf{r} = +\int_{\mathbf{r}}^\infty \mathbf{F}_G \cdot d\mathbf{r}. \tag{1.132}$$

By use of $\mathbf{F}_G = -\mathbf{F}_{\text{applied}}$, a comparison with Eq. (1.95a) shows that the potential is the work done in bringing the unit mass in from infinity. (We can define only potential difference. Here we arbitrarily assign infinity to be a zero of potential.) The integral on the right-hand side of Eq. (1.132) is negative, meaning that $\varphi_G(r)$ is negative. Since $\mathbf{F}_G$ is radial, we obtain a contribution to φ only when $d\mathbf{r}$ is radial, or

$$\varphi_G(r) = -\int_r^\infty \frac{k\,dr}{r^2} = -\frac{k}{r} = -\frac{Gm_1m_2}{r}.$$

The final negative sign is a consequence of the attractive force of gravity. ■

Example 1.13.2 CENTRIFUGAL POTENTIAL

Calculate the scalar potential for the **centrifugal** force per unit mass, $\mathbf{F}_C = \omega^2 r\hat{\mathbf{r}}$, radially **outward**. Physically, you might feel this on a large horizontal spinning disk at an amusement park. Proceeding as in Example 1.13.1 but integrating from the origin outward and taking $\varphi_C(0) = 0$, we have

$$\varphi_C(r) = -\int_0^r \mathbf{F}_C \cdot d\mathbf{r} = -\frac{\omega^2 r^2}{2}.$$

If we reverse signs, taking $\mathbf{F}_{\text{SHO}} = -k\mathbf{r}$, we obtain $\varphi_{\text{SHO}} = \frac{1}{2}kr^2$, the simple harmonic oscillator potential.

The gravitational, centrifugal, and simple harmonic oscillator potentials are shown in Fig. 1.34. Clearly, the simple harmonic oscillator yields stability and describes a restoring force. The centrifugal potential describes an unstable situation. ■

Thermodynamics — Exact Differentials

In thermodynamics, which is sometimes called a search for exact differentials, we encounter equations of the form

$$df = P(x, y)\,dx + Q(x, y)\,dy. \tag{1.133a}$$

The usual problem is to determine whether $\int(P(x, y)\,dx + Q(x, y)\,dy)$ depends only on the endpoints, that is, whether df is indeed an exact differential. The necessary and sufficient condition is that

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{1.133b}$$

or that

$$P(x, y) = \partial f/\partial x,\qquad Q(x, y) = \partial f/\partial y. \tag{1.133c}$$

Equations (1.133c) depend on satisfying the relation

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}. \tag{1.133d}$$

This, however, is exactly analogous to Eq. (1.119), the requirement that $\mathbf{F}$ be irrotational. Indeed, the z-component of Eq. (1.119) yields

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \tag{1.133e}$$

with

$$F_x = \frac{\partial f}{\partial x}, \qquad F_y = \frac{\partial f}{\partial y}.$$

Vector Potential

In some branches of physics, especially electrodynamics, it is convenient to introduce a vector potential $\mathbf{A}$ such that a (force) field $\mathbf{B}$ is given by

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.134}$$

Clearly, if Eq. (1.134) holds, $\nabla \cdot \mathbf{B} = 0$ by Eq. (1.84) and $\mathbf{B}$ is solenoidal. Here we want to develop a converse, to show that when $\mathbf{B}$ is solenoidal a vector potential $\mathbf{A}$ exists. We demonstrate the existence of $\mathbf{A}$ by actually calculating it. Suppose $\mathbf{B} = \hat{\mathbf{x}}b_1 + \hat{\mathbf{y}}b_2 + \hat{\mathbf{z}}b_3$ and our unknown $\mathbf{A} = \hat{\mathbf{x}}a_1 + \hat{\mathbf{y}}a_2 + \hat{\mathbf{z}}a_3$. By Eq. (1.134),

$$\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} = b_1, \tag{1.135a}$$

$$\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} = b_2, \tag{1.135b}$$

$$\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} = b_3. \tag{1.135c}$$

Let us assume that the coordinates have been chosen so that $\mathbf{A}$ is parallel to the yz-plane; that is, $a_1 = 0$.[24] Then

$$b_2 = -\frac{\partial a_3}{\partial x}\qquad b_3 = \frac{\partial a_2}{\partial x}. \tag{1.136}$$

[24]Clearly, this can be done at any one point. It is not at all obvious that this assumption will hold at all points; that is, $\mathbf{A}$ will be two-dimensional. The justification for the assumption is that it works; Eq. (1.141) satisfies Eq. (1.134).

Integrating, we obtain

$$\begin{aligned} a_2 &= \int_{x_0}^{x} b_3\,dx + f_2(y,z), \\ a_3 &= -\int_{x_0}^{x} b_2\,dx + f_3(y,z), \end{aligned} \tag{1.137}$$

where f_2 and f_3 are arbitrary functions of y and z but **not** functions of x. These two equations can be checked by differentiating and recovering Eq. (1.136). Equation (1.135a) becomes[25]

$$\begin{aligned} \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} &= -\int_{x_0}^{x}\left(\frac{\partial b_2}{\partial y} + \frac{\partial b_3}{\partial z}\right)dx + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ &= \int_{x_0}^{x} \frac{\partial b_1}{\partial x}\,dx + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \end{aligned} \tag{1.138}$$

using $\nabla \cdot \mathbf{B} = 0$. Integrating with respect to x, we obtain

$$\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} = b_1(x,y,z) - b_1(x_0,y,z) + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}. \tag{1.139}$$

Remembering that f_3 and f_2 are arbitrary functions of y and z, we choose

$$\begin{aligned} f_2 &= 0, \\ f_3 &= \int_{y_0}^{y} b_1(x_0,y,z)\,dy, \end{aligned} \tag{1.140}$$

so that the right-hand side of Eq. (1.139) reduces to $b_1(x,y,z)$, in agreement with Eq. (1.135a). With f_2 and f_3 given by Eq. (1.140), we can construct $\mathbf{A}$:

$$\mathbf{A} = \hat{\mathbf{y}}\int_{x_0}^{x} b_3(x,y,z)\,dx + \hat{\mathbf{z}}\left[\int_{y_0}^{y} b_1(x_0,y,z)\,dy - \int_{x_0}^{x} b_2(x,y,z)\,dx\right]. \tag{1.141}$$

However, this is not quite complete. We may add any constant since $\mathbf{B}$ is a derivative of $\mathbf{A}$. What is much more important, we may add any gradient of a scalar function $\nabla\varphi$ without affecting $\mathbf{B}$ at all. Finally, the functions f_2 and f_3 are not unique. Other choices could have been made. Instead of setting $a_1 = 0$ to get Eq. (1.136) any cyclic permutation of $1, 2, 3, x, y, z, x_0, y_0, z_0$ would also work.

Example 1.13.3 A Magnetic Vector Potential for a Constant Magnetic Field

To illustrate the construction of a magnetic vector potential, we take the special but still important case of a constant magnetic induction

$$\mathbf{B} = \hat{\mathbf{z}}B_z, \tag{1.142}$$

[25] Leibniz' formula in Exercise 9.6.13 is useful here.

in which B_z is a constant. Equations (1.135a to c) become

$$\begin{aligned}\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}&=0,\\ \frac{\partial a_1}{\partial z}-\frac{\partial a_3}{\partial x}&=0,\\ \frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}&=B_z.\end{aligned}\tag{1.143}$$

If we assume that $a_1=0$, as before, then by Eq. (1.141)

$$\mathbf{A}=\hat{\mathbf{y}}\int^x B_z\,dx=\hat{\mathbf{y}}xB_z,\tag{1.144}$$

setting a constant of integration equal to zero. It can readily be seen that this $\mathbf{A}$ satisfies Eq. (1.134).

To show that the choice $a_1=0$ was not sacred or at least not required, let us try setting $a_3=0$. From Eq. (1.143)

$$\frac{\partial a_2}{\partial z}=0,\tag{1.145a}$$

$$\frac{\partial a_1}{\partial z}=0,\tag{1.145b}$$

$$\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}=B_z.\tag{1.145c}$$

We see a_1 and a_2 are independent of z, or

$$a_1=a_1(x,y),\qquad a_2=a_2(x,y).\tag{1.146}$$

Equation (1.145c) is satisfied if we take

$$a_2=p\int^x B_z\,dx=pxB_z\tag{1.147}$$

and

$$a_1=(p-1)\int^y B_z\,dy=(p-1)yB_z,\tag{1.148}$$

with p any constant. Then

$$\mathbf{A}=\hat{\mathbf{x}}(p-1)yB_z+\hat{\mathbf{y}}pxB_z.\tag{1.149}$$

Again, Eqs. (1.134), (1.142), and (1.149) are seen to be consistent. Comparison of Eqs. (1.144) and (1.149) shows immediately that $\mathbf{A}$ is not unique. The difference between Eqs. (1.144) and (1.149) and the appearance of the parameter p in Eq. (1.149) may be accounted for by rewriting Eq. (1.149) as

$$\begin{aligned}\mathbf{A}&=-\frac{1}{2}(\hat{\mathbf{x}}y-\hat{\mathbf{y}}x)B_z+\left(p-\frac{1}{2}\right)(\hat{\mathbf{x}}y+\hat{\mathbf{y}}x)B_z\\ &=-\frac{1}{2}(\hat{\mathbf{x}}y-\hat{\mathbf{y}}x)B_z+\left(p-\frac{1}{2}\right)B_z\nabla\varphi\end{aligned}\tag{1.150}$$

with

$$\varphi = xy. \tag{1.151}$$

■

The first term in **A** corresponds to the usual form

$$\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r}) \tag{1.152}$$

for **B**, a constant.

