Chapter 5
Shape Operators

In Chapter 2 we measured the shape of a curve in $\mathbf{R}^3$ by its curvature and torsion functions. Now we consider the analogous measurement problem for surfaces. It turns out that the shape of a surface M in $\mathbf{R}^3$ is described infinitesimally by a certain linear operator S defined on each of the tangent planes of M. As with curves, to say that two surfaces in $\mathbf{R}^3$ have the same shape means simply that they are congruent. And just as with curves, we shall justify our infinitesimal measurements by proving that two surfaces with "the same" shape operators are, in fact, congruent. The *algebraic* invariants (determinant, trace, . . .) of its shape operators thus have *geometric* meaning for the surface M. We investigate this matter in detail and find efficient ways to compute these invariants, which we test on a number of geometrically interesting surfaces.

From now on, the notation $M \subset \mathbf{R}^3$ means a connected surface M in $\mathbf{R}^3$ as defined in Chapter 4.

5.1 The Shape Operator of $M \subset \mathbf{R}^3$

Suppose that Z is a Euclidean vector field (Definition 3.7 of Chapter 4) on a surface M in $\mathbf{R}^3$. Although Z is defined only at points of M, the covariant derivative $\nabla_v Z$ (Chapter 2, Section 5) still makes sense *as long as* **v** *is tangent to* M. As usual, $\nabla_v Z$ is the rate of change of Z in the **v** direction, and there are two main ways to compute it.

Method 1. Let α be a curve in M that has initial velocity $\alpha'(0) = \mathbf{v}$. Let Z_α be the restriction of Z to α, that is, the vector field $t \to Z(\alpha(t))$ on α. Then

$$\nabla_v Z = (Z_\alpha)'(0),$$

where the derivative is that of Chapter 2, Section 2.

Method 2. Express Z in terms of the natural frame field of $\mathbf{R}^3$ by

$$Z = \sum z_i U_i.$$

Then

$$\nabla_v Z = \sum \mathbf{v}[z_i] U_i,$$

where the directional derivative is that of Definition 3.10 in Chapter 4.

It is easy to show that these two methods give the same result. In fact, since $Z = \sum z_i U_i$,

$$(Z_\alpha)'(0) = \sum z_i(\alpha)'(0) U_i = \sum \mathbf{v}[z_i] U_i.$$

(Compare Lemma 5.2 of Chapter 2.) Note that even if Z is a tangent vector field, the covariant derivative $\nabla_v Z$ need not be tangent to M.

If M is an *orientable* surface, Proposition 7.5 of Chapter 4 shows that there is always a (differentiable) unit normal vector field U on the entire surface, and in fact—since M is now assumed connected—there are exactly two, $\pm U$. Even if M is not orientable, unit normals $\pm U$ are available *locally*, since a small region around any point is orientable. In fact, we will find explicit formulas for U on the image $\mathrm{x}(D) \subset M$ of any patch.

We are now in a position to find a mathematical measurement of the shape of a surface in $\mathbf{R}^3$.

1.1 Definition If $\mathbf{p}$ is a point of M, then for each tangent vector $\mathbf{v}$ to M at $\mathbf{p}$, let

$$S_p(\mathbf{v}) = -\nabla_v U,$$

where U is a unit normal vector field on a neighborhood of $\mathbf{p}$ in M. S_p is called the *shape operator* of M at $\mathbf{p}$ derived from U.† (Fig. 5.1.)

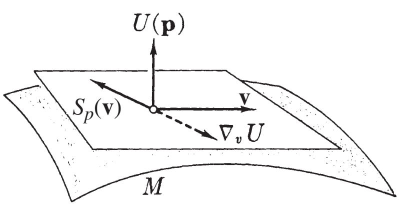

FIG. 5.1

† The minus sign artificially introduced in this definition will sharply reduce the total number of minus signs needed later on.

The tangent plane of M at any point $\mathbf{q}$ consists of all Euclidean vectors orthogonal to $U(\mathbf{q})$. Thus the rate of change $\nabla_v U$ of U in the $\mathbf{v}$ direction tells how the tangent planes of M are varying in the $\mathbf{v}$ direction—and this gives an infinitesimal description of the way M itself is curving in $\mathbf{R}^3$.

Note that if U is replaced by $-U$, then S_p changes to $-S_p$.

1.2 Lemma For each point $\mathbf{p}$ of $M \subset \mathbf{R}^3$, the shape operator is a linear operator

$$S_p\colon T_p(M) \to T_p(M)$$

on the tangent plane of M at $\mathbf{p}$.

Proof. In Definition 1.1, U is a unit vector field, so $U \bullet U = 1$. Thus by a Leibnizian property of covariant derivatives,

$$0 = \mathbf{v}[U \bullet U] = 2\nabla_v U \bullet U(\mathbf{p}) = -2S_p(\mathbf{v}) \bullet U(\mathbf{p}),$$

where $\mathbf{v}$ is tangent to M at $\mathbf{p}$. Since U is also a *normal* vector field, it follows that $S_p(\mathbf{v})$ is tangent to M at $\mathbf{p}$. Thus S_p is a function from $T_p(M)$ to $T_p(M)$. (It is to emphasize this that we use the term "operator" instead of "transformation.")

The linearity of S_p is a consequence of a linearity property of covariant derivatives:

$$\begin{aligned} S_p(a\mathbf{v} + b\mathbf{w}) &= -\nabla_{av+bw} U = -(a\nabla_v U + b\nabla_w U) \\ &= aS_p(\mathbf{v}) + bS_p(\mathbf{w}). \end{aligned}$$

◆

At each point $\mathbf{p}$ of $M \subset \mathbf{R}^3$ there are actually two shape operators, $\pm S_p$, derived from the two unit normals $\pm U$ near $\mathbf{p}$. We shall refer to all of these, collectively, as *the shape operator S of M*. Thus if a choice of unit normal is not specified, there is a relatively harmless ambiguity of sign.

1.3 Example Shape operators of some surfaces in $\mathbf{R}^3$.

(1) Let Σ be the sphere of radius r consisting of all points $\mathbf{p}$ of $\mathbf{R}^3$ with $\| \mathbf{p} \| = r$. Let U be the outward normal on Σ. Now as U moves away from any point $\mathbf{p}$ in the direction $\mathbf{v}$, evidently U topples forward in the exact direction of $\mathbf{v}$ itself (Fig. 5.2). Thus $S(\mathbf{v})$ must have the form $-c\mathbf{v}$.

In fact, using *gradients* as in Example 3.9 of Chapter 4, we find

$$U = \frac{1}{r}\sum x_i U_i.$$

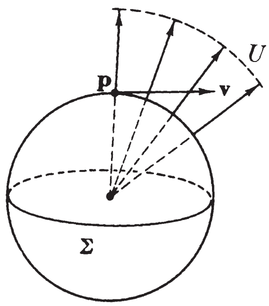

FIG. 5.2

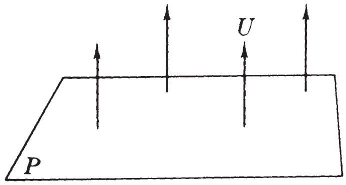

FIG. 5.3

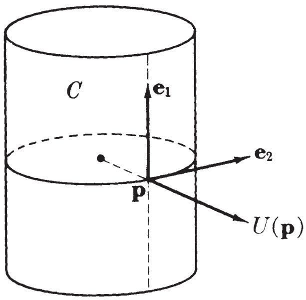

FIG. 5.4

But then

$$\nabla_v U = \frac{1}{r}\sum \mathbf{v}[x_i]U_i(\mathbf{p}) = -\frac{\mathbf{v}}{r}.$$

Thus $S(\mathbf{v}) = -\mathbf{v}/r$ for all $\mathbf{v}$. So the shape operator S is merely scalar multiplication by $-1/r$. This uniformity in S reflects the roundness of spheres: They bend the same way in all directions at all points.

(2) Let P be a plane in $\mathbf{R}^3$. A unit normal vector field U on P is evidently *parallel* in $\mathbf{R}^3$ (constant Euclidean coordinates) (Fig. 5.3). Hence

$$S(\mathbf{v}) = -\nabla_v U = 0$$

for all tangent vectors $\mathbf{v}$ to P. Thus the shape operator is identically zero, which is to be expected, since planes do not bend at all.

(3) Let C be the circular cylinder $x^2 + y^2 = r^2$ in $\mathbf{R}^3$. At any point $\mathbf{p}$ of C, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit tangent vectors, with $\mathbf{e}_1$ tangent to the ruling of the cylinder through $\mathbf{p}$, and $\mathbf{e}_2$ tangent to the cross-sectional circle. Use the outward normal U as indicated in Fig. 5.4.

Now, when U moves from $\mathbf{p}$ in the $\mathbf{e}_1$ direction, it stays parallel to itself just as on a plane; hence $S(\mathbf{e}_1) = 0$. When U moves in the $\mathbf{e}_2$ direction, it topples forward exactly as on a sphere of radius r; hence $S(\mathbf{e}_2) = -\mathbf{e}_2/r$. In this way S describes the "half-flat, half-round" shape of a cylinder.

(4) The *saddle surface M: z = xy*. For the moment we investigate S only at $\mathbf{p} = (0, 0, 0)$ in M. Since the x and y axes of $\mathbf{R}^3$ lie in M, the vectors

$$\mathbf{u}_1 = (1, 0, 0) \text{ and } \mathbf{u}_2 = (0, 1, 0)$$

are tangent to M at $\mathbf{p}$. We use the "upward" unit normal U, which at $\mathbf{p}$ is $(0, 0, 1)$. Along the x axis, U stays orthogonal to the x axis, and as it proceeds in the $\mathbf{u}_1$ direction, U swings from left to right (Fig. 5.5).

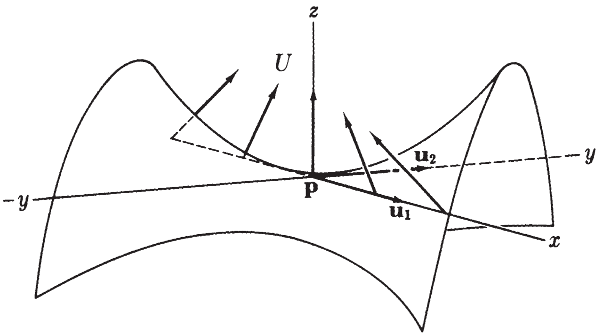

FIG. 5.5

In fact, a routine computation (Exercise 3) shows that $\nabla_{u_1} U = -\mathbf{u}_2$. Similarly, we find $\nabla_{\mathbf{u}_2} U = -\mathbf{u}_1$. Thus the shape operator of M at $\mathbf{p}$ is given by the formula

$$S(a\mathbf{u}_1 + b\mathbf{u}_2) = b\mathbf{u}_1 + a\mathbf{u}_2.$$

These examples clarify the analogy between the shape operator of a surface and the curvature and torsion of a curve. In the case of a curve, there is only one direction to move, and κ and τ measure the rate of change of the unit vector fields T and B (hence N). For a surface, only one unit vector field is intrinsically determined—the unit normal U. Furthermore, at each point, there is now a whole plane of directions in which U can move, so that rates of change of U are measured, not numerically, but by the linear operators S.

1.4 Lemma For each point $\mathbf{p}$ of $M \subset \mathbf{R}^3$, the shape operator

$$S: T_p(M) \to T_p(M)$$

is a *symmetric* linear operator; that is,

$$S(\mathbf{v}) \bullet \mathbf{w} = S(\mathbf{w}) \bullet \mathbf{v}$$

for any pair of tangent vectors to M at $\mathbf{p}$.

We postpone the proof of this crucial fact to Section 4, where it occurs naturally in the course of general computations.

From the viewpoint of linear algebra, a symmetric linear operator on a two-dimensional vector space is a very simple object indeed. For a shape operator, its eigenvalues and eigenvectors, its trace and determinant, all turn out to have geometric meaning of first importance for the surface $M \subset \mathbf{R}^3$.

Exercises

1. Let α be a curve in $M \subset \mathbf{R}^3$. If U is a unit normal of M restricted to the curve α, show that $S(\alpha') = -U'$.

2. Consider the surface M: $z = f(x, y)$, where

$$f(0,0) = f_x(0,0) = f_y(0,0) = 0.$$

(The subscripts indicate partial derivatives.) Show that

(a) The vectors $\mathbf{u}_1 = U_1(\mathbf{0})$ and $\mathbf{u}_2 = U_2(\mathbf{0})$ are tangent to M at the origin $\mathbf{0}$, and

$$U = \frac{-f_x U_1 - f_y U_2 + U_3}{\sqrt{1 + f_x^2 + f_y^2}}$$

is a unit normal vector field on M.

(b) $S(\mathbf{u}_1) = f_{xx}(0, 0)\mathbf{u}_1 + f_{xy}(0, 0)\mathbf{u}_2$,
$S(\mathbf{u}_2) = f_{yx}(0, 0)\mathbf{u}_1 + f_{yy}(0, 0)\mathbf{u}_2$.

(*Note:* The square root in the denominator is no real problem here because of the special character of f at (0, 0). In general, direct computation of S is awkward, and in Section 4 we shall establish indirect ways of getting at it.)

3. (*Continuation.*) In each case, express $S(a\mathbf{u}_1 + b\mathbf{u}_2)$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$, and determine the rank of S at $\mathbf{0}$ (rank S is dimension of image S: 0, 1, or 2).

(a) $z = xy$.
(b) $z = 2x^2 + y^2$.
(c) $z = (x + y)^2$.
(d) $z = xy^2$.

4. Let M be a surface in $\mathbf{R}^3$ oriented by a unit normal vector field

$$U = g_1 U_1 + g_2 U_2 + g_3 U_3.$$

Then the *Gauss map* G: $M \to \Sigma$ of M sends each point $\mathbf{p}$ of M to the point $(g_1(\mathbf{p}), g_2(\mathbf{p}), g_3(\mathbf{p}))$ of the unit sphere Σ. Pictorially: Move $U(\mathbf{p})$ to the origin by parallel motion; there it points to $G(\mathbf{p})$ (Fig. 5.6).

Thus G completely describes the turning of U as it traverses M.

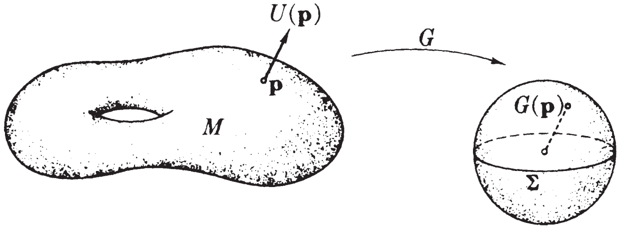

FIG. 5.6

For each of the following surfaces, describe the image $G(M)$ of the Gauss map in the sphere Σ (use either normal):

(a) Cylinder, $x^2 + y^2 = r^2$.

(b) Cone, $z = \sqrt{x^2 + y^2}$.

(c) Plane, $x + y + z = 0$.

(d) Sphere, $(x - 1)^2 + y^2 + (z + 2)^2 = 1$.

5. Let G: $T \to \Sigma$ be the Gauss map of the torus T (Example 2.5 of Ch. 4) derived from its outward unit normal U. What are the image curves under G of the meridians and parallels of T? Which points of Σ are the image of exactly two points of T?

6. Let G: $M \to \Sigma$ be the Gauss map of the saddle surface M: $z = xy$ derived from the unit normal U obtained as in Exercise 2. What is the image under G of one of the straight lines, $y =$ constant, in M? How much of the sphere is covered by the entire image $G(M)$?

7. Show that the shape operator of M is (minus) the tangent map of its Gauss map: If S and G: $M \to \Sigma$ are both derived from U, then $S(\mathbf{v})$ and $-G_*(\mathbf{v})$ are parallel for every tangent vector $\mathbf{v}$ to M.