Adding a gradient of a scalar function, Λ say, to the vector potential **A** does not affect **B**, by Eq. (1.82); this is known as a gauge transformation (see Exercises 1.13.9 and 4.6.4):

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda. \tag{1.153}$$

Suppose now that the wave function ψ_0 solves the Schrödinger equation of quantum mechanics without magnetic induction field **B**,

$$\left\{\frac{1}{2m}(-i\hbar\nabla)^2 + V - E\right\}\psi_0 = 0, \tag{1.154}$$

describing a particle with mass m and charge e. When **B** is switched on, the wave equation becomes

$$\boxed{\left\{\frac{1}{2m}(-i\hbar\nabla - e\mathbf{A})^2 + V - E\right\}\psi = 0.} \tag{1.155}$$

Its solution ψ picks up a phase factor that depends on the coordinates in general,

$$\boxed{\psi(\mathbf{r}) = \exp\left[\frac{ie}{\hbar}\int^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'\right]\psi_0(\mathbf{r}).} \tag{1.156}$$

From the relation

$$\begin{aligned}(-i\hbar\nabla - e\mathbf{A})\psi &= \exp\left[\frac{ie}{\hbar}\int \mathbf{A} \cdot d\mathbf{r}'\right]\left\{(-i\hbar\nabla - e\mathbf{A})\psi_0 - i\hbar\psi_0\frac{ie}{\hbar}\mathbf{A}\right\}\\ &= \exp\left[\frac{ie}{\hbar}\int \mathbf{A} \cdot d\mathbf{r}'\right](-i\hbar\nabla\psi_0),\end{aligned} \tag{1.157}$$

it is obvious that ψ solves Eq. (1.155) if ψ_0 solves Eq. (1.154). The **gauge covariant derivative** $\nabla - i(e/\hbar)\mathbf{A}$ describes the coupling of a charged particle with the magnetic field. It is often called **minimal substitution** and plays a central role in quantum electromagnetism, the first and simplest gauge theory in physics.

To summarize this discussion of the vector potential: *When a vector* **B** *is solenoidal, a vector potential* **A** *exists such that* $\mathbf{B} = \nabla \times \mathbf{A}$. **A** *is undetermined to within an additive gradient. This corresponds to the arbitrary zero of a potential, a constant of integration for the scalar potential*.

In many problems the magnetic vector potential **A** will be obtained from the current distribution that produces the magnetic induction **B**. This means solving Poisson's (vector) equation (see Exercise 1.14.4).

Exercises

1.13.1 If a force $\mathbf{F}$ is given by

$$\mathbf{F} = \left(x^2 + y^2 + z^2\right)^n (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z),$$

find

(a) $\nabla \cdot \mathbf{F}$.
(b) $\nabla \times \mathbf{F}$.
(c) A scalar potential $\varphi(x, y, z)$ so that $\mathbf{F} = -\nabla\varphi$.
(d) For what value of the exponent n does the scalar potential diverge at both the origin and infinity?

ANS. (a) $(2n+3)r^{2n}$, (b) 0,
(c) $-\frac{1}{2n+2}r^{2n+2}, \ n \neq -1$, (d) $n = -1$,
$\varphi = -\ln r$.

1.13.2 A sphere of radius a is uniformly charged (throughout its volume). Construct the electrostatic potential $\varphi(r)$ for $0 \leqslant r < \infty$.
Hint. In Section 1.14 it is shown that the Coulomb force on a test charge at $r = r_0$ depends only on the charge at distances less than r_0 and is independent of the charge at distances greater than r_0. Note that this applies to a **spherically symmetric** charge distribution.

1.13.3 The usual problem in classical mechanics is to calculate the motion of a particle given the potential. For a uniform density (ρ_0), nonrotating massive sphere, Gauss' law of Section 1.14 leads to a gravitational force on a unit mass m_0 at a point r_0 produced by the attraction of the mass at $r \leqslant r_0$. The mass at $r > r_0$ contributes nothing to the force.

(a) Show that $\mathbf{F}/m_0 = -(4\pi G\rho_0/3)\mathbf{r}, \ 0 \leqslant r \leqslant a$, where a is the radius of the sphere.
(b) Find the corresponding gravitational potential, $0 \leqslant r \leqslant a$.
(c) Imagine a vertical hole running completely through the center of the Earth and out to the far side. Neglecting the rotation of the Earth and assuming a uniform density $\rho_0 = 5.5\ \mathrm{gm/cm}^3$, calculate the nature of the motion of a particle dropped into the hole. What is its period?

Note. $\mathbf{F} \propto \mathbf{r}$ is actually a very poor approximation. Because of varying density, the approximation $\mathbf{F} =$ constant along the outer half of a radial line and $\mathbf{F} \propto \mathbf{r}$ along the inner half is a much closer approximation.

1.13.4 The origin of the Cartesian coordinates is at the Earth's center. The moon is on the z-axis, a fixed distance R away (center-to-center distance). The tidal force exerted by the moon on a particle at the Earth's surface (point x, y, z) is given by

$$F_x = -GMm\frac{x}{R^3}, \qquad F_y = -GMm\frac{y}{R^3}, \qquad F_z = +2GMm\frac{z}{R^3}.$$

Find the potential that yields this tidal force.

ANS. $-\dfrac{GMm}{R^3}\left(z^2 - \dfrac{1}{2}x^2 - \dfrac{1}{2}y^2\right).$

In terms of the Legendre polynomials of Chapter 12 this becomes

$-\dfrac{GMm}{R^3} r^2 P_2(\cos\theta).$

1.13.5 A long, straight wire carrying a current I produces a magnetic induction $\mathbf{B}$ with components

$$\mathbf{B} = \frac{\mu_0 I}{2\pi}\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right).$$

Find a magnetic vector potential $\mathbf{A}$.

ANS. $\mathbf{A} = -\hat{\mathbf{z}}(\mu_0 I/4\pi)\ln(x^2+y^2)$. (This solution is not unique.)

1.13.6 If

$$\mathbf{B} = \frac{\hat{\mathbf{r}}}{r^2} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right),$$

find a vector $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$. One possible solution is

$$\mathbf{A} = \frac{\hat{\mathbf{x}} yz}{r(x^2+y^2)} - \frac{\hat{\mathbf{y}} xz}{r(x^2+y^2)}.$$

1.13.7 Show that the pair of equations

$$\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r}), \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

is satisfied by any constant magnetic induction $\mathbf{B}$.

1.13.8 Vector $\mathbf{B}$ is formed by the product of two gradients

$$\mathbf{B} = (\nabla u) \times (\nabla v),$$

where u and v are scalar functions.

(a) Show that $\mathbf{B}$ is solenoidal.
(b) Show that

$$\mathbf{A} = \frac{1}{2}(u\,\nabla v - v\,\nabla u)$$

is a vector potential for $\mathbf{B}$, in that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

1.13.9 The magnetic induction $\mathbf{B}$ is related to the magnetic vector potential $\mathbf{A}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. By Stokes' theorem

$$\int \mathbf{B} \cdot d\boldsymbol{\sigma} = \oint \mathbf{A} \cdot d\mathbf{r}.$$

Show that each side of this equation is invariant under the **gauge transformation**, $\mathbf{A} \to \mathbf{A} + \nabla\varphi$.

Note. Take the function φ to be single-valued. The complete gauge transformation is considered in Exercise 4.6.4.

1.13.10 With $\mathbf{E}$ the electric field and $\mathbf{A}$ the magnetic vector potential, show that $[\mathbf{E} + \partial\mathbf{A}/\partial t]$ is irrotational and that therefore we may write

$$E = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}.$$

1.13.11 The total force on a charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Using the scalar and vector potentials, show that

$$\mathbf{F} = q\left[-\nabla\varphi - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{A}\cdot\mathbf{v})\right].$$

Note that we now have a total time derivative of $\mathbf{A}$ in place of the partial derivative of Exercise 1.13.10.

1.14 Gauss' Law, Poisson's Equation

Gauss' Law

Consider a point electric charge q at the origin of our coordinate system. This produces an electric field $\mathbf{E}$ given by[26]

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}. \tag{1.158}$$

We now derive Gauss' law, which states that the surface integral in Fig. 1.35 is q/ε_0 if the closed surface $S = \partial V$ includes the origin (where q is located) and zero if the surface does not include the origin. The surface S is any closed surface; it need not be spherical.

Using Gauss' theorem, Eqs. (1.101a) and (1.101b) (and neglecting the $q/4\pi\varepsilon_0$), we obtain

$$\int_S \frac{\hat{\mathbf{r}}\cdot d\boldsymbol{\sigma}}{r^2} = \int_V \nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau = 0 \tag{1.159}$$

by Example 1.7.2, provided the surface S does not include the origin, where the integrands are not defined. This proves the second part of Gauss' law.

The first part, in which the surface S must include the origin, may be handled by surrounding the origin with a small sphere $S' = \partial V'$ of radius δ (Fig. 1.36). So that there will be no question what is inside and what is outside, imagine the volume outside the outer surface S and the volume inside surface $S'(r < \delta)$ connected by a small hole. This

[26] The electric field $\mathbf{E}$ is defined as the force per unit charge on a small stationary test charge q_t: $\mathbf{E} = \mathbf{F}/q_t$. From Coulomb's law the force on q_t due to q is $\mathbf{F} = (qq_t/4\pi\varepsilon_0)(\hat{\mathbf{r}}/r^2)$. When we divide by q_t, Eq. (1.158) follows.

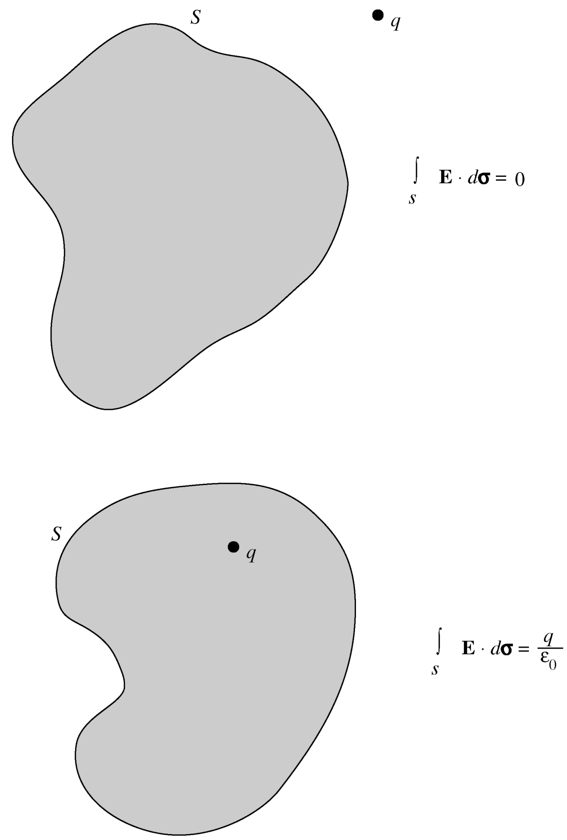

FIGURE 1.35 Gauss' law.

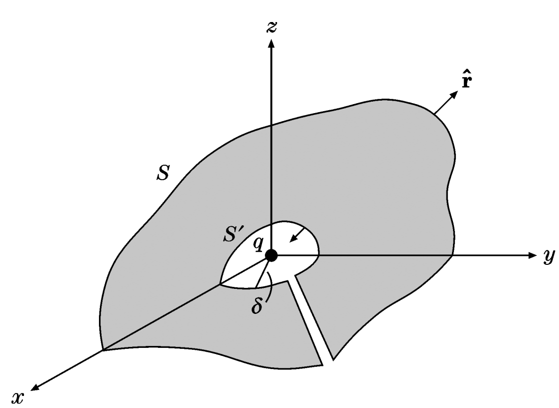

FIGURE 1.36 Exclusion of the origin.

joins surfaces S and S', combining them into one single simply connected closed surface. Because the radius of the imaginary hole may be made vanishingly small, there is no additional contribution to the surface integral. The inner surface is deliberately chosen to be spherical so that we will be able to integrate over it. Gauss' theorem now applies to the volume between S and S' without any difficulty. We have

$$\int_S \frac{\hat{\mathbf{r}} \cdot d\boldsymbol{\sigma}}{r^2} + \int_{S'} \frac{\hat{\mathbf{r}} \cdot d\boldsymbol{\sigma}'}{\delta^2} = 0. \tag{1.160}$$

We may evaluate the second integral, for $d\boldsymbol{\sigma}' = -\hat{\mathbf{r}}\delta^2\, d\Omega$, in which $d\Omega$ is an element of solid angle. The minus sign appears because we agreed in Section 1.10 to have the positive normal $\hat{\mathbf{r}}'$ **outward** from the volume. In this case the outward $\hat{\mathbf{r}}'$ is in the negative radial direction, $\hat{\mathbf{r}}' = -\hat{\mathbf{r}}$. By integrating over all angles, we have

$$\int_{S'} \frac{\hat{\mathbf{r}} \cdot d\boldsymbol{\sigma}'}{\delta^2} = -\int_{S'} \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\delta^2\, d\Omega}{\delta^2} = -4\pi, \tag{1.161}$$

independent of the radius δ. With the constants from Eq. (1.158), this results in

$$\int_S \mathbf{E} \cdot d\boldsymbol{\sigma} = \frac{q}{4\pi\varepsilon_0} 4\pi = \frac{q}{\varepsilon_0}, \tag{1.162}$$

completing the proof of Gauss' law. Notice that although the surface S may be spherical, it **need not** be spherical. Going just a bit further, we consider a distributed charge so that

$$q = \int_V \rho\, d\tau. \tag{1.163}$$

Equation (1.162) still applies, with q now interpreted as the total distributed charge enclosed by surface S:

$$\int_S \mathbf{E} \cdot d\boldsymbol{\sigma} = \int_V \frac{\rho}{\varepsilon_0} d\tau. \tag{1.164}$$

Using Gauss' theorem, we have

$$\int_V \nabla \cdot \mathbf{E}\, d\tau = \int_V \frac{\rho}{\varepsilon_0} d\tau. \tag{1.165}$$

Since our volume is completely arbitrary, the integrands must be equal, or

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1.166}$$

one of Maxwell's equations. If we reverse the argument, Gauss' law follows immediately from Maxwell's equation.

Poisson's Equation

If we replace $\mathbf{E}$ by $-\nabla\varphi$, Eq. (1.166) becomes

$$\nabla \cdot \nabla\varphi = -\frac{\rho}{\varepsilon_0}, \tag{1.167a}$$

which is Poisson's equation. For the condition $\rho = 0$ this reduces to an even more famous equation,

$$\nabla \cdot \nabla\varphi = 0, \tag{1.167b}$$

Laplace's equation. We encounter Laplace's equation frequently in discussing various coordinate systems (Chapter 2) and the special functions of mathematical physics that appear as its solutions. Poisson's equation will be invaluable in developing the theory of Green's functions (Section 9.7).

From direct comparison of the Coulomb electrostatic force law and Newton's law of universal gravitation,

$$\mathbf{F}_E = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}, \qquad \mathbf{F}_G = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}.$$

All of the potential theory of this section applies equally well to gravitational potentials. For example, the gravitational Poisson equation is

$$\nabla \cdot \nabla\varphi = +4\pi G\rho, \tag{1.168}$$

with ρ now a mass density.

Exercises

1.14.1 Develop Gauss' law for the two-dimensional case in which

$$\varphi = -q\frac{\ln\rho}{2\pi\varepsilon_0}, \qquad \mathbf{E} = -\nabla\varphi = q\frac{\hat{\boldsymbol{\rho}}}{2\pi\varepsilon_0\rho}.$$

Here q is the charge at the origin or the line charge per unit length if the two-dimensional system is a unit thickness slice of a three-dimensional (circular cylindrical) system. The variable ρ is measured radially outward from the line charge. $\hat{\boldsymbol{\rho}}$ is the corresponding unit vector (see Section 2.4).

1.14.2 (a) Show that Gauss' law follows from Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Here ρ is the usual charge density.

(b) Assuming that the electric field of a point charge q is spherically symmetric, show that Gauss' law implies the Coulomb inverse square expression

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}.$$

1.14.3 Show that the value of the electrostatic potential φ at any point P is equal to the average of the potential over any spherical surface centered on P. There are no electric charges on or within the sphere.

Hint. Use Green's theorem, Eq. (1.104), with $u^{-1} = r$, the distance from P, and $v = \varphi$. Also note Eq. (1.170) in Section 1.15.