8. An orientable surface has two Gauss maps derived from its two unit normals. Show that they differ only by the antipodal mapping of Σ (Example 8.2 of Ch. 4). Define a Gauss-type mapping for a nonorientable surface in $\mathbf{R}^3$.

9. If V is a tangent vector field on M (with unit normal U), then $S(V)$ is the tangent vector field on M whose value at each point $\mathbf{p}$ is $S_p(V(\mathbf{p}))$. Show that if W is also tangent to M, then

$$S(V) \bullet W = \nabla_V W \bullet U.$$

Deduce that the symmetry of S is equivalent to the assertion that the *bracket*

$$[V, W] = \nabla_V W - \nabla_W V$$

of two tangent vector fields is again a tangent vector field.

5.2 Normal Curvature

Throughout this section we shall work in a region of $M \subset \mathbf{R}^3$ that has been *oriented* by the choice of a unit normal vector field U, and we use the shape operator S derived from U.

The shape of a surface in $\mathbf{R}^3$ influences the shape of the curves in M.

2.1 Lemma If α is a curve in $M \subset \mathbf{R}^3$, then

$$\alpha'' \bullet U = S(\alpha') \bullet \alpha'.$$

Proof. Since α is in M, its velocity α' is always tangent to M. Thus

$$\alpha' \bullet U = 0,$$

where U is restricted to the curve α. Differentiation yields

$$\alpha'' \bullet U + \alpha' \bullet U' = 0.$$

But from Section 1, we know that $S(\alpha') = -U'$. Hence

$$\alpha'' \bullet U = -U' \bullet \alpha' = S(\alpha') \bullet \alpha' \qquad \blacklozenge$$

Geometric interpretation: at each point, $\alpha'' \cdot U$ is the component of acceleration α'' normal to the surface M (Fig. 5.7). The lemma shows that this component depends only on the velocity α' and the shape operator of M. Thus *all curves in M with a given velocity* **v** *at point* **p** *will have the same normal component of acceleration at* **p**, *namely*, $S(\mathbf{v}) \bullet \mathbf{v}$. This is the component of acceleration that *the bending of M in* $\mathbf{R}^3$ *forces them to have*.

Thus if **v** is standardized by reducing it to a unit vector **u**, we get a measurement of the way M is bent in the **u** direction.

2.2 Definition Let **u** be a unit vector tangent to $M \subset \mathbf{R}^3$ at a point **p**. Then the number $k(\mathbf{u}) = S(\mathbf{u}) \bullet \mathbf{u}$ is called the *normal curvature of M in the* **u** *direction*.

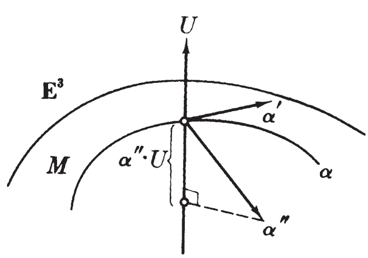

FIG. 5.7

In this context, we define a *tangent direction to M at* $\mathbf{p}$ to be a one-dimensional subspace L of $T_p(M)$, that is, a line through the zero vector (located for intuitive purposes at $\mathbf{p}$). Any nonzero tangent vector at $\mathbf{p}$ determines a direction L, but we prefer to use one of the two unit vectors $\pm\mathbf{u}$ in L. Note that

$$k(\mathbf{u}) = S(\mathbf{u}) \bullet \mathbf{u} = S(-\mathbf{u}) \bullet (-\mathbf{u}) = k(-\mathbf{u}).$$

Thus, although we evaluate k on unit vectors, it is, in effect, a real-valued function on the set of all tangent directions to M.

Given a unit tangent vector $\mathbf{u}$ to M at $\mathbf{p}$, let α be a unit-speed curve in M with initial velocity $\alpha'(0) = \mathbf{u}$. Using the Frenet apparatus of α, the preceding lemma gives

$$\begin{aligned} k(\mathbf{u}) &= S(\mathbf{u}) \bullet \mathbf{u} = \alpha''(0) \bullet U(\mathbf{p}) = \kappa(0)N(0) \bullet U(\mathbf{p}) \\ &= \kappa(0)\cos\vartheta. \end{aligned}$$

Thus the normal curvature of M in the $\mathbf{u}$ direction is $\kappa(0)\cos\vartheta$, where $\kappa(0)$ is the curvature of α at $\alpha(0) = \mathbf{p}$, and ϑ is the angle between the principal normal $N(0)$ and the surface normal $U(\mathbf{p})$, as in Fig. 5.8.

Given $\mathbf{u}$, there is a natural way to choose the curve so that ϑ is 0 or π. In fact, if P is the plane determined by $\mathbf{u}$ and $U(\mathbf{p})$, then P cuts from M (near $\mathbf{p}$) a curve σ called the *normal section* of M in the $\mathbf{u}$ direction. If we give σ unit-speed parametrization with $\sigma'(0) = \mathbf{u}$, then $N(0) = \pm U(p)$, since

$$\sigma''(0) = \kappa(0)N(0)$$

is orthogonal to $\sigma'(0) = \mathbf{u}$ and tangent to P. So for a normal section in the $\mathbf{u}$ direction (Fig. 5.9),

$$k(\mathbf{u}) = \kappa_\sigma(0)N(0) \bullet U(\mathbf{p}) = \pm\kappa_\sigma(0).$$

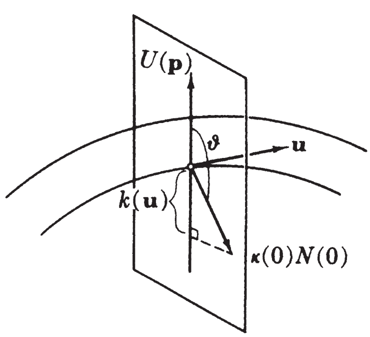

FIG. 5.8

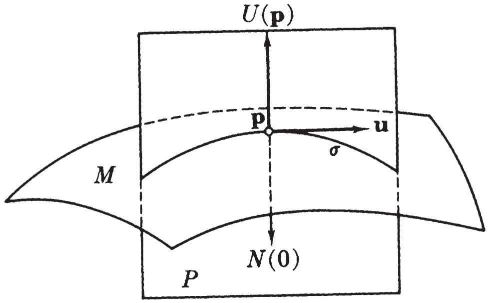

FIG. 5.9

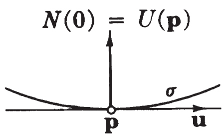

FIG. 5.10

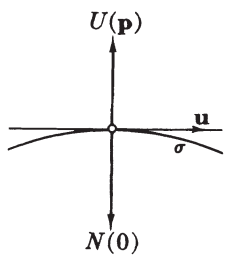

FIG. 5.11

Thus it is possible to make a reasonable estimate of the normal curvatures in various directions on a surface $M \subset \mathbf{R}^3$ by picturing what the corresponding normal sections would look like. We know that the principal normal N of a curve tells in which direction it is turning. Thus the preceding discussion gives geometric meaning to the sign of the normal curvature $k(\mathbf{u})$ (relative to our fixed choice of U).

(1) If $k(\mathbf{u}) > 0$, then $N(0) = U(\mathbf{p})$, so the normal section σ is bending toward $U(\mathbf{p})$ at $\mathbf{p}$ (Fig. 5.10). Thus in the $\mathbf{u}$ direction the surface M is bending *toward* $U(\mathbf{p})$.

(2) If $k(\mathbf{u}) < 0$, then $N(0) = -U(\mathbf{p})$, so the normal section σ is bending *away from* $U(\mathbf{p})$ at $\mathbf{p}$. Thus in the $\mathbf{u}$ direction M is bending away from $U(\mathbf{p})$ (Fig. 5.11).

(3) If $k(\mathbf{u}) = 0$, then $k_\sigma(0) = 0$ and $N(0)$ is undefined. Here the normal section σ is not turning at $\sigma(0) = \mathbf{p}$. We cannot conclude that in the $\mathbf{u}$ direction M is not bending at all, since κ might be zero only at $\sigma(0) = \mathbf{p}$. But we can conclude that its rate of bending is unusually small.

In different directions at a fixed point $\mathbf{p}$, the surface may bend in quite different ways. For example, consider the saddle surface $z = xy$ in Example 1.3. If we identify the tangent plane of M at $\mathbf{p} = (0, 0, 0)$ with the xy plane of

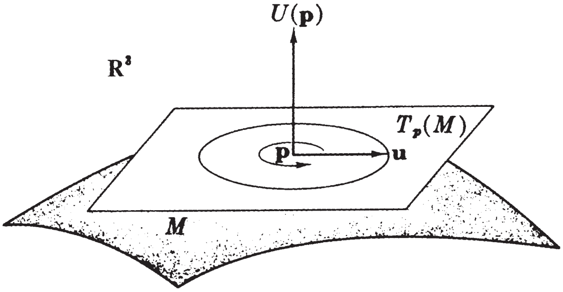

FIG. 5.12

$\mathbf{R}^3$, then clearly the normal curvature in the direction of the x and y axes is zero, since the normal sections are straight lines. However, Fig. 5.5 shows that in the tangent direction given by the line $y = x$, the normal curvature is positive, for the normal section is a parabola bending upward. ($U(\mathbf{p}) = (0, 0, 1)$ is "upward.") But in the direction of the line $y = -x$, normal curvature is negative, since this parabola bends downward.

Let us now fix a point $\mathbf{p}$ of $M \subset \mathbf{R}^3$ and imagine that a unit tangent vector $\mathbf{u}$ at $\mathbf{p}$ revolves, sweeping out the unit circle in the tangent plane $T_p(M)$. From the corresponding normal sections, we get a moving picture of the way M is bending in *every* direction at $\mathbf{p}$ (Fig. 5.12).

2.3 Definition Let $\mathbf{p}$ be a point of $M \subset \mathbf{R}^3$. The maximum and minimum values of the normal curvature $k(\mathbf{u})$ of M at $\mathbf{p}$ are called the *principal curvatures* of M at $\mathbf{p}$, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called *principal directions* of M at $\mathbf{p}$. Unit vectors in these directions are called *principal vectors* of M at $\mathbf{p}$.

Using the normal-section scheme discussed above, it is often fairly easy to pick out the directions of maximum and minimum bending. For example, if we use the outward normal (U) on a circular cylinder C as in Fig. 5.4, then the normal sections of C all bend away from U, so $k(\mathbf{u}) \leqq 0$. Furthermore, it is reasonably clear that the maximum value $k_1 = 0$ occurs only in the direction $\mathbf{e}_1$ of a ruling; minimum value $k_2 < 0$ occurs only in the direction $\mathbf{e}_2$ tangent to a cross-section.

An interesting special case occurs at points $\mathbf{p}$ for which $k_1 = k_2$. The maximum and minimum normal curvature being equal, it follows that $k(\mathbf{u})$ is constant: *M bends the same amount in all directions at* $\mathbf{p}$ (so all directions are principal).

2.4 Definition A point $\mathbf{p}$ of $M \subset \mathbf{R}^3$ is *umbilic* provided the normal curvature $k(\mathbf{u})$ is constant on all unit tangent vectors $\mathbf{u}$ at $\mathbf{p}$.

For example, what we found in (1) of Example 1.3 was that every point of the sphere Σ is umbilic, with $k_1 = k_2 = -1/r$.

2.5 Theorem (1) If **p** is an umbilic point of $M \subset \mathbf{R}^3$, then the shape operator S at **p** is just scalar multiplication by $k = k_1 = k_2$.

(2) If **p** is a nonumbilic point, $k_1 \neq k_2$, then there are exactly two principal directions, and these are orthogonal. Furthermore, if $\mathbf{e}_1$ and $\mathbf{e}_2$ are principal vectors in these directions, then

$$S(\mathbf{e}_1) = k_1\mathbf{e}_1, \quad S(\mathbf{e}_2) = k_2\mathbf{e}_2.$$

In short, the principal curvatures of M at **p** are the *eigenvalues* of S, and the principal vectors of M at **p** are the *eigenvectors* of S.

Proof. Suppose that k takes on its maximum value k_1 at $\mathbf{e}_1$, so

$$k_1 = k(\mathbf{e}_1) = S(\mathbf{e}_1) \bullet \mathbf{e}_1.$$

Let $\mathbf{e}_2$ be merely a unit tangent vector orthogonal to $\mathbf{e}_1$ (presently we shall show that it is also a principal vector).

If **u** is any unit tangent vector at **p**, we write

$$\mathbf{u} = \mathbf{u}(\vartheta) = c\mathbf{e}_1 + s\mathbf{e}_2,$$

where $c = \cos\vartheta$, $s = \sin\vartheta$ (Fig. 5.13). Thus normal curvature k at **p** becomes a function on the real line: $k(\vartheta) = k(\mathbf{u}(\vartheta))$.

For $1 \leqq i, j \leqq 2$, let S_{ij} be the number $S(\mathbf{e}_i) \cdot \mathbf{e}_j$. Note that $S_{11} = k_1$, and by the symmetry of the shape operator, $S_{12} = S_{21}$. We compute

$$\begin{aligned} k(\vartheta) &= S(c\mathbf{e}_1 + s\mathbf{e}_2) \bullet (c\mathbf{e}_1 + s\mathbf{e}_2) \\ &= c^2 S_{11} + 2scS_{12} + s^2 S_{22}. \end{aligned} \tag{1}$$

Hence

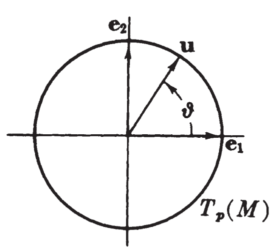

FIG. 5.13

$$\frac{dk}{d\vartheta}(\vartheta) = 2sc(S_{22} - S_{11}) + 2(c^2 - S^2)S_{12}. \tag{2}$$

If $\vartheta = 0$, then $c = 1$ and $s = 0$, so $\mathbf{u}(0) = \mathbf{e}_1$. Thus, by assumption, $k(\vartheta)$ is a maximum at $\vartheta = 0$, so $(dk/d\vartheta)(0) = 0$. It follows immediately from (2) that $S_{12} = 0$.

Since $\mathbf{e}_1$, $\mathbf{e}_2$ is an orthonormal basis for $T_p(M)$, we deduce by orthonormal expansion that

$$S(\mathbf{e}_1) = S_{11}\mathbf{e}_1, \quad \mathrm{S}(\mathbf{e}_2) = S_{22}\mathbf{e}_2. \tag{3}$$

Now if $\mathbf{p}$ is umbilic, then $S_{22} = k(\mathbf{e}_2)$ is the same as $S_{11} = k(\mathbf{e}_1) = k_1$, so (3) shows that S is scalar multiplication by $k_1 = k_2$.

If $\mathbf{p}$ is *not* umbilic, we look back at (1), which has become

$$k(\vartheta) = c^2 k_1 + s^2 S_{22}. \tag{4}$$

Since k_1 is the maximum value of $k(\vartheta)$, and $k(\vartheta)$ is now nonconstant, it follows that $k_1 > S_{22}$. But then (4) shows: (a) the maximum value k_1 is taken on *only* when $c = \pm 1$, $s = 0$, that is, in the $\mathbf{e}_1$ direction; and (b) the minimum value k_2 is S_{22}, and is taken on *only* when $c = 0$, $s = \pm 1$ that is, in the $\mathbf{e}_2$ direction. This proves the second assertion in the theorem, since (3) now reads:

$$S(\mathbf{e}_1) = k_1\mathbf{e}_1, \quad \mathrm{S}(\mathbf{e}_2) = k_2\mathbf{e}_2.$$ ◆

Contained in the preceding proof is Euler's formula for the normal curvature of M in *all* directions at $\mathbf{p}$.