1.14.4 Using Maxwell's equations, show that for a system (steady current) the magnetic vector potential $\mathbf{A}$ satisfies a vector Poisson equation,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J},$$

provided we require $\nabla \cdot \mathbf{A} = 0$.

1.15 Dirac Delta Function

From Example 1.6.1 and the development of Gauss' law in Section 1.14,

$$\int \nabla \cdot \nabla \left(\frac{1}{r}\right) d\tau = -\int \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau = \begin{cases} -4\pi \\ 0, \end{cases} \tag{1.169}$$

depending on whether or not the integration includes the origin $\mathbf{r} = 0$. This result may be conveniently expressed by introducing the Dirac delta function,

$$\boxed{\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}) \equiv -4\pi\delta(x)\delta(y)\delta(z).} \tag{1.170}$$

This Dirac delta function is **defined** by its assigned properties

$$\delta(x) = 0, \qquad x \neq 0 \tag{1.171a}$$

$$\boxed{f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\, dx,} \tag{1.171b}$$

where $f(x)$ is any well-behaved function and the integration includes the origin. As a special case of Eq. (1.171b),

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{1.171c}$$

From Eq. (1.171b), $\delta(x)$ must be an infinitely high, infinitely thin spike at $x = 0$, as in the description of an impulsive force (Section 15.9) or the charge density for a point charge.[27] The problem is that **no such function exists**, in the usual sense of function. However, the crucial property in Eq. (1.171b) can be developed rigorously as the limit of a **sequence** of functions, a distribution. For example, the delta function may be approximated by the

[27]The delta function is frequently invoked to describe very short-range forces, such as nuclear forces. It also appears in the normalization of continuum wave functions of quantum mechanics. Compare Eq. (1.193c) for plane-wave eigenfunctions.

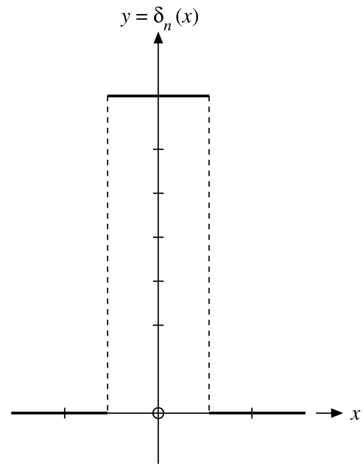

FIGURE 1.37 δ-Sequence function.

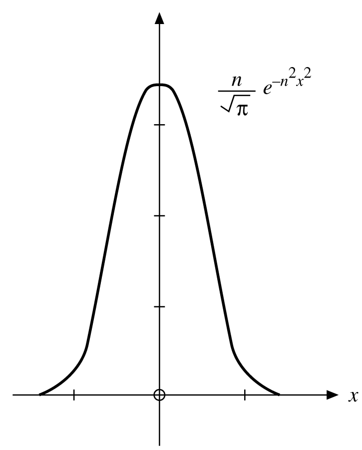

FIGURE 1.38 δ-Sequence function.

sequences of functions, Eqs. (1.172) to (1.175) and Figs. 1.37 to 1.40:

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & x > \frac{1}{2n} \end{cases} \tag{1.172}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2x^2\right) \tag{1.173}$$

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1+n^2x^2} \tag{1.174}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\, dt. \tag{1.175}$$

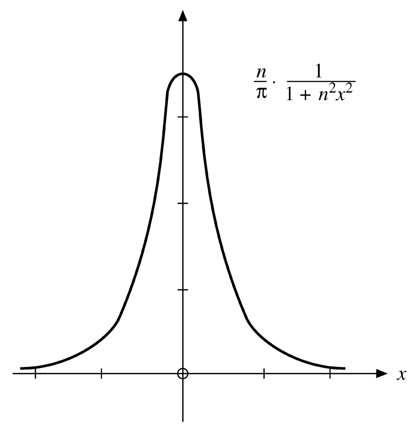

FIGURE 1.39 δ-Sequence function.

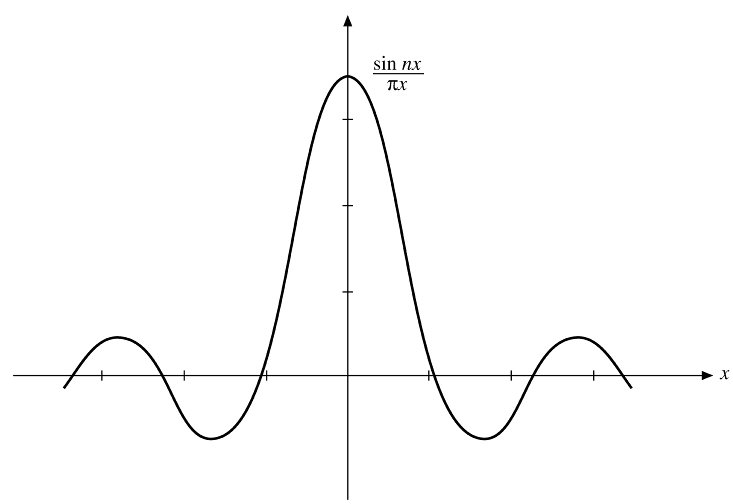

FIGURE 1.40 δ-Sequence function.

These approximations have varying degrees of usefulness. Equation (1.172) is useful in providing a simple derivation of the integral property, Eq. (1.171b). Equation (1.173) is convenient to differentiate. Its derivatives lead to the Hermite polynomials. Equation (1.175) is particularly useful in Fourier analysis and in its applications to quantum mechanics. In the theory of Fourier series, Eq. (1.175) often appears (modified) as the Dirichlet kernel:

$$\delta_n(x) = \frac{1}{2\pi} \frac{\sin[(n + \frac{1}{2})x]}{\sin(\frac{1}{2}x)}. \tag{1.176}$$

In using these approximations in Eq. (1.171b) and later, we assume that $f(x)$ is well behaved — it offers no problems at large x.

For most physical purposes such approximations are quite adequate. From a mathematical point of view the situation is still unsatisfactory: The limits

$$\lim_{n\to\infty} \delta_n(x)$$

do not exist.

A way out of this difficulty is provided by the theory of distributions. Recognizing that Eq. (1.171b) is the fundamental property, we focus our attention on it rather than on $\delta(x)$ itself. Equations (1.172) to (1.175) with $n = 1, 2, 3, \ldots$ may be interpreted as **sequences** of normalized functions:

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1. \tag{1.177}$$

The sequence of integrals has the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = f(0). \tag{1.178}$$

Note that Eq. (1.178) is the limit of a sequence of integrals. Again, the limit of $\delta_n(x)$, $n \to \infty$, does not exist. (The limits for all four forms of $\delta_n(x)$ diverge at $x = 0$.)

We may treat $\delta(x)$ consistently in the form

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx. \tag{1.179}$$

$\delta(x)$ is labeled a distribution (not a function) defined by the sequences $\delta_n(x)$ as indicated in Eq. (1.179). We might emphasize that the integral on the left-hand side of Eq. (1.179) is not a Riemann integral.[28] It is a limit.

This distribution $\delta(x)$ is only one of an infinity of possible distributions, but it is the one we are interested in because of Eq. (1.171b).

From these sequences of functions we see that Dirac's delta function must be even in x, $\delta(-x) = \delta(x)$.

The integral property, Eq. (1.171b), is useful in cases where the argument of the delta function is a function $g(x)$ with simple zeros on the real axis, which leads to the rules

$$\delta(ax) = \frac{1}{a}\delta(x), \qquad a > 0, \tag{1.180}$$

$$\delta\bigl(g(x)\bigr) = \sum_{\substack{a,\\ g(a)=0,\\ g'(a)\neq 0}} \frac{\delta(x-a)}{|g'(a)|}. \tag{1.181a}$$

Equation (1.180) may be written

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\, dx = \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\, dy = \frac{1}{a} f(0),$$

[28]It can be treated as a Stieltjes integral if desired. $\delta(x)\,dx$ is replaced by $du(x)$, where $u(x)$ is the Heaviside step function (compare Exercise 1.15.13).

applying Eq. (1.171b). Equation (1.180) may be written as $\delta(ax) = \frac{1}{|a|}\delta(x)$ for $a < 0$. To prove Eq. (1.181a) we decompose the integral

$$\int_{-\infty}^{\infty} f(x)\delta\big(g(x)\big)\,dx = \sum_a \int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta\big((x-a)g'(a)\big)\,dx \tag{1.181b}$$

into a sum of integrals over small intervals containing the zeros of $g(x)$. In these intervals, $g(x) \approx g(a) + (x-a)g'(a) = (x-a)g'(a)$. Using Eq. (1.180) on the right-hand side of Eq. (1.181b) we obtain the integral of Eq. (1.181a).

Using integration by parts we can also **define the derivative** $\delta'(x)$ of the Dirac delta function by the relation

$$\int_{-\infty}^{\infty} f(x)\delta'(x-x')\,dx = -\int_{-\infty}^{\infty} f'(x)\delta(x-x')\,dx = -f'(x'). \tag{1.182}$$

We use $\delta(x)$ frequently and call it the Dirac delta function[29] — for historical reasons. Remember that it is not really a function. It is essentially a shorthand notation, defined implicitly as the limit of integrals in a sequence, $\delta_n(x)$, according to Eq. (1.179). It should be understood that our Dirac delta function has significance only as part of an integrand. In this spirit, the linear operator $\int dx\,\delta(x-x_0)$ operates on $f(x)$ and yields $f(x_0)$:

$$\mathcal{L}(x_0)f(x) \equiv \int_{-\infty}^{\infty} \delta(x-x_0)f(x)\,dx = f(x_0). \tag{1.183}$$

It may also be classified as a linear mapping or simply as a generalized function. Shifting our singularity to the point $x = x'$, we write the Dirac delta function as $\delta(x-x')$. Equation (1.171b) becomes

$$\int_{-\infty}^{\infty} f(x)\delta(x-x')\,dx = f(x'). \tag{1.184}$$

As a description of a singularity at $x = x'$, the Dirac delta function may be written as $\delta(x-x')$ or as $\delta(x'-x)$. Going to three dimensions and using spherical polar coordinates, we obtain

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(\mathbf{r})r^2\,dr\sin\theta\,d\theta\,d\varphi = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = 1. \tag{1.185}$$

This corresponds to a singularity (or source) at the origin. Again, if our source is at $\mathbf{r} = \mathbf{r_1}$, Eq. (1.185) becomes

$$\iiint \delta(\mathbf{r}_2 - \mathbf{r}_1)r_2^2\,dr_2\sin\theta_2\,d\theta_2\,d\varphi_2 = 1. \tag{1.186}$$

[29] Dirac introduced the delta function to quantum mechanics. Actually, the delta function can be traced back to Kirchhoff, 1882. For further details see M. Jammer, *The Conceptual Development of Quantum Mechanics*. New York: McGraw–Hill (1966), p. 301.

Example 1.15.1 TOTAL CHARGE INSIDE A SPHERE

Consider the total electric flux $\oint \mathbf{E} \cdot d\boldsymbol{\sigma}$ out of a sphere of radius R around the origin surrounding n charges e_j, located at the points $\mathbf{r}_j$ with $r_j < R$, that is, inside the sphere. The electric field strength $\mathbf{E} = -\nabla\varphi(\mathbf{r})$, where the potential

$$\varphi = \sum_{j=1}^{n} \frac{e_j}{|\mathbf{r} - \mathbf{r}_j|} = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

is the sum of the Coulomb potentials generated by each charge and the total charge density is $\rho(\mathbf{r}) = \sum_j e_j \delta(\mathbf{r} - \mathbf{r}_j)$. The delta function is used here as an abbreviation of a pointlike density. Now we use Gauss' theorem for

$$\oint \mathbf{E} \cdot d\boldsymbol{\sigma} = -\oint \nabla\varphi \cdot d\boldsymbol{\sigma} = -\int \nabla^2\varphi \, d\tau = \int \frac{\rho(\mathbf{r})}{\varepsilon_0} d\tau = \frac{\sum_j e_j}{\varepsilon_0}$$

in conjunction with the differential form of Gauss's law, $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$, and

$$\sum_j e_j \int \delta(\mathbf{r} - \mathbf{r}_j) \, d\tau = \sum_j e_j.$$

■

Example 1.15.2 PHASE SPACE

In the scattering theory of relativistic particles using Feynman diagrams, we encounter the following integral over energy of the scattered particle (we set the velocity of light $c = 1$):

$$\begin{aligned}\int d^4p\,\delta\big(p^2 - m^2\big) f(p) &\equiv \int d^3p \int dp_0\, \delta\big(p_0^2 - \mathbf{p}^2 - m^2\big) f(p) \\ &= \int_{E>0} \frac{d^3p\, f(E,\mathbf{p})}{2\sqrt{m^2 + \mathbf{p}^2}} + \int_{E<0} \frac{d^3p\, f(E,\mathbf{p})}{2\sqrt{m^2 + \mathbf{p}^2}},\end{aligned}$$

where we have used Eq. (1.181a) at the zeros $E = \pm\sqrt{m^2 + \mathbf{p}^2}$ of the argument of the delta function. The physical meaning of $\delta(p^2 - m^2)$ is that the particle of mass m and four-momentum $p^\mu = (p_0, \mathbf{p})$ is on its mass shell, because $p^2 = m^2$ is equivalent to $E = \pm\sqrt{m^2 + \mathbf{p}^2}$. Thus, the on-mass-shell volume element in momentum space is the Lorentz invariant $\frac{d^3p}{2E}$, in contrast to the nonrelativistic d^3p of momentum space. The fact that a negative energy occurs is a peculiarity of relativistic kinematics that is related to the antiparticle. ■

Delta Function Representation by Orthogonal Functions

Dirac's delta function[30] can be expanded in terms of any basis of real orthogonal functions $\{\varphi_n(x), n = 0, 1, 2, \ldots\}$. Such functions will occur in Chapter 10 as solutions of ordinary differential equations of the Sturm–Liouville form.

[30]This section is optional here. It is not needed until Chapter 10.

They satisfy the orthogonality relations

$$\int_a^b \varphi_m(x)\varphi_n(x)\,dx = \delta_{mn}, \tag{1.187}$$

where the interval (a, b) may be infinite at either end or both ends. [For convenience we assume that φ_n has been defined to include $(w(x))^{1/2}$ if the orthogonality relations contain an additional positive weight function $w(x)$.] We use the φ_n to expand the delta function as

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x), \tag{1.188}$$

where the coefficients a_n are functions of the variable t. Multiplying by $\varphi_m(x)$ and integrating over the orthogonality interval (Eq. (1.187)), we have

$$a_m(t) = \int_a^b \delta(x-t)\varphi_m(x)\,dx = \varphi_m(t) \tag{1.189}$$

or

$$\delta(x-t) = \sum_{n=0}^{\infty} \varphi_n(t)\varphi_n(x) = \delta(t-x). \tag{1.190}$$

This series is assuredly not uniformly convergent (see Chapter 5), but it may be used as part of an integrand in which the ensuing integration will make it convergent (compare Section 5.5).

Suppose we form the integral $\int F(t)\delta(t-x)\,dx$, where it is assumed that $F(t)$ can be expanded in a series of orthogonal functions $\varphi_p(t)$, a property called *completeness*. We then obtain

$$\begin{aligned}\int F(t)\delta(t-x)\,dt &= \int \sum_{p=0}^{\infty} a_p\varphi_p(t) \sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(t)\,dt \\ &= \sum_{p=0}^{\infty} a_p\varphi_p(x) = F(x),\end{aligned} \tag{1.191}$$

the cross products $\int \varphi_p\varphi_n\,dt\,(n \neq p)$ vanishing by orthogonality (Eq. (1.187)). Referring back to the definition of the Dirac delta function, Eq. (1.171b), we see that our series representation, Eq. (1.190), satisfies the defining property of the Dirac delta function and therefore is a representation of it. This representation of the Dirac delta function is called **closure**. The assumption of completeness of a set of functions for expansion of $\delta(x-t)$ yields the closure relation. The converse, that closure implies completeness, is the topic of Exercise 1.15.16.

Integral Representations for the Delta Function

Integral transforms, such as the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(i\omega t)\,dt$$

of Chapter 15, lead to the corresponding integral representations of Dirac's delta function. For example, take

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi}\int_{-n}^{n} \exp\bigl(i\omega(t-x)\bigr)\,d\omega, \tag{1.192}$$

using Eq. (1.175). We have

$$f(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)\delta_n(t-x)\,dt, \tag{1.193a}$$

where $\delta_n(t-x)$ is the sequence in Eq. (1.192) defining the distribution $\delta(t-x)$. Note that Eq. (1.193a) assumes that $f(t)$ is continuous at $t = x$. If we substitute Eq. (1.192) into Eq. (1.193a) we obtain

$$f(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-n}^{n}\exp\bigl(i\omega(t-x)\bigr)\,d\omega\,dt. \tag{1.193b}$$

Interchanging the order of integration and then taking the limit as $n \to \infty$, we have the Fourier integral theorem, Eq. (15.20).

With the understanding that it belongs under an integral sign, as in Eq. (1.193a), the identification

$$\boxed{\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\bigl(i\omega(t-x)\bigr)\,d\omega} \tag{1.193c}$$

provides a very useful integral representation of the delta function.