2.6 Corollary Let k_1, k_2 and $\mathbf{e}_1$, $\mathbf{e}_2$ be the principal curvatures and vectors of $M \subset \mathbf{R}^3$ at $\mathbf{p}$. Then if $\mathbf{u} = \cos\vartheta\mathbf{e}_1 + \sin\vartheta\mathbf{e}_2$, the normal curvature of M in the $\mathbf{u}$ direction is (Fig. 5.13)

$$k(\mathbf{u}) = k_1\cos^2\vartheta + k_2\sin^2\vartheta.$$

Here is another way to show how the principal curvatures k_1 and k_2 control the shape of M near an arbitrary point $\mathbf{p}$. Since the position of M in $\mathbf{R}^3$ is of no importance, we can assume that (1) $\mathbf{p}$ is at the origin of $\mathbf{R}^3$, (2) the tangent plane $T_p(M)$ is the xy plane of $\mathbf{R}^3$, and (3) the x and y axes are the principal directions. *Near* $\mathbf{p}$, M can be expressed as M: $z = f(x, y)$, as shown in Fig. 5.14, and the idea is to construct an *approximation* of M near $\mathbf{p}$ by using only terms up to quadratic in the Taylor expansion of the function f. Now (1) and (2) imply $f^0 = f_x^0 = f_y^0 = 0$, where the subscripts indicate partial derivatives and the superscript zero denotes evaluation at $x = 0$, $y = 0$. Thus the quadratic approximation of f near $(0, 0)$ reduces to

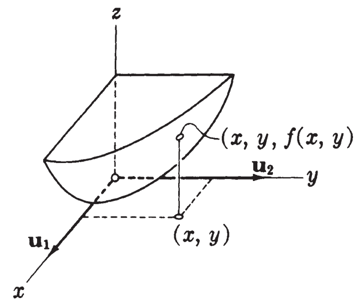

FIG. 5.14

$$f(x, y) \sim \frac{1}{2}\left(f_{xx}^0 x^2 + 2 f_{xy}^0 xy + f_{yy}^0 y^2\right).$$

In Exercise 1.2 we found that for the tangent vectors

$$\mathbf{u}_1 = (1, 0, 0) \quad \text{and} \quad \mathbf{u}_2 = (0, 1, 0)$$

at $\mathbf{p} = 0$,

$$S(\mathbf{u}_1) = -\nabla_{\mathbf{u}_1} U = f_{xx}^0 \mathbf{u}_1 + f_{xy}^0 \mathbf{u}_2,$$

$$S(\mathbf{u}_2) = -\nabla_{\mathbf{u}_2} U = f_{xy}^0 \mathbf{u}_1 + f_{yy}^0 \mathbf{u}_2.$$

By condition (3) above, $\mathbf{u}_1$ and $\mathbf{u}_2$ are *principal* vectors, so it follows from Theorem 2.5 that $k_1 = f_{xx}^0$, $k_2 = f_{yy}^0$, and $f_{xy}^0 = 0$.

Substituting these values in the quadratic approximation of f, we conclude that *the shape of M near* $\mathbf{p}$ *is approximately the same as that of the surface*

$$M'\colon z = \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right)$$

near $\mathbf{0}$. M' is called the *quadratic approximation* of *M near* $\mathbf{p}$. It is an analogue for surfaces of a Frenet approximation of a curve.

From Definition 2.2 through Corollary 2.6 we have been concerned with the geometry of $M \subset \mathbf{R}^3$ near one of its points $\mathbf{p}$. These results thus apply simultaneously to all the points of the oriented region $\mathscr{O}$ on which, by our initial assumption, the unit normal U is defined. In particular then, we have actually defined principal curvature *functions* k_1 and k_2 on $\mathscr{O}$, where at each point $\mathbf{p}$ of $\mathscr{O}$, $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ are the principal curvatures of M at $\mathbf{p}$. Note that these functions are only defined "modulo sign": If U is replaced by $-U$, they become $-k_1$ and $-k_2$.

Exercises

1. Use the results of Example 1.3 to find the principal curvatures and principal vectors of

(a) The cylinder, at every point.
(b) The saddle surface, at the origin.

2. If $\mathbf{v} \neq \mathbf{0}$ is a tangent vector (not necessarily of unit length), show that the normal curvature of M in the direction of $\mathbf{v}$ is $k(\mathbf{v}) = S(\mathbf{v}) \bullet \mathbf{v} / \mathbf{v} \bullet \mathbf{v}$.

3. For each integer $n \geqq 2$, let α_n be the curve $t \to (r\cos t,\ r\sin t,\ \pm t^n)$ in the cylinder M: $x^2 + y^2 = r^2$. These curves all have the same velocity at $t = 0$; test Lemma 2.1 by showing that they all have the same normal component of acceleration at $t = 0$.

4. For each of the following surfaces, find the quadratic approximation near the origin:

(a) $z = \exp(x^2 + y^2) - 1$. (b) $z = \log\cos x - \log\cos y$.
(c) $z = (x + 3y)^3$.

5.3 Gaussian Curvature

The preceding section found the geometrical meaning of the eigenvalues and eigenvectors of the shape operator. Now we examine the determinant and trace of S.

3.1 Definition The *Gaussian curvature* of $M \subset \mathbf{R}^3$ is the real-valued function $K = \det S$ on M. Explicitly, for each point $\mathbf{p}$ of M, the Gaussian curvature $K(\mathbf{p})$ of M at $\mathbf{p}$ is the determinant of the shape operator S of M at $\mathbf{p}$.

The *mean curvature* of $M \subset \mathbf{R}^3$ is the function $H = 1/2$ trace S. Gaussian and mean curvature are expressed in terms of principal curvature by

3.2 Lemma $K = k_1 k_2,\ H = \dfrac{1}{2}(k_1 + k_2).$

Proof. The determinant (and trace) of a linear operator may be defined as the common value of the determinant (and trace) of all its matrices. If $\mathbf{e}_1$ and $\mathbf{e}_2$ are principal vectors at a point $\mathbf{p}$, then by Theorem 2.5, we have $S(\mathbf{e}_1) = k_1(\mathbf{p})\mathbf{e}_1$ and $S(\mathbf{e}_2) = k_2(\mathbf{p})\mathbf{e}_2$. Thus the matrix of S at $\mathbf{p}$ with respect to $\mathbf{e}_1$, $\mathbf{e}_2$ is

$$\begin{pmatrix} k_1(\mathbf{p}) & 0 \\ 0 & k_2(\mathbf{p}) \end{pmatrix}.$$

This immediately gives the required result. ◆

A significant fact about the Gaussian curvature: It is independent of the choice of the unit normal U. If U is changed to $-U$, then the signs of *both* k_1 and k_2 change, so $K = k_1k_2$ is unaffected. This is obviously not the case with mean curvature $H = (k_1 + k_2)/2$, which has the same ambiguity of sign as the principal curvatures themselves.

The normal section method in Section 2 lets us tell, by inspection, approximately what the principal curvatures of M are at each point. Thus we get a reasonable idea of what the Gaussian curvature $K = k_1k_2$ is at each point $\mathbf{p}$ by merely *looking* at the surface M. In particular, we can usually tell what the sign of $K(\mathbf{p})$ is—and this sign has an important geometric meaning, which we now illustrate.

3.3 Remark *The sign of Gaussian curvature at a point* $\mathbf{p}$.

(1) ***Positive.*** If $K(\mathbf{p}) > 0$, then by Lemma 3.2, the principal curvatures $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have the same sign. By Corollary 2.6, either $k(\mathbf{u}) > 0$ for all unit vectors $\mathbf{u}$ at $\mathbf{p}$ or $k(\mathbf{u}) < 0$. Thus *M is bending away from its tangent plane $T_p(M)$ in all tangent directions at* $\mathbf{p}$ (Fig. 5.15)

The quadratic approximation of M near $\mathbf{p}$ is the paraboloid

$$z = \frac{1}{2}k_1(\mathbf{p})x^2 + \frac{1}{2}k_2(\mathbf{p})y^2.$$

(2) ***Negative.*** If $K(\mathbf{p}) < 0$, then by Lemma 3.2, the principal curvatures $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have opposite signs. Thus the quadratic approximation of M near $\mathbf{p}$ is a hyperboloid, so M is also saddle-shaped *near* $\mathbf{p}$ (Fig. 5.16).

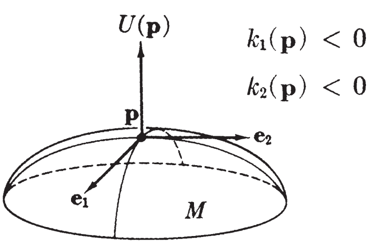

FIG. 5.15

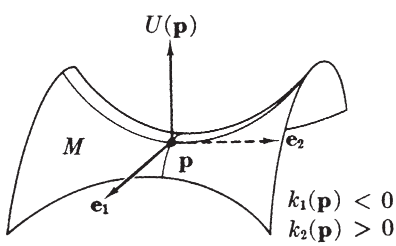

FIG. 5.16

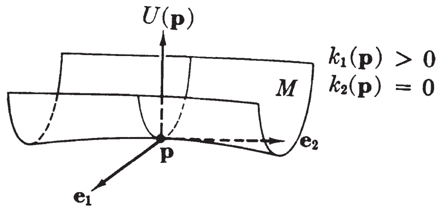

FIG. 5.17

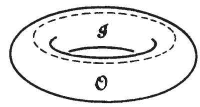

FIG. 5.18

(3) ***Zero.*** If $K(\mathbf{p}) = 0$, then by Lemma 3.2 there are two cases:

(a) If only one principal curvature is zero, say

$$k_1(\mathbf{p}) \neq 0, \quad k_2(\mathbf{p}) = 0.$$

(b) If both principal curvatures are zero, say

$$k_1(\mathbf{p}) = k_2(\mathbf{p}) = 0.$$

In case (a) the quadratic approximation is the cylinder $2z = k_1(\mathbf{p})x^2$, so M is trough-shaped near $\mathbf{p}$ (Fig. 5.17).

In case (b), the quadratic approximation reduces simply to the plane $z = 0$, so we get no information about the shape of M near $\mathbf{p}$.

A torus of revolution T provides a good example of these different cases. At points on the outer half $\mathcal{O}$ of T, the torus bends away from its tangent plane as one can see from Fig. 5.18; hence $K > 0$ on $\mathcal{O}$. But near each point $\mathbf{p}$ of the inner half $\mathcal{J}$, T is saddle-shaped and cuts through $T_p(M)$. Hence $K < 0$ on $\mathcal{J}$.

Near each point on the two circles (top and bottom) that separate $\mathcal{O}$ and $\mathcal{J}$, the torus is trough-shaped; hence $K = 0$ there. (A quantitative check of these qualitative results is given in Section 7.)

In case 3(b) above, where both principal curvatures vanish, $\mathbf{p}$ is called a *planar* point of M. (There are no planar points on the torus.) For example, the central point $\mathbf{p}$ of a *monkey saddle*, say

$$M: z = x\left(x + \sqrt{3}y\right)\left(x - \sqrt{3}y\right),$$

is planar. Here three hills and valleys meet, as shown in Fig. 5.19. Thus $\mathbf{p}$ *must* be a planar point—the shape of M near $\mathbf{p}$ is too complicated for the other three possibilities in Remark 3.3.

We consider now some ways to compute Gaussian and mean curvature.

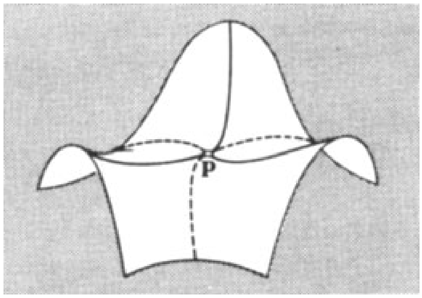

FIG. 5.19

3.4 Lemma If **v** and **w** are linearly independent tangent vectors at a point **p** of $M \subset \mathbf{R}^3$, then

$$S(\mathbf{v}) \times S(\mathbf{w}) = K(\mathbf{p})\mathbf{v} \times \mathbf{w},$$

$$S(\mathbf{v}) \times \mathbf{w} + \mathbf{v} \times S(\mathbf{w}) = 2H(\mathbf{p})\mathbf{v} \times \mathbf{w}.$$

Proof. Since **v**, **w** is a basis for the tangent plane $T_p(M)$, we can write

$$S(\mathbf{v}) = a\mathbf{v} + b\mathbf{w},$$

$$S(\mathbf{w}) = c\mathbf{v} + d\mathbf{w}.$$

This shows that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix of S with respect to the basis **v**, **w**. Hence

$$K(\mathbf{p}) = \det S = ad - bc, \quad H(\mathbf{p}) = \frac{1}{2}\operatorname{trace} S = \frac{1}{2}(a + d).$$

Using standard properties of the cross product, we compute

$$S(\mathbf{v}) \times S(\mathbf{w}) = (a\mathbf{v} + b\mathbf{w}) \times (c\mathbf{v} + d\mathbf{w})$$

$$= (ad - bc)\mathbf{v} \times \mathbf{w} = K(\mathbf{p})\mathbf{v} \times \mathbf{w},$$

and a similar calculation gives the formula for $H(\mathbf{p})$. ◆

Thus if V and W are tangent vector fields that are linearly independent at each point of an oriented region, we have vector field equations

$$S(V) \times S(W) = KV \times W,$$

$$S(V) \times W + V \times S(W) = 2HV \times W.$$

These may be solved for K and H by dotting each side with the normal vector field $V \times W$, and using the Lagrange identity (Exercise 6). We then find

$$K = \frac{\begin{vmatrix} S(V) \bullet V & S(V) \bullet W \\ S(W) \bullet V & S(W) \bullet V \end{vmatrix}}{\begin{vmatrix} V \bullet V & V \bullet W \\ W \bullet V & W \bullet W \end{vmatrix}},$$

$$H = \frac{\begin{vmatrix} S(V) \bullet V & S(V) \bullet W \\ W \bullet V & W \bullet W \end{vmatrix} + \begin{vmatrix} V \bullet V & V \bullet W \\ S(W) \bullet V & S(W) \bullet W \end{vmatrix}}{2\begin{vmatrix} V \bullet V & V \bullet W \\ W \bullet V & W \bullet W \end{vmatrix}}.$$

The denominators are never zero, since the independence of V and W is equivalent to $(V \times W) \bullet (V \times W) > 0$. In particular, the functions K and H are *differentiable*.

Once K and H are known, it is a simple matter to find k_1 and k_2.

3.5 Corollary On an oriented region $\mathscr{O}$ in M, the principal curvature functions are

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

Proof. To verify the formula, it suffices to substitute

$$K = k_1 k_2 \quad \text{and} \quad H = \frac{k_1 + k_2}{2}$$

and note that

$$H^2 - K = \frac{(k_1 + k_2)^2}{4} - k_1 k_2 = \frac{(k_1 - k_2)^2}{4}. \qquad \blacklozenge$$

A more enlightening derivation (Exercise 4) uses the characteristic polynomial of S.

This formula shows only that k_1 and k_2 are *continuous* functions on $\mathscr{O}$; they need not be differentiable since the square-root function is badly behaved at zero. The identity in the proof shows that $H^2 - K$ is zero only at umbilic points, however, so *k_1 and k_2 are differentiable on any oriented region free of umbilics*.

A natural way to single out special types of surfaces in $\mathbf{R}^3$ is by restrictions on Gaussian and mean curvature.

3.6 Definition A surface M in $\mathbf{R}^3$ is *flat* provided its Gaussian curvature is zero, and *minimal* provided its mean curvature is zero.