When the Laplace transform (see Sections 15.1 and 15.9)

$$L_\delta(s) = \int_0^{\infty}\exp(-st)\delta(t-t_0) = \exp(-st_0), \qquad t_0 > 0 \tag{1.194}$$

is inverted, we obtain the complex representation

$$\delta(t-t_0) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\bigl(s(t-t_0)\bigr)\,ds, \tag{1.195}$$

which is essentially equivalent to the previous Fourier representation of Dirac's delta function.

Exercises

1.15.1 Let

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n}, \\ n, & -\frac{1}{2n} < x < \frac{1}{2n}, \\ 0, & \frac{1}{2n} < x. \end{cases}$$

Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx = f(0),$$

assuming that $f(x)$ is continuous at $x = 0$.

1.15.2 Verify that the sequence $\delta_n(x)$, based on the function

$$\delta_n(x) = \begin{cases} 0, & x < 0, \\ ne^{-nx}, & x > 0, \end{cases}$$

is a delta sequence (satisfying Eq. (1.178)). Note that the singularity is at $+0$, the positive side of the origin.
Hint. Replace the upper limit (∞) by c/n, where c is large but finite, and use the mean value theorem of integral calculus.

1.15.3 For

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1 + n^2x^2},$$

(Eq. (1.174)), show that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1.$$

1.15.4 Demonstrate that $\delta_n = \sin nx/\pi x$ is a delta distribution by showing that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\frac{\sin nx}{\pi x}dx = f(0).$$

Assume that $f(x)$ is continuous at $x = 0$ and vanishes as $x \to \pm\infty$.
Hint. Replace x by y/n and take $\lim n \to \infty$ **before** integrating.

1.15.5 Fejer's method of summing series is associated with the function

$$\delta_n(t) = \frac{1}{2\pi n}\left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^2.$$

Show that $\delta_n(t)$ is a delta distribution, in the sense that

$$\lim_{n\to\infty} \frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t)\left[\frac{\sin(nt/2)}{\sin(t/2)}\right]^2 dt = f(0).$$

1.15.6 Prove that

$$\delta\big[a(x - x_1)\big] = \frac{1}{a}\delta(x - x_1).$$

Note. If $\delta[a(x - x_1)]$ is considered even, relative to x_1, the relation holds for negative a and $1/a$ may be replaced by $1/|a|$.

1.15.7 Show that

$$\delta\big[(x - x_1)(x - x_2)\big] = \big[\delta(x - x_1) + \delta(x - x_2)\big]/|x_1 - x_2|.$$

Hint. Try using Exercise 1.15.6.

1.15.8 Using the Gauss error curve delta sequence ($\delta_n = \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2}$), show that

$$x\frac{d}{dx}\delta(x) = -\delta(x),$$

treating $\delta(x)$ and its derivative as in Eq. (1.179).

1.15.9 Show that

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = -f'(0).$$

Here we assume that $f'(x)$ is continuous at $x = 0$.

1.15.10 Prove that

$$\delta\big(f(x)\big) = \left|\frac{df(x)}{dx}\right|_{x=x_0}^{-1} \delta(x - x_0),$$

where x_0 is chosen so that $f(x_0) = 0$.
Hint. Note that $\delta(f)\, df = \delta(x)\, dx$.

1.15.11 Show that in spherical polar coordinates $(r, \cos\theta, \varphi)$ the delta function $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ becomes

$$\frac{1}{r_1^2}\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\varphi_1 - \varphi_2).$$

Generalize this to the curvilinear coordinates (q_1, q_2, q_3) of Section 2.1 with scale factors h_1, h_2, and h_3.

1.15.12 A rigorous development of Fourier transforms[31] includes as a theorem the relations

$$\lim_{a\to\infty} \frac{2}{\pi}\int_{x_1}^{x_2} f(u + x)\frac{\sin ax}{x}dx$$
$$= \begin{cases} f(u + 0) + f(u - 0), & x_1 < 0 < x_2 \\ f(u + 0), & x_1 = 0 < x_2 \\ f(u - 0), & x_1 < 0 = x_2 \\ 0, & x_1 < x_2 < 0 \text{ or } 0 < x_1 < x_2. \end{cases}$$

Verify these results using the Dirac delta function.

[31] I. N. Sneddon, *Fourier Transforms*. New York: McGraw-Hill (1951).

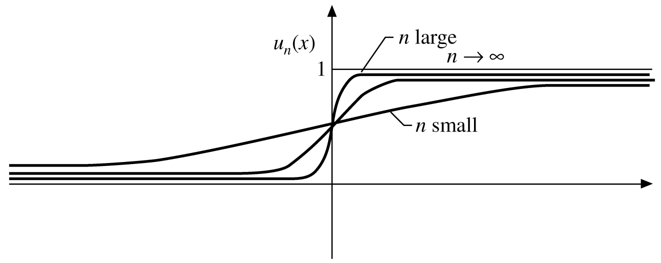

FIGURE 1.41 $\frac{1}{2}[1 + \tanh nx]$ and the Heaviside unit step function.

1.15.13 (a) If we define a sequence $\delta_n(x) = n/(2\cosh^2 nx)$, show that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1, \qquad \text{independent of } n.$$

(b) Continuing this analysis, show that[32]

$$\int_{-\infty}^{x} \delta_n(x)\,dx = \frac{1}{2}[1 + \tanh nx] \equiv u_n(x),$$

$$\lim_{n\to\infty} u_n(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

This is the Heaviside unit step function (Fig. 1.41).

1.15.14 Show that the unit step function $u(x)$ may be represented by

$$u(x) = \frac{1}{2} + \frac{1}{2\pi i} P \int_{-\infty}^{\infty} e^{ixt} \frac{dt}{t},$$

where P means Cauchy principal value (Section 7.1).

1.15.15 As a variation of Eq. (1.175), take

$$\delta_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt - |t|/n}\,dt.$$

Show that this reduces to $(n/\pi)1/(1 + n^2x^2)$, Eq. (1.174), and that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1.$$

Note. In terms of integral transforms, the initial equation here may be interpreted as either a Fourier exponential transform of $e^{-|t|/n}$ or a Laplace transform of e^{ixt}.

[32]Many other symbols are used for this function. This is the AMS-55 (see footnote 4 on p. 330 for the reference) notation: u for unit.

1.15.16 (a) The Dirac delta function representation given by Eq. (1.190),

$$\delta(x-t)=\sum_{n=0}^{\infty}\varphi_n(x)\varphi_n(t),$$

is often called the **closure relation**. For an orthonormal set of real functions, φ_n, show that closure implies completeness, that is, Eq. (1.191) follows from Eq. (1.190).

Hint. One can take

$$F(x)=\int F(t)\delta(x-t)\,dt.$$

(b) Following the hint of part (a) you encounter the integral $\int F(t)\varphi_n(t)\,dt$. How do you know that this integral is finite?

1.15.17 For the finite interval $(-\pi,\pi)$ write the Dirac delta function $\delta(x-t)$ as a series of sines and cosines: $\sin nx, \cos nx, n=0,1,2,\ldots$. Note that although these functions are orthogonal, they are not normalized to unity.

1.15.18 In the interval $(-\pi,\pi)$, $\delta_n(x)=\frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$.

(a) Write $\delta_n(x)$ as a Fourier cosine series.
(b) Show that your Fourier series agrees with a Fourier expansion of $\delta(x)$ in the limit as $n\to\infty$.
(c) Confirm the delta function nature of your Fourier series by showing that for any $f(x)$ that is finite in the interval $[-\pi,\pi]$ and continuous at $x=0$,

$$\int_{-\pi}^{\pi} f(x)\big[\text{Fourier expansion of } \delta_\infty(x)\big]\,dx=f(0).$$

1.15.19 (a) Write $\delta_n(x)=\frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$ in the interval $(-\infty,\infty)$ as a Fourier integral and compare the limit $n\to\infty$ with Eq. (1.193c).
(b) Write $\delta_n(x)=n\exp(-nx)$ as a Laplace transform and compare the limit $n\to\infty$ with Eq. (1.195).

Hint. See Eqs. (15.22) and (15.23) for (a) and Eq. (15.212) for (b).

1.15.20 (a) Show that the Dirac delta function $\delta(x-a)$, expanded in a Fourier sine series in the half-interval $(0,L)$, $(0<a<L)$, is given by

$$\delta(x-a)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Note that this series actually describes

$$-\delta(x+a)+\delta(x-a)\qquad \text{in the interval } (-L,L).$$

(b) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x)=\begin{cases}0, & 0\leqslant x<a\\ 1, & a<x<L,\end{cases}$$

is, for $0 \leqslant x < L$,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

(c) Verify that the term

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) \qquad \text{is} \qquad \langle f(x)\rangle \equiv \frac{1}{L}\int_0^L f(x)\,dx.$$

1.15.21 Verify the Fourier cosine expansion of the square wave, Exercise 1.15.20(b), by direct calculation of the Fourier coefficients.

1.15.22 We may define a sequence

$$\delta_n(x) = \begin{cases} n, & |x| < 1/2n, \\ 0, & |x| > 1/2n. \end{cases}$$

(This is Eq. (1.172).) Express $\delta_n(x)$ as a Fourier integral (via the Fourier integral theorem, inverse transform, etc.). Finally, show that we may write

$$\delta(x) = \lim_{n\to\infty}\delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk.$$

1.15.23 Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp\left(-n^2x^2\right),$$

show that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk.$$

Note. Remember that $\delta(x)$ is defined in terms of its behavior as part of an integrand—especially Eqs. (1.178) and (1.189).

1.15.24 Derive sine and cosine representations of $\delta(t-x)$ that are comparable to the exponential representation, Eq. (1.193c).

ANS. $\frac{2}{\pi}\int_0^{\infty}\sin\omega t\sin\omega x\,d\omega$, $\frac{2}{\pi}\int_0^{\infty}\cos\omega t\cos\omega x\,d\omega$.

1.16 HELMHOLTZ'S THEOREM

In Section 1.13 it was emphasized that the choice of a magnetic vector potential $\mathbf{A}$ was not unique. The divergence of $\mathbf{A}$ was still undetermined. In this section two theorems about the divergence and curl of a vector are developed. The first theorem is as follows:

A vector is uniquely specified by giving its divergence and its curl within a simply connected region (without holes) and its normal component over the boundary.

Note that the subregions, where the divergence and curl are defined (often in terms of Dirac delta functions), are part of our region and are not supposed to be removed here or in Helmholtz's theorem, which follows. Let us take

$$\begin{aligned} \nabla \cdot \mathbf{V}_1 &= s, \\ \nabla \times \mathbf{V}_1 &= \mathbf{c}, \end{aligned} \tag{1.196}$$

where s may be interpreted as a source (charge) density and $\mathbf{c}$ as a circulation (current) density. Assuming also that the normal component V_{1n} on the boundary is given, we want to show that $\mathbf{V}_1$ is unique. We do this by assuming the existence of a second vector, $\mathbf{V}_2$, which satisfies Eq. (1.196) and has the same normal component over the boundary, and then showing that $\mathbf{V}_1 - \mathbf{V}_2 = 0$. Let

$$\mathbf{W} = \mathbf{V}_1 - \mathbf{V}_2.$$

Then

$$\nabla \cdot \mathbf{W} = 0 \tag{1.197}$$

and

$$\nabla \times \mathbf{W} = 0. \tag{1.198}$$

Since $\mathbf{W}$ is irrotational we may write (by Section (1.13))

$$\mathbf{W} = -\nabla\varphi. \tag{1.199}$$

Substituting this into Eq. (1.197), we obtain

$$\nabla \cdot \nabla\varphi = 0, \tag{1.200}$$

Laplace's equation.

Now we draw upon Green's theorem in the form given in Eq. (1.105), letting u and v each equal φ. Since

$$W_n = V_{1n} - V_{2n} = 0 \tag{1.201}$$

on the boundary, Green's theorem reduces to

$$\int_V (\nabla\varphi) \cdot (\nabla\varphi)\, d\tau = \int_V \mathbf{W} \cdot \mathbf{W}\, d\tau = 0. \tag{1.202}$$

The quantity $\mathbf{W} \cdot \mathbf{W} = W^2$ is nonnegative and so we must have

$$\mathbf{W} = \mathbf{V}_1 - \mathbf{V}_2 = 0 \tag{1.203}$$

everywhere. Thus $\mathbf{V}_1$ is unique, proving the theorem.

For our magnetic vector potential $\mathbf{A}$ the relation $\mathbf{B} = \nabla \times \mathbf{A}$ specifies the curl of $\mathbf{A}$. Often for convenience we set $\nabla \cdot \mathbf{A} = 0$ (compare Exercise 1.14.4). Then (with boundary conditions) $\mathbf{A}$ is fixed.

This theorem may be written as a uniqueness theorem for solutions of Laplace's equation, Exercise 1.16.1. In this form, this uniqueness theorem is of great importance in solving electrostatic and other Laplace equation boundary value problems. If we can find a solution of Laplace's equation that satisfies the necessary boundary conditions, then our solution is the complete solution. Such boundary value problems are taken up in Sections 12.3 and 12.5.

Helmholtz's Theorem

The second theorem we shall prove is Helmholtz's theorem.

A vector **V** *satisfying Eq.* (1.196) *with both source and circulation densities vanishing at infinity may be written as the sum of two parts, one of which is irrotational, the other of which is solenoidal*.

Note that our region is simply connected, being all of space, for simplicity. Helmholtz's theorem will clearly be satisfied if we may write **V** as

$$\mathbf{V} = -\nabla\varphi + \nabla \times \mathbf{A}, \tag{1.204a}$$

$-\nabla\varphi$ being irrotational and $\nabla \times A$ being solenoidal. We proceed to justify Eq. (1.204a).

V is a known vector. We take the divergence and curl

$$\nabla \cdot \mathbf{V} = s(\mathbf{r}) \tag{1.204b}$$

$$\nabla \times \mathbf{V} = \mathbf{c}(\mathbf{r}) \tag{1.204c}$$

with $s(\mathbf{r})$ and $\mathbf{c}(\mathbf{r})$ now known functions of position. From these two functions we construct a scalar potential $\varphi(\mathbf{r}_1)$,

$$\varphi(\mathbf{r}_1) = \frac{1}{4\pi}\int \frac{s(\mathbf{r}_2)}{r_{12}} d\tau_2, \tag{1.205a}$$

and a vector potential $\mathbf{A}(\mathbf{r}_1)$,

$$\mathbf{A}(\mathbf{r}_1) = \frac{1}{4\pi}\int \frac{\mathbf{c}(\mathbf{r}_2)}{r_{12}} d\tau_2. \tag{1.205b}$$

If $s = 0$, then **V** is solenoidal and Eq. (1.205a) implies $\varphi = 0$. From Eq. (1.204a), $\mathbf{V} = \nabla \times \mathbf{A}$, with **A** as given in Eq. (1.141), which is consistent with Section 1.13. Further, if $\mathbf{c} = 0$, then **V** is irrotational and Eq. (1.205b) implies $\mathbf{A} = 0$, and Eq. (1.204a) implies $\mathbf{V} = -\nabla\varphi$, consistent with scalar potential theory of Section 1.13.

Here the argument $\mathbf{r}_1$ indicates (x_1, y_1, z_1), the field point; $\mathbf{r}_2$, the coordinates of the source point (x_2, y_2, z_2), whereas

$$r_{12} = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{1/2}. \tag{1.206}$$

When a direction is associated with r_{12}, the positive direction is taken to be away from the source and toward the field point. Vectorially, $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, as shown in Fig. 1.42. Of course, s and **c** must vanish sufficiently rapidly at large distance so that the integrals exist. The actual expansion and evaluation of integrals such as Eqs. (1.205a) and (1.205b) is treated in Section 12.1.

From the uniqueness theorem at the beginning of this section, **V** is uniquely specified by its divergence, s, and curl, **c** (and boundary conditions). Returning to Eq. (1.204a), we have

$$\nabla \cdot \mathbf{V} = -\nabla \cdot \nabla\varphi, \tag{1.207a}$$

the divergence of the curl vanishing, and

$$\nabla \times \mathbf{V} = \nabla \times (\nabla \times \mathbf{A}), \tag{1.207b}$$

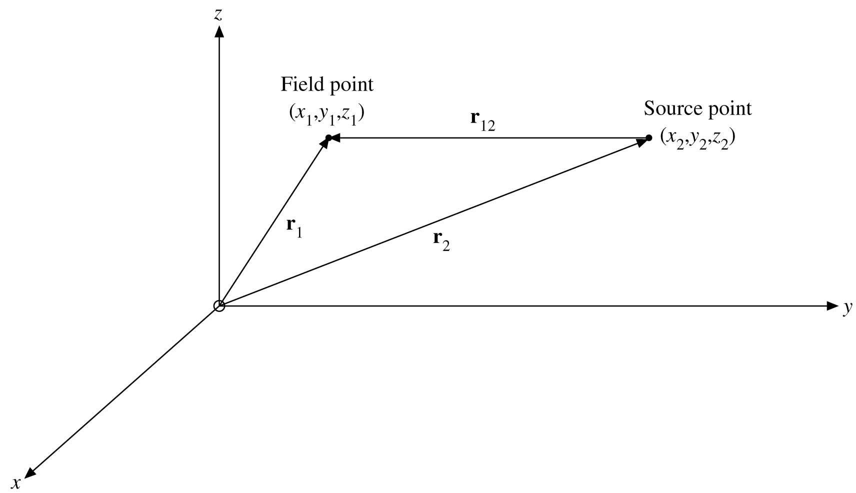

FIGURE 1.42 Source and field points.