As expected, a plane is flat, for by Example 1.3 its shape operators are all zero, so $K = \det S = 0$. On a circular cylinder, (3) of Example 1.3 shows that S is *singular* at each point $\mathbf{p}$, that is, has rank less than the dimension of the tangent plane $T_p(M)$. Thus, although S itself is never zero, its determinant is always zero, so cylinders are also flat. This terminology seems odd at first for a surface so obviously curved, but it will be amply justified in later work.

Note that minimal surfaces have Gaussian curvature $K \leqq 0$, because if

$$H = \frac{k_1 + k_2}{2} = 0,$$

then $k_1 = -k_2$, so $K = k_1 k_2 \leqq 0$.

Another notable class of surfaces consists of those with *constant* Gaussian curvature. As mentioned earlier, Example 1.3 shows that a sphere of radius r has $k_1 = k_2 = -1/r$ (for U outward). Thus the sphere Σ has constant positive curvature $K = 1/r^2$: The smaller the sphere, the larger its curvature.

We shall find many examples of these various special types of surface as we proceed through this chapter.

Exercises

1. Show that there are no umbilics on a surface with $K < 0$, and that when $K \leqq 0$, umbilic points are planar.

2. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be orthonormal tangent vectors at a point $\mathbf{p}$ of M. What geometric information can be deduced from each of the following conditions on S at $\mathbf{p}$?

(a) $S(\mathbf{u}_1) \bullet \mathbf{u}_2 = 0$. (b) $S(\mathbf{u}_1) + S(\mathbf{u}_2) = 0$.
(c) $S(\mathbf{u}_1) \times S(\mathbf{u}_2) = 0$. (d) $S(\mathbf{u}_1) \bullet S(\mathbf{u}_2) = 0$.

3. (*Mean curvature.*) Prove that

(a) the average value of the normal curvature in *any* two orthogonal directions at $\mathbf{p}$ is $H(\mathbf{p})$. (The analogue for K is false.)

(b) $H(\mathbf{p}) = 1/2\pi \int_0^{2\pi} k(\vartheta)\, d\vartheta$,

where $k(\vartheta)$ is the normal curvature, as in Corollary 2.6.

4. The *characteristic polynomial* of an arbitrary linear operator S is

$$p(k) = \det(A - kI),$$

where A is any matrix of S.

(a) Show that the characteristic polynomial of the shape operator is

$$k^2 - 2Hk + K.$$

(b) Every linear operator satisfies its characteristic equation; that is, $p(S)$ is the zero operator when S is formally substituted in $p(k)$. Prove this in the case of the shape operator by showing that

$$S(\mathbf{v}) \bullet S(\mathbf{w}) - 2HS(\mathbf{v}) \bullet \mathbf{w} + K\mathbf{v} \bullet \mathbf{w} = 0$$

for any pair of tangent vectors to M.

The real-valued functions

$$\mathrm{I}(\mathbf{v}, \mathbf{w}) = \mathbf{v} \bullet \mathbf{w}, \quad \mathrm{II}(\mathbf{v}, \mathbf{w}) = S(\mathbf{v}) \bullet \mathbf{w},$$
$$\mathrm{III}(\mathbf{v}, \mathbf{w}) = S^2(\mathbf{v}) \bullet \mathbf{w} = S(\mathbf{v}) \bullet S(\mathbf{w}),$$

defined for all pairs of tangent vectors to an oriented surface, are traditionally called the *first, second*, and *third fundamental forms* of M. They are not differential forms; in fact, they are symmetric in **v** and **w** rather than alternating. The shape operator does not appear explicitly in the classical treatment of this subject; it is replaced by the second fundamental form.

5. (*Dupin curve.*) For a point **p** of an oriented region of M, let C_0 be the intersection of M near **p** with its tangent plane $T_p(M)$; specifically, C_0 consists of those points of M *near* **p** that lie in the plane through **p** orthogonal to $U(\mathbf{p})$. C_0 may be approximated by substituting for M its quadratic approximation $\hat{M}$; thus C_0 is approximated by the curve

$$\hat{C}_0\colon k_1x^2 + k_2y^2 = 0, \quad \text{near } (0,\ 0).$$

(a) Describe $\hat{C}_0$ in each of the three cases $K(\mathbf{p}) > 0$, $K(\mathbf{p}) < 0$, and $K(\mathbf{p}) = 0$ (not planar).

(b) Repeat (a) with C_0 replaced by C_ε and $C_{-\varepsilon}$, where the tangent plane has been replaced by the two parallel planes at distance $\pm\varepsilon$ from it.

(c) This scheme fails for planar points since the quadratic approximation becomes $\hat{M}$: $z = 0$. For the monkey saddle, sketch C_0, C_ε, and $C_{-\varepsilon}$.

6. For vectors **x**, **y**, **v**, **w** in $\mathbf{R}^3$, prove the *Lagrange identity*

$$(\mathbf{x} \times \mathbf{y}) \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{x} \bullet \mathbf{v} & \mathbf{x} \bullet \mathbf{w} \\ \mathbf{y} \bullet \mathbf{v} & \mathbf{y} \bullet \mathbf{w} \end{vmatrix}.$$

(a) By hand. (*Hint:* Since both sides are linear in each vector separately, it suffices to prove the identity when each vector is one of the unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.)

(b) By computer. (For dot and cross products, see the Appendix.)

7. (*Parallel surfaces.*) Let M be a surface oriented by U; for a fixed number ε (positive or negative) let F: M $\rightarrow \mathbf{R}^3$ be the mapping such that

$$F(\mathbf{p}) = \mathbf{p} + \varepsilon U(\mathbf{p}).$$

(a) If $\mathbf{v}$ is tangent to M at $\mathbf{p}$, show that $\bar{\mathbf{v}} = F_*(\mathbf{v})$ is $\mathbf{v} - \varepsilon S(\mathbf{v})$. Deduce that

$$\bar{\mathbf{v}} \times \bar{\mathbf{w}} = J(\mathbf{p})\mathbf{v} \times \mathbf{w},$$

where

$$J = 1 - 2\varepsilon H + \varepsilon^2 K = (1 - \varepsilon k_1)(1 - \varepsilon k_2).$$

When the function J does not vanish on M (for example, if M is compact and $|\varepsilon|$ small), this shows that F is a regular mapping, so the image

$$\overline{M} = F(M)$$

is at least an immersed surface in $\mathbf{R}^3$ (Ex. 16 in Sec. 4.8). $\overline{M}$ is said to be *parallel* to M at distance ε (Fig. 5.20).

(b) Show that the canonical isomorphisms of $\mathbf{R}^3$ make U a unit normal on $\overline{M}$ for which $\overline{S}(\bar{\mathbf{v}}) = S(\mathbf{v})$.

(c) Derive the following formulas for the Gaussian and mean curvatures of M:

$$\overline{K}(F) = \frac{K}{J}; \quad \overline{H}(F) = \frac{H - \varepsilon K}{J}.$$

8. (*Continuation.*)

(a) Check the results in (c) in the case of a sphere of radius r oriented by the outward normal U. Describe the mapping $F = F_\varepsilon$ when ε is 0, $-r$, and $-2r$.

(b) Starting from an orientable surface with constant positive Gaussian curvature, construct a surface with constant mean curvature.

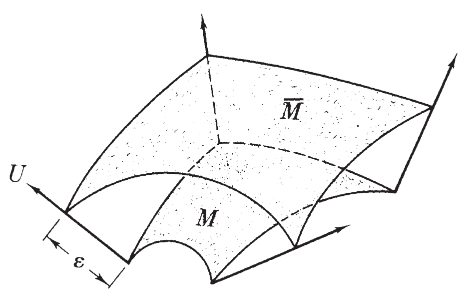

FIG. 5.20

5.4 Computational Techniques

We have defined the shape operators S of a surface M in $\mathbf{R}^3$ and found geometrical meaning for its main algebraic invariants: Gaussian curvature K, mean curvature H, principal curvatures k_1 and k_2, and (at each point) principal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. We shall now see how to express these invariants in terms of patches in M.

If $\mathbf{x}$: $D \to M$ is a patch in $M \subset \mathbf{R}^3$, we have already used the three real-valued functions

$$E = \mathbf{x}_u \bullet \mathbf{x}_u, \quad F = \mathbf{x}_u \bullet \mathbf{x}_v = \mathbf{x}_v \bullet \mathbf{x}_u, \quad G = \mathbf{x}_v \bullet \mathbf{x}_v$$

on D. Here $E > 0$ and $G > 0$ are the squares of the speeds of the u- and v-parameter curves of $\mathbf{x}$, and F measures the *coordinate angle* ϑ between $\mathbf{x}_u$ and $\mathbf{x}_v$, since

$$F = \mathbf{x}_u \bullet \mathbf{x}_v = \|\mathbf{x}_u\| \, \|\mathbf{x}_v\| \cos\vartheta = \sqrt{EG}\,\cos\vartheta$$

(Fig. 5.21). E, F, and G are the "warping functions" of the patch $\mathbf{x}$: They measure the way $\mathbf{x}$ distorts the flat region D in $\mathbf{R}^2$ in order to apply it to the curved region $\mathbf{x}(D)$ in M. These functions completely determine the dot product of tangent vectors at points of $\mathbf{x}(D)$, for if

$$\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v \quad \text{and} \quad \mathbf{w} = w_1\mathbf{x}_u + w_2\mathbf{x}_u,$$

then

$$\mathbf{v} \bullet \mathbf{w} = Ev_1w_1 + F(v_1w_2 + v_2w_1) + Gv_2w_2.$$

(In such equations we understand that $\mathbf{x}_u$, $\mathbf{x}_v$, E, F, and G are evaluated at (u, v) where $\mathbf{x}(u, v)$ is the point of application of $\mathbf{v}$ and $\mathbf{w}$.)

Now $\mathbf{x}_u \times \mathbf{x}_v$ is a function on D whose value at each point (u, v) of D is a vector orthogonal to both $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$—and hence normal to M at the point $\mathbf{x}(u, v)$. Furthermore, by Exercise 6 of Section 3,

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = EG - F^2.$$

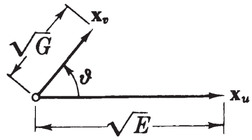

FIG. 5.21

Since $\mathbf{x}$ is, by definition, regular, this real-valued function on D is never zero. Thus we can construct the *unit normal function*

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

on D, which assigns to each (u, v) in D a unit normal vector to M at $\mathbf{x}(u, v)$. We emphasize that in this context, U, like $\mathbf{x}_u$ and $\mathbf{x}_v$, is not a vector field on $\mathbf{x}(D)$, but merely a vector-valued function on D. Nevertheless, we may regard the system $\mathbf{x}_u$, $\mathbf{x}_v$, U as a kind of defective frame field. At least U has unit length and is orthogonal to both $\mathbf{x}_u$ and $\mathbf{x}_v$, even though $\mathbf{x}_u$ and $\mathbf{x}_v$ are generally not orthonormal.

In this context, covariant derivatives are usually computed along the parameter curves of $\mathbf{x}$, where by the discussion in Section 1, they reduce to partial differentiation with respect to u and v. As in the case of $\mathbf{x}_u$ and $\mathbf{x}_v$, these partial derivatives are again denoted by subscripts u and v. If

$$\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)),$$

then just as for $\mathbf{x}_u$ and $\mathbf{x}_v$ on page 140, we have

$$\mathbf{x}_{uu} = \left(\frac{\partial^2 x_1}{\partial u^2}, \frac{\partial^2 x_2}{\partial u^2}, \frac{\partial^2 x_3}{\partial u^2}\right)_{\mathbf{x}},$$
$$\mathbf{x}_{vu} = \left(\frac{\partial^2 x_1}{\partial u\,\partial v}, \frac{\partial^2 x_2}{\partial u\,\partial v}, \frac{\partial^2 x_3}{\partial u\,\partial v}\right)_{\mathbf{x}},$$
$$\mathbf{x}_{vv} = \left(\frac{\partial^2 x_1}{\partial v^2}, \frac{\partial^2 x_2}{\partial v^2}, \frac{\partial^2 x_3}{\partial v^2}\right)_{\mathbf{x}}.$$

Evidently $\mathbf{x}_{uu}$ and $\mathbf{x}_{vv}$ give the accelerations of the u- and v-parameter curves. Since order of partial differentiation is immaterial, $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, which gives both the covariant derivative of $\mathbf{x}_u$ in the $\mathbf{x}_v$ direction and of $\mathbf{x}_v$ in the $\mathbf{x}_u$ direction.

Now if S is the shape operator derived from U, we define three more real-valued functions on D:

$$\text{L} = S(\mathbf{x}_u) \bullet \mathbf{x}_u,$$
$$\text{M} = S(\mathbf{x}_u) \bullet \mathbf{x}_v = S(\mathbf{x}_v) \bullet \mathbf{x}_u,$$
$$\text{N} = S(\mathbf{x}_v) \bullet \mathbf{x}_v.$$

Because $\mathbf{x}_u$, $\mathbf{x}_v$ gives a basis for the tangent space of M at each point of $\mathbf{x}(D)$, it is clear that these functions uniquely determine the shape operator. Since this basis is generally not orthonormal, L, M, and N do not lead to simple expressions for $S(\mathbf{x}_u)$ and $S(\mathbf{x}_v)$ in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$. In the formulas preceding Corollary 3.5, however, they *do* provide simple expressions for Gaussian and mean curvature.

4.1 Corollary If $\mathbf{x}$ is a patch in $M \subset \mathbf{R}^3$, then

$$K(\mathbf{x}) = \frac{\mathrm{LN} - \mathrm{M}^2}{EG - F^2}, \quad H(\mathbf{x}) = \frac{G\mathrm{L} + E\mathrm{N} - 2F\mathrm{M}}{2(EG - F^2)}.$$

Proof. At a point $\mathbf{p}$ of $\mathbf{x}(D)$, the formulas on page 220 express $K(\mathbf{p})$ and $H(\mathbf{p})$ in terms of tangent vectors $V(\mathbf{p})$ and $W(\mathbf{p})$ at $\mathbf{p}$. If $V(\mathbf{p})$ and $W(\mathbf{p})$ are replaced by the tangent vectors $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ at $\mathbf{x}(u, v)$, we find the required formulas for K and H at $\mathbf{x}(u, v)$. ◆

When the patch $\mathbf{x}$ is clear from context, we shall usually abbreviate the composite functions $K(\mathbf{x})$ and $H(\mathbf{x})$ to merely K and H.

By a device like that used in Lemma 2.1, we can find a simple way to *compute* L, M, and N—and thereby K and H. For example, since $U \bullet \mathbf{x}_u = 0$, partial differentiation with respect to v—that is, ordinary differentiation along v-parameter curves—yields

$$0 = \frac{\partial}{\partial v}(U \bullet \mathbf{x}_u) = U_v \bullet \mathbf{x}_u + U \bullet \mathbf{x}_{uv}.$$

(Recall that U_v is the covariant derivative of the vector field $v \to U(u_0, v)$ on each v-parameter curve $u = u_0$.) Since $\mathbf{x}_v$ gives the velocity vectors of such curves, Exercise 1.1 shows that $U_v = -S(\mathbf{x}_v)$. Thus the preceding equation becomes

$$S(\mathbf{x}_v) \bullet \mathbf{x}_u = U \bullet \mathbf{x}_{uv}$$

(Fig. 5.22). Three similar equations may be found by replacing u by v, and v by u. In particular,

$$S(\mathbf{x}_u) \bullet \mathbf{x}_v = U \bullet \mathbf{x}_{vu} = U \bullet \mathbf{x}_{uv} = S(\mathbf{x}_v) \bullet \mathbf{x}_u.$$

Again, since $\mathbf{x}_u$ and $\mathbf{x}_v$ give a basis for the tangent space at each point, this is sufficient to prove that *S is symmetric* (Lemma 1.4).

4.2 Lemma If $\mathbf{x}$ is a patch in $M \subset \mathbf{R}^3$, then

$$\mathrm{L} = S(\mathbf{x}_u) \bullet \mathbf{x}_u = U \bullet \mathbf{x}_{uu},$$
$$\mathrm{M} = S(\mathbf{x}_u) \bullet \mathbf{x}_v = U \bullet \mathbf{x}_{uv},$$
$$\mathrm{N} = S(\mathbf{x}_v) \bullet \mathbf{x}_v = U \bullet \mathbf{x}_{vv}.$$

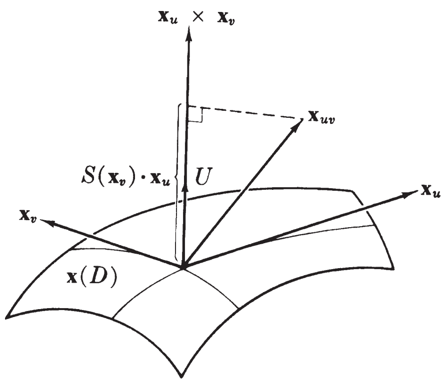

FIG. 5.22

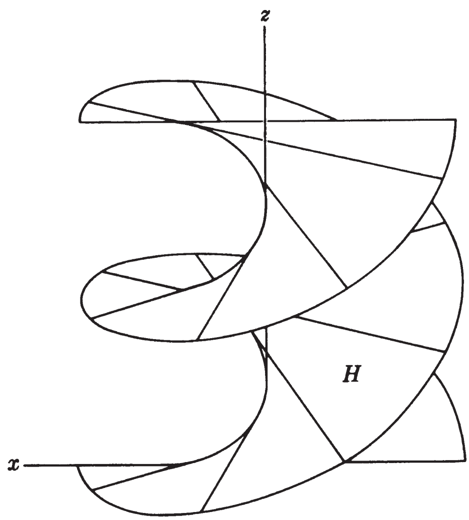

FIG. 5.23

The first equation in each case is just the definition, and *u and v* may be reversed in the formulas for M.

4.3 Example *Computation of Gaussian and mean curvature*

(1) ***Helicoid*** (Exercise 5 of Section 4.2). This surface H, shown in Fig. 5.23, is covered by a single patch

$$\mathbf{x}(u,v) = (u\cos v,\ u\sin v,\ bv),\quad b \neq 0,$$

for which

$$\begin{aligned} \mathbf{x}_u &= (\cos v, \sin v, 0), & E &= 1, \\ \mathbf{x}_v &= (-u \sin v, u \cos v, b), & F &= 0, \\ & & G &= b^2 + u^2. \end{aligned}$$

Hence

$$\mathbf{x}_u \times \mathbf{x}_v = (b \sin v, -b \cos v, u).$$

Because coordinate patches are, by definition, regular mappings, we have seen in Chapter 4 that the function

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2}$$

is never zero. For any patch we denote this useful function by W, that is,

$$W = \sqrt{EG - F^2}.$$

In the case at hand, $W = W(u, v) = \sqrt{b^2 + u^2}$, so the unit normal function is

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{W} = \frac{(b \sin v, -b \cos v, u)}{\sqrt{b^2 + u^2}}.$$

(A computation of U can always be checked by verifying that the result is a unit vector orthogonal to both $\mathbf{x}_u$ and $\mathbf{x}_v$.)

Next we find

$$\begin{aligned} \mathbf{x}_{uu} &= 0, \\ \mathbf{x}_{uv} &= (-\sin v, \cos v, 0), \\ \mathbf{x}_{vv} &= (-u \cos v, -u \sin v, 0). \end{aligned}$$

Here $\mathbf{x}_{uu} = 0$ is obvious, since the u-parameter curves are straight lines. The v-parameter curves are helices, and this formula for the acceleration $\mathbf{x}_{vv}$ was found already in Chapter 2. Now by Lemma 4.2,

$$\begin{aligned} \mathrm{L} &= \mathbf{x}_{uu} \bullet \frac{(\mathbf{x}_u \times \mathbf{x}_v)}{W} = 0, \\ \mathrm{M} &= \mathbf{x}_{uv} \bullet \frac{(\mathbf{x}_u \times \mathbf{x}_v)}{W} = -\frac{b}{W}, \\ \mathrm{N} &= \mathbf{x}_{vv} \bullet \frac{(\mathbf{x}_u \times \mathbf{x}_v)}{W} = 0. \end{aligned}$$

Hence by Corollary 4.1 and the results above,

$$K = \frac{\text{LN} - \text{M}^2}{EG - F^2} = \frac{-(b/W)^2}{W^2} = \frac{-b^2}{W^4} = \frac{-b^2}{(b^2 + u^2)^2},$$

$$H = \frac{G\text{L} + E\text{N} - 2F\text{M}}{2(EG - F^2)} = 0.$$

Thus the helicoid is a minimal surface with Gaussian curvature

$$-\frac{1}{b^2} \leq K < 0.$$

The minimum value $K = -1/b^2$ occurs on the central axis ($u = 0$) of the helicoid, and $K \to 0$ as distance $|u|$ from the axis increases to infinity.

(2) ***The saddle surface M***: $z = xy$ (Example 1.3). This time we use the Monge patch $\mathbf{x}(u, v) = (u, v, uv)$ and with the same format as above, compute

$$\begin{aligned}
\mathbf{x}_u &= (1, 0, v), & E &= 1 + v^2, \\
\mathbf{x}_v &= (0, 1, u), & F &= uv, \\
& & G &= 1 + u^2, \\
U &= (-v, -u, 1)/W, & W &= \sqrt{1 + u^2 + v^2}, \\
\mathbf{x}_{uu} &= 0, & \text{L} &= 0, \\
\mathbf{x}_{uv} &= (0, 0, 1), & \text{M} &= 1/W, \\
\mathbf{x}_{vv} &= 0, & \text{N} &= 0.
\end{aligned}$$

Hence

$$K = \frac{-1}{(1 + u^2 + v^2)^2}, \quad H = \frac{-uv}{(1 + u^2 + v^2)^{3/2}}.$$

Strictly speaking, these functions are $K(\mathbf{x})$ and $H(\mathbf{x})$ defined on the domain $\mathbf{R}^2$ of $\mathbf{x}$. In this case, it is easy to express K and H directly as functions on M by using the cylindrical coordinate functions $r = \sqrt{x^2 + y^2}$ and z. Note from Fig. 5.24 that

$$r(\mathbf{x}(u, v)) = \sqrt{u^2 + v^2}$$

and

$$z(\mathbf{x}(u, v)) = uv;$$

hence on M,

$$K = \frac{-1}{(1 + r^2)^2}, \quad H = \frac{-z}{(1 + r^2)^{3/2}}.$$

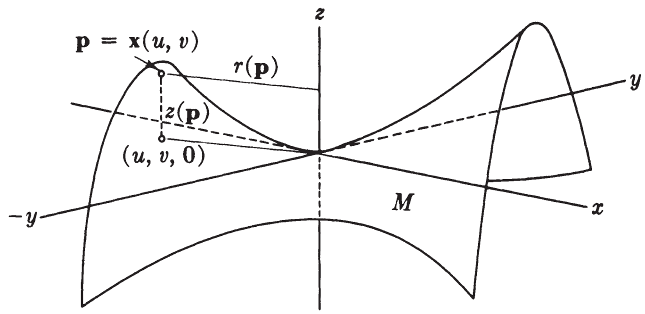

FIG. 5.24

Thus the Gaussian curvature of M depends only on distance to the z axis, rising from $K = -1$ (at the origin) toward zero as r goes to infinity, while H varies more radically.

Like all simple (that is, one-patch) surfaces, the helicoid and saddle surfaces are orientable, since computations as above provide a unit normal on the whole surface. Thus the principal curvature functions $k_1 \geq k_2$ can be defined unambiguously on each surface. These can always be found from K and H by Corollary 3.5. Since the helicoid is a minimal surface, we get the simple result

$$k_1, k_2 = \frac{\pm b}{(b^2 + u^2)}.$$

For the saddle surface,

$$k_1, k_2 = \frac{-z \pm \sqrt{1 + r^2 + z^2}}{(1 + r^2)^{3/2}}.$$

Techniques for computing principal vectors are left to the exercises.

The computational results in this section, though stated for coordinate patches, remain valid for arbitrary regular mappings $\mathbf{x}$: $D \to \mathbf{R}^3$ since the restriction of $\mathbf{x}$ to any small enough open set in D is a patch.

Exercises

1. For the geographical parametrization

$\mathbf{x}(u, v) = (r \cos v \cos u,\ r \cos v \sin u,\ r \sin v)$, where $-\pi/2 < v < \pi/2$, $\pi < u < \pi$, of the sphere Σ of radius r, find E, F, G, and W, and then the unit normal U, Gaussian curvature K, and mean curvature H.

2. For a Monge patch $\mathbf{x}(u, v) = (u, v, f(u, v))$, show that,

$$E = 1 + f_u^2, \quad \text{L} = \frac{f_{uu}}{W},$$
$$F = f_u f_v, \quad \text{M} = \frac{f_{uv}}{W},$$
$$G = 1 + f_v^2, \quad \text{N} = \frac{f_{vv}}{W},$$

where

$$W = \sqrt{EG - F^2} = \left(1 + f_u^2 + f_v^2\right)^{1/2}$$

Then find formulas for K and H.

3. (*Continuation.*) Deduce that the image of $\mathbf{x}$ is
(a) flat if and only if

$$f_{uu} f_{vv} - f_{uv}^2 = 0;$$

(b) minimal if and only if

$$\left(1 + f_u^2\right) f_{vv} - 2 f_u f_v f_{uv} + \left(1 + f_v^2\right) f_{uu} = 0.$$

4. Let $\mathbf{x}$ be the patch

$$\mathbf{x}(u, v) = (u,\ v,\ \log\cos v - \log\cos u)$$

defined on $-\pi/2 < u,\ \ v < \pi/2$. Show that the image of $\mathbf{x}$ is a minimal surface with Gaussian curvature

$$K = \frac{-\sec^2 u \sec^2 v}{W^4},$$

where $W^2 = 1 + \tan^2 u + \tan^2 v$. (This patch is in Scherk's surface, Ex. 5 of Sec. 5.5.)

5. Show that a curve segment

$$\alpha(t) = \mathbf{x}(a_1(t), a_2(t)), \quad a \leqq t \leqq b,$$

has length

$$L(\alpha) = \int_a^b \left(E a_1'^2 + 2F a_1' a_2' + G a_2'^2\right)^{\frac{1}{2}} dt,$$

where E, F, and G are evaluated on a_1, a_2.

6. Find the Gaussian curvature of the elliptic and hyperbolic paraboloids

$$M\colon\ z = \frac{x^2}{a^2} + \varepsilon\frac{y^2}{b^2},$$

where $\varepsilon = \pm 1$.

7. Find the curvature of the monkey saddle M: $z = x^3 - 3xy^2$, and express it in terms of $r = \sqrt{x^2 + y^2}$.

8. A patch $\mathbf{x}$ in M is *orthogonal* provided $F = 0$ (so $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal at each point). Show that in this case

$$S(\mathbf{x}_u) = \frac{\text{L}}{E}\mathbf{x}_u + \frac{\text{M}}{G}\mathbf{x}_v,$$

$$S(\mathbf{x}_v) = \frac{\text{M}}{E}\mathbf{x}_u + \frac{\text{N}}{G}\mathbf{x}_v.$$

(b) A patch $\mathbf{x}$ in M is *principal* provided $F = \text{M} = 0$. Prove that $\mathbf{x}_u$ and $\mathbf{x}_v$ are principal vectors at each point, with corresponding principal curvatures L/E and N/G.

9. Prove that a nonzero tangent vector $\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$ is a principal vector if and only if

$$\begin{vmatrix} v_2^2 & -v_1v_2 & v_1^2 \\ E & F & G \\ \text{L} & \text{M} & \text{N} \end{vmatrix} = 0.$$

(*Hint:* $\mathbf{v}$ is principal if and only if $S(\mathbf{v}) \times \mathbf{v} = 0$.)

10. Show that on the saddle surface $z = xy$ the two vector fields

$$\left(\sqrt{1+x^2} \pm \sqrt{1+y^2},\ \ y\sqrt{1+x^2} \pm x\sqrt{1+y^2}\right)$$

are principal at each point. Check that they are orthogonal and tangent to M.

11. If $\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$ is tangent to M at $\mathbf{x}(u, v)$, show that the normal curvature in the direction $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ is

$$k(\mathbf{u}) = \frac{\text{L}v_1^2 + 2\text{M}v_1v_2 + \text{N}v_2^2}{Ev_1^2 + 2Fv_1v_2 + Gv_2^2},$$

where the various functions are evaluated at (u, v).

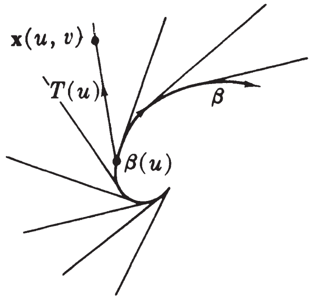

FIG. 5.25

12. Show that a ruled surface $\mathbf{x}(u, v) = \beta(u) + v\delta(u)$ has Gaussian curvature

$$K = \frac{-\text{M}^2}{EG - F^2} = \frac{-(\beta' \cdot \delta \times \delta')^2}{W^4},$$

where $W = \|\beta' \times \delta + v\delta' \times \delta\|$.

13. (*Flat ruled surfaces.*)

(a) Show that generalized cones and cylinders are flat (Exs. 3 and 4 of Sec. 4.2).

(b) If β is a unit-speed curve in $\mathbf{R}^3$ with $\kappa > 0$, the ruled surface

$$\mathbf{x}(u, v) = \beta(u) + vT(u), \quad v \neq 0,$$

where $T(u) = \beta'(u)$, is called the *tangent surface* of β. Prove that $\mathbf{x}$ is regular and the tangent surface is flat. (The surface is separated into two pieces by the curve; Fig. 5.25 shows the $v > 0$ half.)

14. (*Patch criterion for umbilics.*)

(a) Show that a point $\mathbf{x}(u, v)$ is umbilic if and only if there is a number k such that at (u, v),

$$\text{L} = kE, \quad \text{M} = kF, \quad \text{N} = kG.$$

(Then k is the principal curvature $k_1 = k_2$.)

15. Find the umbilic points, if any, on the following surfaces:

(a) Monkey saddle (Ex. 7).

(b) Elliptic paraboloid (Ex. 6), assuming $a \geqq b$.

(*Hint*: Compute the "vectors" (E, F, G) and $(\text{L}, \text{M}, \text{N})$ for arbitrary (u, v), discarding common factors if convenient. Then solve $(E, F, G) \times (\text{L}, \text{M}, \text{N}) = 0$ for (u, v).)

16. (*Loxodromes.*) For $a \neq 0$, let f_a: $(-\pi/2, \pi/2) \to \mathbf{R}$ be the unique function such that

$$f_a'(t) = \frac{a}{\cos t} \quad \text{and} \quad f(0) = 0.$$

If $\mathbf{x}$ is the geographical parametrization of a sphere, the curve $\lambda_a(t) = \mathbf{x}(f_a(t), t)$ is a *loxodrome.*

(a) Prove that λ_a' always makes a constant angle with the due-north vector field $\mathbf{x}_v$. Thus λ_a represents a trip with constant (idealized) compass bearing.

(b) Show that the length of λ_a from the south pole (0, 0, $-r$) to the north pole (0, 0, r) (limit values) is $\sqrt{1+a^2}\,\pi r$.

(c) (*Computer.*) Verify that $f_a(t) = a \log\tan(t/2 + \pi/4)$, and plot λ_{10} from near the south pole to near the north pole on a unit sphere. (Require smoothness, and keep the same scale on each axis.)