the curl of the gradient vanishing. If we can show that

$$-\nabla \cdot \nabla\varphi(\mathbf{r}_1) = s(\mathbf{r}_1) \tag{1.207c}$$

and

$$\nabla \times \big(\nabla \times \mathbf{A}(\mathbf{r}_1)\big) = \mathbf{c}(\mathbf{r}_1), \tag{1.207d}$$

then $\mathbf{V}$ as given in Eq. (1.204a) will have the proper divergence and curl. Our description will be internally consistent and Eq. (1.204a) justified.[33]

First, we consider the divergence of $\mathbf{V}$:

$$\nabla \cdot \mathbf{V} = -\nabla \cdot \nabla\varphi = -\frac{1}{4\pi}\nabla \cdot \nabla \int \frac{s(\mathbf{r}_2)}{r_{12}} d\tau_2. \tag{1.208}$$

The Laplacian operator, $\nabla \cdot \nabla$, or ∇^2, operates on the field coordinates (x_1, y_1, z_1) and so commutes with the integration with respect to (x_2, y_2, z_2). We have

$$\nabla \cdot \mathbf{V} = -\frac{1}{4\pi}\int s(\mathbf{r}_2)\nabla_1^2\left(\frac{1}{r_{12}}\right) d\tau_2. \tag{1.209}$$

We must make two minor modifications in Eq. (1.169) before applying it. First, our source is at r_2, not at the origin. This means that a nonzero result from Gauss' law appears if and only if the surface S includes the point $\mathbf{r} = \mathbf{r}_2$. To show this, we rewrite Eq. (1.170):

$$\nabla^2\left(\frac{1}{r_{12}}\right) = -4\pi\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.210}$$

[33] Alternatively, we could solve Eq. (1.207c), Poisson's equation, and compare the solution with the constructed potential, Eq. (1.205a). The solution of Poisson's equation is developed in Section 9.7.

This shift of the source to $\mathbf{r}_2$ may be incorporated in the defining equation (1.171b) as

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = 0, \qquad \mathbf{r}_1 \neq \mathbf{r}_2, \tag{1.211a}$$

$$\int f(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)\, d\tau_1 = f(\mathbf{r}_2). \tag{1.211b}$$

Second, noting that differentiating r_{12}^{-1} twice with respect to x_2, y_2, z_2 is the same as differentiating **twice** with respect to x_1, y_1, z_1, we have

$$\begin{aligned}\nabla_1^2\left(\frac{1}{r_{12}}\right) = \nabla_2^2\left(\frac{1}{r_{12}}\right) &= -4\pi\,\delta(\mathbf{r}_1 - \mathbf{r}_2)\\ &= -4\pi\,\delta(\mathbf{r}_2 - \mathbf{r}_1).\end{aligned} \tag{1.212}$$

Rewriting Eq. (1.209) and using the Dirac delta function, Eq. (1.212), we may integrate to obtain

$$\begin{aligned}\nabla \cdot \mathbf{V} &= -\frac{1}{4\pi}\int s(\mathbf{r}_2)\nabla_2^2\left(\frac{1}{r_{12}}\right) d\tau_2\\ &= -\frac{1}{4\pi}\int s(\mathbf{r}_2)(-4\pi)\delta(\mathbf{r}_2 - \mathbf{r}_1)\, d\tau_2\\ &= s(\mathbf{r}_1).\end{aligned} \tag{1.213}$$

The final step follows from Eq. (1.211b), with the subscripts 1 and 2 exchanged. Our result, Eq. (1.213), shows that the assumed forms of $\mathbf{V}$ and of the scalar potential φ are in agreement with the given divergence (Eq. (1.204b)).

To complete the proof of Helmholtz's theorem, we need to show that our assumptions are consistent with Eq. (1.204c), that is, that the curl of $\mathbf{V}$ is equal to $\mathbf{c}(\mathbf{r}_1)$. From Eq. (1.204a),

$$\begin{aligned}\nabla \times \mathbf{V} &= \nabla \times (\nabla \times \mathbf{A})\\ &= \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}.\end{aligned} \tag{1.214}$$

The first term, $\nabla\nabla \cdot \mathbf{A}$, leads to

$$4\pi\,\nabla\nabla \cdot \mathbf{A} = \int \mathbf{c}(\mathbf{r}_2) \cdot \nabla_1\nabla_1\left(\frac{1}{r_{12}}\right) d\tau_2 \tag{1.215}$$

by Eq. (1.205b). Again replacing the second derivatives with respect to x_1, y_1, z_1 by second derivatives with respect to x_2, y_2, z_2, we integrate each component[34] of Eq. (1.215) by parts:

$$\begin{aligned}4\pi\,\nabla\nabla \cdot \mathbf{A}|_x &= \int \mathbf{c}(\mathbf{r}_2) \cdot \nabla_2 \frac{\partial}{\partial x_2}\left(\frac{1}{r_{12}}\right) d\tau_2\\ &= \int \nabla_2 \cdot \left[\mathbf{c}(\mathbf{r}_2)\frac{\partial}{\partial x_2}\left(\frac{1}{r_{12}}\right)\right] d\tau_2\\ &\quad - \int \left[\nabla_2 \cdot \mathbf{c}(\mathbf{r}_2)\right]\frac{\partial}{\partial x_2}\left(\frac{1}{r_{12}}\right) d\tau_2.\end{aligned} \tag{1.216}$$

[34] This avoids creating the **tensor** $\mathbf{c}(\mathbf{r}_2)\nabla_2$.

The second integral vanishes because the circulation density $\mathbf{c}$ is solenoidal.[35] The first integral may be transformed to a surface integral by Gauss' theorem. If $\mathbf{c}$ is bounded in space or vanishes faster that $1/r$ for large r, so that the integral in Eq. (1.205b) exists, then by choosing a sufficiently large surface the first integral on the right-hand side of Eq. (1.216) also vanishes.

With $\nabla\nabla\cdot\mathbf{A} = 0$, Eq. (1.214) now reduces to

$$\nabla\times\mathbf{V} = -\nabla^2\mathbf{A} = -\frac{1}{4\pi}\int \mathbf{c}(\mathbf{r}_2)\nabla_1^2\left(\frac{1}{r_{12}}\right)d\tau_2. \qquad (1.217)$$

This is exactly like Eq. (1.209) except that the scalar $s(\mathbf{r}_2)$ is replaced by the vector circulation density $\mathbf{c}(\mathbf{r}_2)$. Introducing the Dirac delta function, as before, as a convenient way of carrying out the integration, we find that Eq. (1.217) reduces to Eq. (1.196). We see that our assumed forms of $\mathbf{V}$, given by Eq. (1.204a), and of the vector potential $\mathbf{A}$, given by Eq. (1.205b), are in agreement with Eq. (1.196) specifying the curl of $\mathbf{V}$.

This completes the proof of Helmholtz's theorem, showing that a vector may be resolved into irrotational and solenoidal parts. Applied to the electromagnetic field, we have resolved our field vector $\mathbf{V}$ into an irrotational electric field $\mathbf{E}$, derived from a scalar potential φ, and a solenoidal magnetic induction field $\mathbf{B}$, derived from a vector potential $\mathbf{A}$. The source density $s(\mathbf{r})$ may be interpreted as an electric charge density (divided by electric permittivity ε), whereas the circulation density $\mathbf{c}(\mathbf{r})$ becomes electric current density (times magnetic permeability μ).

Exercises

1.16.1 Implicit in this section is a proof that a function $\psi(\mathbf{r})$ is uniquely specified by requiring it to (1) satisfy Laplace's equation and (2) satisfy a complete set of boundary conditions. Develop this proof explicitly.

1.16.2 (a) Assuming that $\mathbf{P}$ is a solution of the vector Poisson equation, $\nabla_1^2\mathbf{P}(\mathbf{r}_1) = -\mathbf{V}(\mathbf{r}_1)$, develop an alternate proof of Helmholtz's theorem, showing that $\mathbf{V}$ may be written as

$$\mathbf{V} = -\nabla\varphi + \nabla\times\mathbf{A},$$

where

$$\mathbf{A} = \nabla\times\mathbf{P},$$

and

$$\varphi = \nabla\cdot\mathbf{P}.$$

(b) Solving the vector Poisson equation, we find

$$\mathbf{P}(\mathbf{r}_1) = \frac{1}{4\pi}\int_V \frac{\mathbf{V}(\mathbf{r}_2)}{r_{12}}d\tau_2.$$

Show that this solution substituted into φ and $\mathbf{A}$ of part (a) leads to the expressions given for φ and $\mathbf{A}$ in Section 1.16.

[35] Remember, $\mathbf{c} = \nabla\times\mathbf{V}$ is known.

Additional Readings

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