17. (*Tubes.*) If β is a curve in $\mathbf{R}^3$ with $0 < \kappa \leqq b$, let

$$\mathbf{x}(u, v) = \beta(u) + \varepsilon(\cos vN(u) + \sin vB(u)).$$

Thus the v-parameter curves are circles of constant radius ε in planes orthogonal to β. Show that

(a) $\mathbf{x}_u \times \mathbf{x}_v = -\varepsilon(1 - \kappa\varepsilon \cos v)(\cos vN(u) + \sin vB(u))$.

(b) If ε is small enough, $\mathbf{x}$ is regular. So $\mathbf{x}$ is at least an immersed surface, called a *tube* around β.

(c) $U = \cos vN(u) + \sin vB(u)$ is a unit normal vector on the tube.

(d) $K = \dfrac{-\kappa(u)\cos v}{\varepsilon(1-\kappa(u)\varepsilon\cos v)}$.

(*Hint:* Use $S(\mathbf{x}_u) \times S(\mathbf{x}_v) = K\, \mathbf{x}_u \times \mathbf{x}_v$.)

The following exercises deal with use of the computer in patch computations.

18. (*Computer.*) The Appendix gives computer commands for the functions E, F, G, W, L, M, N derived from a patch.

(a) Write the computer commands, based on Corollary 4.1, that give the Gaussian curvature and mean curvature of a patch in terms of these functions.

(b) To test these commands, find E, F, G, W, L, M, N, K, H for each of the cases in Example 4.3. Compare with the text computations.

19. (*Computer.*) Make a save file containing the following patches or parametrizations. (See Appendix for "save files" and format for parameters.)

(a) the patch in Exercise 4.

(b) a single Monge patch—with parameters *a, b*, ε—for the hyperboloids in Exercise 6.
(c) a Monge patch for the monkey saddle (Ex. 7), in terms of (i) rectangular coordinates u, v and (ii) polar coordinates r, ϑ on $\mathbf{R}^2$.
(d) the parametrization of Enneper's surface in Exercise 16.
(e) the geographical parametrization of a sphere of radius r.

20. (*Computer formulas.*)
(a) For a patch $\mathbf{x}$ in $\mathbf{R}^3$, show that Gaussian curvature can be expressed directly in terms of $\mathbf{x}$ as

$$K(u,v) = \frac{(\mathbf{x}_{uu} \bullet \mathbf{x}_u \times \mathbf{x}_v)(\mathbf{x}_{vv} \bullet \mathbf{x}_u \times \mathbf{x}_v) - (\mathbf{x}_{uv} \bullet \mathbf{x}_u \times \mathbf{x}_v)^2}{\left((\mathbf{x}_u \bullet \mathbf{x}_u)(\mathbf{x}_v \bullet \mathbf{x}_v) - (\mathbf{x}_u \bullet \mathbf{x}_v)^2\right)^2}.$$

This formula gives the fastest general computer computation of K. The Appendix has computer commands for it in the *Mathematica* and *Maple* systems.
(b) Test this command on the two cases in Example 4.3 and the patches in Exercise 19.

The derivation of the corresponding formula for *mean* curvature is rather tedious. This formula may be found in Alfred Gray's book [G].
(c) Find a computer formula for the Gaussian curvature of the graph of a function f: $\mathbf{R}^2 \to \mathbf{R}$. (*Hint:* use Ex. 2.) Test this command on the Monge patches referenced in Exercises 18 and 19.

21. (*Computer.*)
(a) Write commands that, given a curve α on some interval, plot the tube of radius r around α. (See Ex. 17.)
(b) Use part (a) to plot the tube of radius $\frac{1}{2}$ around two turns of the helix $t \to (3 \cos t, 3 \sin t, t/2)$.
(c) Plot the tube of radius $\frac{1}{2}$ around the curve τ in Exercise 19 of Sec. 2.4. (This makes it clear that τ is a trefoil knot.)

5.5 The Implicit Case

In this brief section we describe a way to compute the geometry of a surface $M \subset \mathbf{R}^3$ that has a nonvanishing normal vector field Z defined on the entire surface. The main case is a surface given in implicit form M: $g = 0$, for then, by Lemma 3.8 of Chapter 4, the gradient

$$\nabla g = \sum \frac{\partial g}{\partial x_i}$$

is such a vector field.

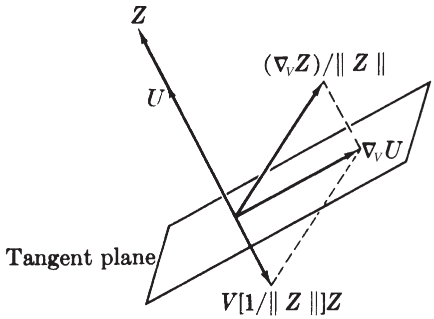

FIG. 5.26

Let S be the shape operator on M derived from the unit normal

$$U = Z/\|Z\|.$$

Write $Z = \sum z_i U_i$. Then if V is a tangent vector field on M, Method 2 in Section 1 gives

$$\nabla_V Z = \sum v[z_i] U_i.$$

Hence, using a Leibnizian property of such derivatives,

$$\nabla_V U = \nabla_V \left(\frac{Z}{\|Z\|} \right) = \frac{\nabla_V Z}{\|Z\|} + V \left[\frac{1}{\|Z\|} \right] Z$$

(Fig. 5.26). The last term here, $V[1/\|Z\|]\, Z$, is evidently a *normal* vector field; we do not care which one it is, so we denote it merely by $-N_V$. Thus

$$S(V) = -\nabla_V U = -\frac{\nabla_V Z}{\|Z\|} + N_V.$$

Note that if W is another tangent vector field on M, then $N_V \times N_W = 0$, while products such as $N_V \times Y$ are tangent to M for *any* Euclidean vector field Y on M. Thus it is a routine matter to deduce the following lemma from Lemma 3.4.

5.1 Lemma Let Z be a nonvanishing normal vector field on M. If V and W are tangent vector fields such that $V \times W = Z$, then

$$K = \frac{Z \bullet \nabla_V Z \times \nabla_W Z}{\|Z\|^4},$$

$$H = -Z \bullet \frac{\nabla_V Z \times W + V \times \nabla_W Z}{2\|Z\|^3}.$$

To compute, say, the Gaussian curvature of a surface M: $g = c$ using patches, one must begin by explicitly finding enough of them to cover all of M; a complete computation of K may thus be tedious, even when g is a rather simple function. The following example shows to advantage the approach just described.

5.2 Example Curvature of the ellipsoid

$$M:\ g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We write $g = \sum \frac{x_i^2}{a_i^2}$, and use the (nonvanishing) normal vector field

$$Z = \frac{1}{2}\nabla g = \sum \frac{x_i}{a_i^2} U_i.$$

Then if $V = \sum v_i U_i$ is a tangent vector field on M,

$$\nabla_V Z = \sum \frac{V[x_i]}{a_i^2} U_i = \sum \frac{v_i}{a_i^2} U_i,$$

since

$$V[x_i] = dx_i(V) = v_i.$$

Similar results for another tangent vector field W yield

$$Z \cdot \nabla_V Z \times \nabla_W Z = \begin{vmatrix} \frac{x_1}{a_1^2} & \frac{x_2}{a_2^2} & \frac{x_3}{a_3^2} \\ \frac{v_1}{a_1^2} & \frac{v_2}{a_2^2} & \frac{v_3}{a_3^2} \\ \frac{w_1}{a_1^2} & \frac{w_2}{a_2^2} & \frac{w_3}{a_3^2} \end{vmatrix} = \frac{1}{a_1^2 a_2^2 a_3^2} X \bullet V \times W,$$

where X is the special vector field $\sum x_i U_i$ used in Example 3.9 of Chapter 4.

It is always possible to choose V and W so that $V \times W = Z$. But then

$$X \bullet V \times W = X \bullet Z = \sum \frac{x_i^2}{a_i^2} = 1.$$

Thus by Lemma 5.1 we have found

$$K = \frac{1}{a_1^2 a_2^2 a_3^2 \|Z\|^4}, \quad \text{where } \|Z\|^4 = \left(\sum \frac{x_i^2}{a_i^4}\right)^2,$$

that is,

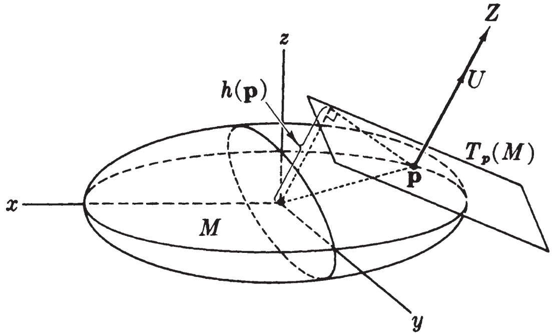

FIG. 5.27

$$K = \frac{1}{a^2b^2c^2\|Z\|^4}, \quad \text{where } \|Z\|^4 = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2.$$

For *any* oriented surface in $\mathbf{R}^3$, its *support function* h assigns to each point $\mathbf{p}$ the orthogonal distance $h(\mathbf{p}) = \mathbf{p} \bullet U(\mathbf{p})$ from the origin to the Euclidean tangent plane $T_p(M)$, as shown in Fig. 5.27 for the ellipsoid. Using the above-mentioned vector field X (whose value at $\mathbf{p}$ is the tangent vector $\mathbf{p}_p$), we find for the ellipsoid that

$$h = X \bullet U = X \bullet \frac{Z}{\|Z\|} = \frac{1}{\|Z\|}.$$

Thus a clearer expression of the Gaussian curvature of the ellipsoid is

$$K = \frac{h^4}{a^2b^2c^2}.$$

Note that if $a = b = c = r$ (so M is a sphere), then $h = 1/\|Z\|$ has constant value r, and this formula reduces to $K = 1/r^2$.

Exercises

1. Show that the elliptic hyperboloids of one and two sheets (Ex. 2.9 of Ch. 4) have Gaussian curvatures $K = -h^4/a^2b^2c^2$ and $K = h^4/a^2b^2c^2$, respectively, where both support functions h are given by the same formula as for the ellipsoid in Example 5.2.

2. If h is the support function of an oriented surface $M \subset \mathbf{R}^3$, show that
(a) A point $\mathbf{p}$ of M is a critical point of h if and only if $\mathbf{p} \bullet S(\mathbf{v}) = 0$ for all tangent vectors to M at $\mathbf{p}$. (*Hint:* Write h as $X \bullet U$, where $X = \sum x_i U_i$.)

(b) When $K(\mathbf{p}) \neq 0$, $\mathbf{p}$ is a critical point of h if and only if $\mathbf{p}$ (considered as a vector) is orthogonal to M at $\mathbf{p}$.

3. (a) Use the preceding exercises to find the critical points of the Gaussian curvature function K on the ellipsoid and on the hyperboloids of one and two sheets (Ex. 2.9 of Ch. 4).

(b) Assuming $a \geqq b \geqq c$ for these surfaces, find their Gaussian curvature intervals.

4. Compute K and H for the saddle surface M: $z = xy$ by the method of this section. (*Hint*: Take V and W tangent to the rulings of M.)

5. *Scherk's minimal surface, M*: $e^z \cos x = \cos y$. Let $\mathscr{R}$ be the region in the xy plane on which $\cos x \cos y > 0$. $\mathscr{R}$ is a checkerboard pattern of open squares, with vertices $(\pi/2 + m\pi,\ \pi/2 + n\pi)$. Show that:

(a) M is a surface.

(b) For each point (u, v) in $\mathscr{R}$ there is exactly one point (u, v, w) in M. The only other points of M are entire vertical lines over each of the vertices of $\mathscr{R}$ (Fig. 5.28).

(c) M is a minimal surface with $K = -e^{2z}/(e^{2z}\sin^2 x + 1)^2$.

(*Hint:* $V = \cos x\, U_1 + \sin x\, U_3$ is a tangent vector field.)

(d) The patch in Exercise 4.4 parametrizes the part of M over a typical open square. Show that the curvature $K(u, v)$ calculated there is consistent with (c).

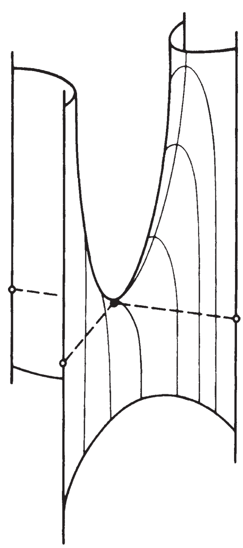

FIG. 5.28

6. Let Z be a nonvanishing normal vector field on M. Show that a tangent vector $\mathbf{v}$ to M at $\mathbf{p}$ is principal if and only if

$$\mathbf{v} \bullet Z(\mathbf{p}) \times \nabla_v Z = 0.$$

(*Hint:* Recall that $\mathbf{v}$ is principal if and only if $S(\mathbf{v}) \times \mathbf{v} = 0$.)

The preceding equation together with the tangency equation $Z(\mathbf{p}) \bullet \mathbf{v} = 0$ can be solved for the principal directions. Thus umbilics can be located using these equations, since $\mathbf{p}$ is umbilic if and only if every tangent vector at $\mathbf{p}$ is principal.

7. For the ellipsoid M: $\sum x_i^2/a_i^2 = 1$, show that:

(a) A tangent vector $\mathbf{v}$ at $\mathbf{p}$ is principal if and only if

$$0 = p_1 v_2 v_3 (a_2^2 - a_3^2) + p_2 v_3 v_1 (a_3^2 - a_1^2) + p_3 v_1 v_2 (a_1^2 - a_2^2).$$

(b) Assuming $a_1 > a_2 > a_3$, there are exactly four umbilics on M, with coordinates

$$p_1 = \pm a_1 \left(\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2} \right)^{1/2}, \quad p_2 = 0, \quad p_3 = \pm a_3 \left(\frac{a_2^2 - a_3^2}{a_1^2 - a_3^2} \right)^{1/2}.$$

5.6 Special Curves in a Surface

In this section we consider three geometrically significant types of curves in a surface $M \subset \mathbf{R}^3$.

6.1 Definition A regular curve α in $M \subset \mathbf{R}^3$ is a *principal curve* provided that the velocity α' of α always points in a principal direction.

Thus principal curves always travel in directions for which the bending of M in $\mathbf{R}^3$ takes its extreme values. Neglecting changes of parametrization, there are exactly two principal curves through each nonumbilic point of M—and these necessarily cut orthogonally across each other. (At an umbilic point $\mathbf{p}$, every direction is principal, and near $\mathbf{p}$ the pattern of principal curves can be quite complicated.)

6.2 Lemma Let α be a regular curve in $M \subset \mathbf{R}^3$, and let U be a unit normal vector field restricted to α. Then

(1) The curve α is principal if and only if U' and α' are collinear at each point.

(2) If α is a principal curve, then the principal curvature of M in the direction of α' is $\alpha'' \bullet U/\alpha' \bullet \alpha'$.

Proof. (1) Exercise 1.1 shows that $S(\alpha') = -U'$. Thus U' and α' are collinear if and only if $S(\alpha')$ and α' are collinear. But by Theorem 2.5, this amounts to saying that α' always points in a principal direction or, equivalently, that α is a principal curve.

(2) Since α is a principal curve, the vector field $\alpha'/\|\alpha'\|$ consists entirely of (unit) principal vectors belonging to, say, the principal curvature k_i. Thus

$$k_i = k(\alpha'/\|\alpha'\|) = S(\alpha'/\|\alpha'\|) \bullet \alpha'/\|\alpha'\|$$
$$= \frac{S(\alpha') \bullet \alpha'}{\alpha' \bullet \alpha'} = \frac{\alpha'' \bullet U}{\alpha' \bullet \alpha'},$$

where the last equality uses Lemma 2.1. ◆

In this lemma, (1) is a simple criterion for a curve to be principal, while (2) gives the principal curvature along a curve known to be principal.

6.3 Lemma Let α be a curve cut from a surface $M \subset \mathbf{R}^3$ by a plane P. If the angle between M and P is constant along α, then α is a principal curve of M.

Proof. Let U and V be unit normal vector fields to M and P (respectively) along the curve α, as shown in Fig. 5.29. Since P is a plane, V is parallel, that is, $V' = 0$. The constant-angle assumption means that $U \bullet V$ is constant; thus

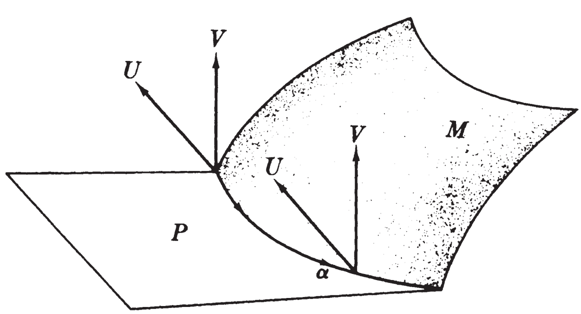

FIG. 5.29

$$0 = (U \bullet V)' = U' \bullet V.$$

Since U is a unit vector, U' is orthogonal to U as well as to V. The same is of course true of α', since α lies in both M and P. If U and V are linearly independent (as in Fig. 5.29) we conclude that U' and α' are collinear; hence by Lemma 6.2, α is principal.

However, linear independence fails only when $U = \pm V$. But then $U' = 0$, so α is (trivially) principal in this case as well. ◆

Using this result, it is easy to see that *the meridians and parallels of a surface of revolution M are its principal curves.* Indeed, each meridian μ is sliced from M by a plane *through* the axis of revolution and hence orthogonal to M along μ, while each parallel π is sliced from M by a plane *orthogonal* to the axis, and by rotational symmetry such a plane makes a constant angle with M along π.

Directions tangent to $M \subset \mathbf{R}^3$ in which the normal curvature is zero are called *asymptotic directions.* Thus a tangent vector $\mathbf{v}$ is *asymptotic* provided $k(\mathbf{v}) = S(\mathbf{v}) \bullet \mathbf{v} = 0$, so in an asymptotic direction, M is (instantaneously, at least) not bending away from its tangent plane.

Using Corollary 2.6 we can get a complete analysis of asymptotic directions in terms of Gaussian curvature.

6.4 Lemma Let $\mathbf{p}$ be a point of $M \subset \mathbf{R}^3$.

(1) If $K(\mathbf{p}) > 0$, then there are no asymptotic directions at $\mathbf{p}$.

(2) If $K(\mathbf{p}) < 0$, then there are exactly two asymptotic directions at $\mathbf{p}$, and these are bisected by the principal directions (Fig. 5.30) at angle ϑ such that

$$\tan^2 \vartheta = \frac{-k_1(\mathbf{p})}{k_2(\mathbf{p})}.$$

(3) If $K(\mathbf{p}) = 0$, then *every* direction is asymptotic if $\mathbf{p}$ is a planar point; otherwise there is exactly one asymptotic direction and it is also principal.

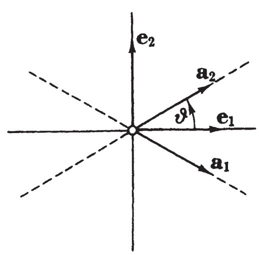

FIG. 5.30

Proof. These cases all derive from Euler's formula

$$k(\mathbf{u}) = k_1(\mathbf{p})\cos^2\vartheta + k_2(\mathbf{p})\sin^2\vartheta$$

in Corollary 2.6.

(1) Since $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have the same sign, $k(\mathbf{u})$ is never zero.

(2) Here $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have opposite signs, and we can solve the equation $0 = k_1(\mathbf{p})\cos^2\vartheta + k_2(\mathbf{p})\sin^2\vartheta$ to obtain the two asymptotic directions.

(3) If $\mathbf{p}$ is planar, then

$$k_1(\mathbf{p}) = k_2(\mathbf{p}) = 0;$$

hence $k(\mathbf{u})$ is identically zero. If just $k_2(\mathbf{p}) = 0$, then

$$k(\mathbf{u}) = k_1(\mathbf{p})\cos^2\vartheta$$

is zero only when $\cos\vartheta = 0$, that is, in the principal direction $\mathbf{u} = \mathbf{e}_2$. ◆

We can get an approximate idea of the asymptotic directions at a point $\mathbf{p}$ of a given surface M by picturing the intersection of the tangent plane $T_p(M)$ with M near $\mathbf{p}$. When $K(\mathbf{p})$ is negative, this intersection will consist of two curves through $\mathbf{p}$ whose tangent lines (at $\mathbf{p}$) are asymptotic directions (Exercise 5 of Section 53).

Figure 5.31 shows the two asymptotic directions A and A' at a point $\mathbf{p}$ on the inner equator of a torus.

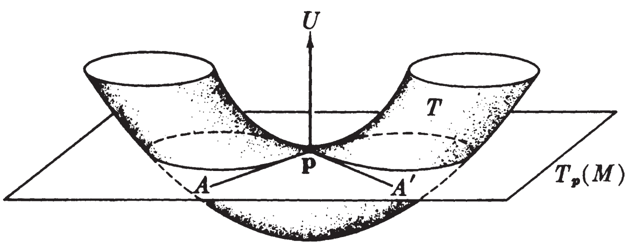

FIG. 5.31

6.5 Definition A regular curve α in $M \subset \mathbf{R}^3$ is an *asymptotic curve* provided its velocity α' always points in an asymptotic direction.

Thus α is asymptotic if and only if

$$k(\alpha') = S(\alpha') \bullet \alpha' = 0.$$

Since $S(\alpha') = -U'$, this gives a criterion, $U' \bullet \alpha' = 0$, for α to be asymptotic. Asymptotic curves are more sensitive to Gaussian curvature than are principal curves: Lemma 6.3 shows that there are none in regions where K is positive, but two cross (at an angle depending on K) at each point of a region where K is negative.

The simplest criterion for a curve in M to be asymptotic is that *its acceleration α'' always be tangent to M.* In fact, differentiation of $U \bullet \alpha' = 0$ gives

$$U' \bullet \alpha' + U \bullet \alpha'' = 0,$$

so $U' \bullet \alpha' = 0$ (α asymptotic) if and only if $U \bullet \alpha'' = 0$.

The analysis of asymptotic directions in Lemma 6.4 has consequences for both flat and minimal surfaces. First, *a surface M in $\mathbf{R}^3$ is minimal if and only if there exist two* orthogonal *asymptotic directions at each of its points.* In fact, $H(\mathbf{p}) = 0$ is equivalent to $k_1(\mathbf{p}) = -k_2(\mathbf{p})$, and an examination of the possibilities in Lemma 5.4 shows that $k_1(\mathbf{p}) = -k_2(\mathbf{p})$ if and only if either (a) $\mathbf{p}$ is planar (so the criterion holds trivially) or (b)

$$K(\mathbf{p}) < 0 \quad \text{with } \vartheta = \pm\pi/4,$$

which means that the two asymptotic directions are orthogonal.

Thus a surface is minimal if and only if through each point there are two asymptotic curves that cross *orthogonally.* This observation gives geometric meaning to the calculations in Example 4.3, which show that the helicoid is a minimal surface. In fact, the u- and v-parameter curves of the patch $\mathbf{x}$ are orthogonal since $F = 0$, and their accelerations are tangent to the surface since $\mathrm{L} = U \bullet \mathbf{x}_{uu} = 0$ and $\mathrm{N} = U \bullet \mathbf{x}_{vv} = 0$.

Recall that a *ruled surface* is swept out by a line moving through $\mathbf{R}^3$ (Definition 2.6 in Chapter 4). We have seen, for example, that the helicoid and saddle surface in Example 4.3 are ruled surfaces. Thus it is no accident that both these surfaces have K negative, since:

6.6 Lemma A ruled surface M has Gaussian curvature $K \leqq 0$. Furthermore, $K = 0$ if and only if the unit normal U is parallel along each ruling of M (so all points $\mathbf{p}$ on a ruling have the same Euclidean tangent plane $T_p(M)$).

Proof. A straight line $t \to \mathbf{p} + t\mathbf{q}$ is certainly *asymptotic* since its acceleration is zero and is thus trivially tangent to M. By definition a ruled surface contains a line through each of its points, so there is an asymptotic direction at each point. Hence, by Lemma 6.4, $K \leqq 0$.

Now let $\alpha(t) = \mathbf{p} + t\mathbf{q}$ be an arbitrary ruling in M. If U is parallel along α, then $S(\alpha') = -U' = 0$. Thus α is a principal curve with principal curvature $k(\alpha') = 0$, so $K = k_1 k_2 = 0$.

Conversely, if $K = 0$ we deduce from Case (3) in Lemma 6.4 that asymptotic directions (and curves) in M are also *principal.* Thus each ruling is principal ($S(\alpha') = k(\alpha')\alpha'$) as well as asymptotic ($k(\alpha') = 0$); hence

$$U' = -S(\alpha') \times 0, \qquad \blacklozenge$$

and U is parallel along each ruling of M.

We come now to the last and most important of the three types of curves under discussion.

6.8 Definition A curve α in $M \subset \mathbf{R}^3$ is a *geodesic* of M provided its acceleration α'' is always normal to M.

Since α'' is normal to M, the inhabitants of M perceive no acceleration at all—for them the geodesic is a "straight line." A full study of geodesics is given in later chapters, where, in particular, we examine their character as shortest routes of travel. Geodesics are far more plentiful in a surface M than are principal or asymptotic curves. Indeed, Theorem 4.2 of Chapter 7 will show that given any tangent vector $\mathbf{v}$ to M there is a (unique) geodesic with initial velocity $\mathbf{v}$.

Because the acceleration α'' of a geodesic is orthogonal to M, it is orthogonal to the velocity α' of α. Thus *geodesics have constant speed*, since differentiation of $\|\alpha'\|^2 = \alpha' \bullet \alpha'$ gives $2\alpha' \bullet \alpha'' = 0$.

A straight line $\alpha(t) = \mathbf{p} + t\mathbf{q}$ contained in M is always a geodesic of M since its acceleration $\alpha'' = 0$ is trivially normal to M. Though they lack any geometric significance, constant curves are also geodesics, but to avoid clutter this case is often neglected.

6.9 Example *Geodesics of some surfaces in* $\mathbf{R}^3$.

(1) *Planes.* If α is a geodesic in a plane P orthogonal to $\mathbf{u}$, then $\alpha' \bullet \mathbf{u} = 0$, hence $\alpha'' \bullet \mathbf{u} = 0$. But α'' is by definition normal to P, hence collinear with $\mathbf{u}$, so $\alpha'' = 0$. Thus α is a straight line. Since as noted above, every such line is a geodesic, we conclude that *the geodesics of P are the straight lines in P.*

(2) *Spheres.* A *great circle* in a sphere $\Sigma \subset \mathbf{R}^3$ is a circle cut from Σ by a plane P through the center (Fig. 5.32). If α is a constant-speed parametrization of any circle, we know that its acceleration α'' points toward the center of the circle. In the case of a great circle that center is also the center of the sphere Σ. Thus α'' is normal to Σ, so α is a geodesic of Σ.

We can find such a geodesic with any given initial velocity $\mathbf{v}_p$ (the required plane P passes through $\mathbf{p}$ orthogonal to $\mathbf{p} \times \mathbf{v}$). Hence by the uniqueness

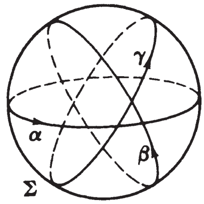

FIG. 5.32

feature mentioned earlier, this construction yields all the geodesics of Σ. Explicitly, *the geodesics of a sphere are the constant-speed parametrizations of its great circles* (Fig. 5.32).

(3) *Cylinders.* The geodesics of, say, the circular cylinder M: $x^2 + y^2 = r^2$ are all curves of the form

$$\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d).$$

To see this, write an arbitrary curve in M as

$$\alpha(t) = (r\cos \vartheta(t), r\sin \vartheta(t), h(t)).$$

A vector normal to M must have z coordinate zero. Thus if α is a geodesic, $h'' = 0$, so $h(t) = ct + d$. Since the speed of a geodesic is constant, the speed $(r^2\vartheta'^2 + h'^2)^{1/2}$ of α is constant, so ϑ' is constant. Hence $\vartheta(t) = at + b$.

When both constants a and c are nonzero, α is a helix on M. In extreme cases, α parametrizes a ruling if $a = 0$ and a cross-sectional circle if $c = 0$.◆

A *closed geodesic* is a geodesic segment α: $[a, b] \to M$ that is smoothly closed ($\gamma'(b) = \gamma'(a)$) and hence extendible by periodicity over the whole real line. Thus closed geodesics and periodic geodesics are effectively the same thing. In the surfaces above, every geodesic of the sphere is closed, while on the cylinder only the cross-sectional circles are closed.

6.10 Remark Here is a simple geometric way to find examples of geodesics. If a unit-speed curve α in M lies in a plane P everywhere orthogonal to M along α, then α is a geodesic of M. *Proof.* Since α has constant speed, α'' is always orthogonal to α', but these two vectors lie in a plane orthogonal to M, and α' is always tangent to M. Hence α'' must be orthogonal to M, so α is geodesic.

Using this remark we could have found all the geodesics in the preceding example except the helices in the cylinder. It shows at once that on a surface of revolution M, *all meridians are geodesics*, since they are cut from M by planes passing through the axis of rotation and hence orthogonal to M.

The essential properties of the three types of curves we have considered can be summarized as follows:

Principal curves	$k(\alpha') = k_1$ or k_2,	$S(\alpha')$ collinear α',	
Asymptotic curves	$k(\alpha') = 0$,	$S(\alpha')$ orthogonal to α',	α'' tangent to M
Geodesics			α'' normal to M

Exercises

1. Prove that a curve α in M is a straight line of $\mathbf{R}^3$ if and only if α is both geodesic and asymptotic.

2. To which of the three types—principal, asymptotic, geodesic—do the following curves belong?
(a) The top circle α of a torus (Fig. 5.33).
(b) The outer equator β of a torus.
(c) The x axis in M: $z = xy$.

(Assume constant-speed parametrizations.)

3. (*Closed geodesics.*) Show:
(a) In a surface of revolution, a parallel through a point $\alpha(t)$ on the profile curve is a (necessarily closed) geodesic if and only if $\alpha'(t)$ is parallel to the axis of revolution.
(b) There are at least three closed geodesics on every ellipsoid (Ex. 9 of Sec. 4.2).

4. Let α be an asymptotic curve in $M \subset \mathbf{R}^3$ with curvature $\kappa > 0$.
(a) Prove that the binormal B of α is normal to the surface along α, and deduce that $S(T) = \tau N$.
(b) Show that along α the surface has Gaussian curvature $K = -\tau^2$.
(c) Use (b) to find the Gaussian curvature of the helicoid (Example 4.3).

5. Suppose that a curve α lies in two surfaces M and N that make a constant angle along α (that is, $U \bullet V$ constant). Show that α is principal in M if and only if principal in N.

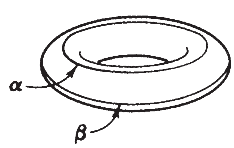

FIG. 5.33

6. If $\mathbf{x}$ is a patch in M, prove that a curve $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$ is
(a) Principal if and only if

$$\begin{vmatrix} a_2'^2 & -a_1'a_2' & a_1'^2 \\ E & F & G \\ \text{L} & \text{M} & \text{N} \end{vmatrix} = 0.$$

(b) Asymptotic if and only if $\text{L}a_1'^2 + 2\text{M}a_1'a_2' + \text{N}a_2'^2 = 0$.

7. Let α be a unit-speed curve in $M \subset \mathbf{R}^3$. Instead of the Frenet frame field on α, consider the *Darboux frame field* T, V, U—where T is the unit tangent of α, U is the surface normal restricted to α, and $V = U \times T$ (Fig. 5.34).
(a) Show that

$$\begin{aligned} T' &= \qquad\quad gV + kU, \\ V' &= -gT \qquad\quad + tU, \\ U' &= -kT - tV, \end{aligned}$$

where $k = S(T) \bullet T$ is the normal curvature $k(T)$ of M in the T direction, and $t = S(T) \bullet V$.
The new function g is called the *geodesic curvature* of α.
(b) Deduce that α is

$$\begin{aligned} \text{geodesic} &\Leftrightarrow g = 0, \\ \text{asymptotic} &\Leftrightarrow k = 0, \\ \text{principal} &\Leftrightarrow t = 0. \end{aligned}$$

8. If α is a (unit speed) curve in M, show that
(a) α is both principal and geodesic if and only if it lies in a plane everywhere orthogonal to M along α.
(b) α is both principal and asymptotic if and only if it lies in a plane everywhere tangent to M along α.

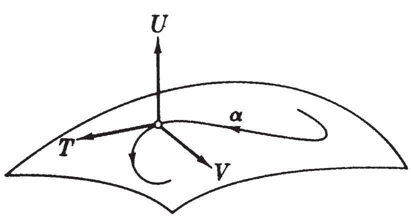

FIG. 5.34

9. On the monkey saddle M (see Fig. 5.19) find *three* asymptotic curves and *three* principal curves passing through the origin **0**. (This is possible only because **0** is a planar umbilic point.)

10. Let α be a regular curve in $M \subset \mathbf{R}^3$, and let U be the unit normal of M along α. Show that α is a principal curve of M if and only if the ruled surface $\mathbf{x}(u, v) = \alpha(u) + vU(u)$ is flat.

11. A ruled surface is *noncylindrical* if its rulings are always changing directions; thus for any director curve, $\delta \times \delta' \neq 0$. Show that:

(a) a noncylindrical ruled surface has a parametrization

$$\mathbf{x}(u, v) = \sigma(u) + v\delta(u)$$

for which $\|\delta\| = 1$ and $\sigma' \bullet \delta' = 0$.

(b) for this parametrization,

$$K = \frac{-p^2(u)}{\left(p^2(u) + v^2\right)^2}, \quad \text{where } p = \frac{\sigma' \bullet \delta \times \delta'}{\delta' \bullet \delta'}.$$

The curve σ is called the *striction curve*, and the function p is the *distribution parameter.*

(c) Deduce from the behavior of K on each ruling that the route of the striction curve is independent of parametrization, and hence that the distribution parameter is essentially a function on the set of rulings.

(*Hint:* For (a), find f such that $\sigma = \alpha + f\delta$. For (b), show that $\sigma' \times \delta = p\delta'$.)

12. In each case below, find the striction curve and distribution parameter, and check the formula for K in (b) of the preceding exercise.

(a) the helicoid in Example 4.3.

(b) the tangent surface of a curve (Ex. 13 of Sec. 4).

(c) both sets of rulings of the saddle surface in Example 4.3.

(*Hint:* In the usual ruled parametrization $(u, 0, 0) + v(0, 1, u)$, this last vector must be replaced by a unit vector in order to apply Ex. 11. The curvature formula in Example 4.3 will then change.)

13. If $\mathbf{x}(u, v) = \alpha(u) + v\delta(u)$ parametrizes a noncylindrical ruled surface, let $L(u)$ be the ruling through $\alpha(u)$. Show that:

(a) If ϑ_ε is the smallest angle from $L(u)$ to $L(u + \varepsilon)$, and d_ε is the orthogonal distance from $L(u)$ to $L(u + \varepsilon)$, then

$$\lim_{\varepsilon \to 0} \frac{d_\varepsilon}{\vartheta_\varepsilon} = p(u).$$

Thus the distribution parameter is the rate of turning of L.

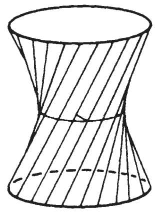

FIG. 5.35

(b) There is a unique point $\mathbf{p}_\varepsilon$ of $L(u)$ that is nearest to $L(u + \varepsilon)$, and

$$\lim_{\varepsilon\to 0} \mathbf{p}_\varepsilon = \sigma(u)$$

(Fig. 5.35). (This gives another characterization of the striction curve σ.)

(*Hint:* The common perpendicular to $L(u)$ and $L(u + \varepsilon)$ is in the direction of $\delta(u) \times \delta(u + \varepsilon) \approx \varepsilon\delta(u) \times \delta'(u)$.)

14. Let $\mathbf{x}(u, v) = \alpha(u) + v\delta(u)$, with $\|\delta\| = 1$, parametrize a flat ruled surface M. Show that:

(a) If α is always zero, then M is a generalized cone.
(b) If δ is always zero, then M is a generalized cylinder.
(c) If both α' and δ' are never zero, then M is the tangent surface of its striction curve. (*Hint:* Parametrize by $\sigma + v\delta$ as in Ex. 11, giving σ unit speed. Use $K = 0$ to show that $T = \sigma'$ and δ are collinear.)

These are only the extreme cases. For example, a flat piece of paper could be bent cylindrically at one end and conically at the other. Note that of the three types, only the cylinder has rulings that are entire straight lines.

15. (*Enneper's minimal surface.*) Prove:

(a) The mapping $\mathbf{x}$: $\mathbf{R}^2 \to \mathbf{R}^3$ given by

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right),$$

though not one-to-one, is regular, and hence defines an immersed surface $\mathscr{E}$.
(b) $\mathbf{x}$ is a principal parametrization of $\mathscr{E}$, that is, the u- and v-parameter curves are principal curves.
(c) $\mathscr{E}$ is a minimal surface.
(d) The asymptotic curves of $\mathscr{E}$ are $u \to \mathbf{x}(u, \pm u)$.

16. (*Continuation by computer graphics.*)

(a) Plot $\mathbf{x}(D) \subset \mathscr{E}$, for D: $-3 \leqq u, v \leqq 3$. (Note that by the preceding exercise the parameter curves are principal.)
(b) Show that the Euclidean isometry $(x, y, z) \to (-y, x, -z)$ carries $\mathscr{E}$ to itself.

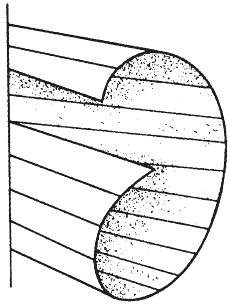

FIG. 5.36

Thus the $z < 0$ half of $\mathscr{E}$ is the mirror image of a 90° rotation of the $z > 0$ half. Further properties of $\mathscr{E}$ are developed in Exercise 10 of Section 6.8.

17. A *right conoid* is a ruled surface whose rulings all pass orthogonally through a fixed axis (Fig. 5.36). Taking this axis as the z axis of $\mathbf{R}^3$, we find the parametrization

$$\mathbf{x}(u, v) = (u\cos\vartheta(v), u\sin\vartheta(v), h(v)),$$

where the u-parameter curves are the rulings. (This reversal of u and v from earlier exercises makes it clear that the helicoid is a conoid.)

(a) Find the Gaussian and mean curvature of $\mathbf{x}$.

(b) Show that the surface is noncylindrical if ϑ' is never zero, and in this case, find the striction curve and parameter of distribution.

18. (*Computer graphics.*)

(a) A right conoid has base curve $\alpha(v) = (0, 0, \cos 2v)$ and director curve $\delta(v) = (\cos v, \sin v, 0)$. For the resulting ruled parametrization (with u-parameter curves as rulings), plot the portion with $-2.5 \leqq u \leqq 2.5$, $0 \leqq v \leqq \pi$.

(b) The axis of a right conoid is the z axis, and its rulings pass through every point of the circle $y^2 + z^2 = r^2$ in the plane $x = c$. Verify that

$$\mathbf{x}(u, v) = (uc, ur\cos v, r\sin v)$$

parametrizes this conoid, and for $r = 2$, $c = 1$ plot the portion between the axis and the circle.

(c) Same as (b) except that the circle is replaced by the curve $y = \sin z$. Find a ruled parametrization, and for $c = 4$, plot the region with $0 \leqq z \leqq 4\pi$ and rulings running from $x = -4$ to $x = +4$.

19. Prove that a surface that is both *ruled* and *minimal* is part of either a plane or a helicoid.

(*Hint:* Flat regions in M are planar; thus arguing as in Thm. 6.2 we can suppose $K < 0$. Use the parametrization in Ex. 11.)

5.7 Surfaces of Revolution

The geometry of a surface of revolution is rather simple, yet these surfaces exhibit a wide variety of geometric behavior; thus they offer a good field for experiment.

We apply the methods of Section 4 to study an arbitrary surface of revolution M, with the usual parametrization, given in Example 2.4 of Chapter 4 by

$$\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v).$$

Here $h(u) > 0$ is the radius of the parallel at distance $g(u)$ along the axis of revolution of M, as shown in Fig. 4.14. This geometric significance for g and h means that our results do not depend on the particular position of M relative to the coordinate axes of $\mathbf{R}^3$.

Because g and h are functions of u alone, we can write

$$\mathbf{x}_u = (g', h' \cos v, h' \sin v),$$
$$\mathbf{x}_v = (0, -h\sin v, h \cos v),$$

and hence

$$E = g'^2 + h'^2, \quad F = 0, \quad G = h^2.$$

Here E is the square of the speed of the profile curve and hence of all the meridians (u-parameter curves), while G is the square of the speed of the parallels (v-parameter curves). Next we find, successively,

$$\mathbf{x}_u \times \mathbf{x}_v = (hh', -hg'\cos v, -hg'\sin v),$$
$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2} = h\sqrt{g'^2 + h'^2},$$
$$U = \frac{1}{\sqrt{g'^2 + h'^2}}(h', -g'\cos v, -g'\sin v).$$

Taking second derivatives gives

$$\mathbf{x}_{uu} = (g'', h'' \cos v, h'' \sin v),$$
$$\mathbf{x}_{uv} = (0, -h'\sin v, h' \cos v),$$
$$\mathbf{x}_{vv} = (0, -h\cos v, -h \sin v),$$

Hence

$$\text{L} = \frac{-g'h'' + g''h'}{\sqrt{g'^2 + h'^2}}, \quad \text{M} = 0, \quad \text{N} = \frac{g'h}{\sqrt{g'^2 + h'^2}}.$$

Since $F = \text{M} = 0$, $\mathbf{x}$ is a principal parametrization (Exercise 8 of Section 4), and for the shape operator S derived from U,

$$S(\mathbf{x}_u) = \frac{\text{L}}{E}\mathbf{x}_u, \quad S(\mathbf{x}_v) = \frac{\text{N}}{G}\mathbf{x}_v.$$

This is an analytical proof that the *meridians* and *parallels* of a surface of revolution are its principal curves. Furthermore, if the corresponding principal curvature functions are denoted by k_μ and k_π, instead of k_1 and k_2, we have

$$k_\mu = \frac{\text{L}}{E} = \frac{-\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{(g'^2 + h'^2)^{3/2}}, \quad k_\pi = \frac{\text{N}}{G} = \frac{g'}{h(g'^2 + h'^2)^{1/2}}. \tag{1}$$

Thus the Gaussian curvature of M is

$$K = k_\mu k_\pi = \frac{-g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h(g'^2 + h'^2)^2}. \tag{2}$$

This formula defines K as a real-valued function on the domain of the profile curve

$$\alpha(u) = (g(u), h(u), 0).$$

By the conventions of Section 4, $K(u)$ is the Gaussian curvature $K(\mathbf{x}(u, v))$ of *M at every point of the parallel through* $\alpha(u)$. The same is true for the other functions above. The rotational symmetry of M about its axis of revolution means that its geometry is "constant on parallels"—completely determined by the profile curve.

In the special case where the profile curve passes at most once over each point of the axis of rotation, we can usually arrange for the function g to be simply $g(u) = u$ (Exercise 2.8 of Chapter 4). Then the formulas (1) and (2) above reduce to

$$k_\mu = \frac{-h''}{(1 + h'^2)^{3/2}}, \quad k_\pi = \frac{1}{h(1 + h'^2)^{1/2}},$$
$$K = \frac{-h''}{h(1 + h'^2)^2}. \tag{3}$$

7.1 Example *Surfaces of revolution.*

(1) *Torus of revolution T.* The usual parametrization **x** in Example 2.5 of Chapter 4 has

$$g(u) = r \sin u, \quad h(u) = R + r \cos u,$$

for constants $0 < r < R$. Although the axis of revolution is now the z axis, formulas (1) and (2) above remain valid, and we compute

$$E = r^2, \quad F = 0, \quad G = (R + r \cos u)^2,$$
$$\text{L} = r, \quad \text{M} = 0, \quad \text{N} = (R + r \cos u)\cos u,$$
$$k_\mu = \frac{1}{r}, \quad k_\pi = \frac{\cos u}{R + r \cos u},$$
$$K = \frac{\cos u}{r(R + r \cos u)}.$$

This gives an analytical proof that the Gaussian curvature of the torus is positive on the outer half and negative on the inner half. K has its maximum value $1/r(R + r)$ on the outer equator ($u = 0$), its minimum value $-1/r(R - r)$ on the inner equator ($u = \pi$), and is zero on the top and bottom circles ($u = \pm\pi/2$).

(2) *Catenoid.* The curve $y = c \cosh(x/c)$ is a *catenary*; its shape is that of a chain hanging under the influence of gravity. The surface obtained by rotating this curve around the x axis is called a *catenoid* (Fig. 5.37). From the formulas (3) we find,

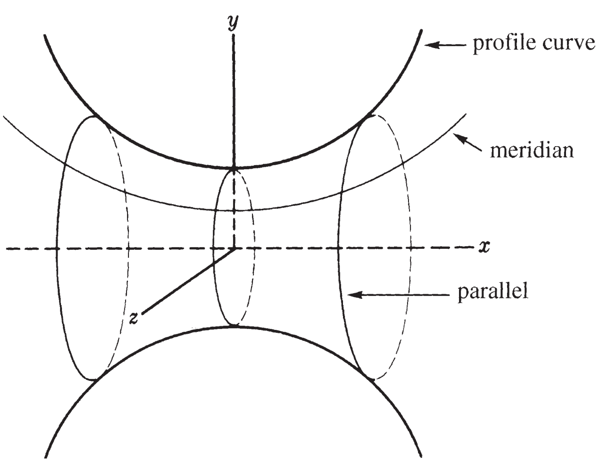

FIG. 5.37

$$-k_\mu = k_\pi = \frac{1}{c\cosh^2(x/c)},$$

and hence

$$H = 0, \quad K = \frac{-1}{c^2\cosh^4(u/c)}.$$

Since its mean curvature H is zero, the catenoid is a minimal surface. Its Gaussian curvature interval is $-1/c^2 \leqq K < 0$, with minimum value $K = -1/c^2$ on the central circle ($u = 0$). ◆

7.2 Theorem If a surface of revolution M is a minimal surface, then M is contained in either a plane or a catenoid.

Proof. M is parametrized as usual by

$$\mathbf{x}(u, v) = (g(u), h(u)\cos v, h(u)\sin v),$$

with u in a (possibly infinite) interval I.

Case 1. g' is identically zero. Then g is constant, so M is part of a plane orthogonal to the axis of revolution.

Case 2. g' is never zero. By Exercise 8 in Section 4.2, M has a parametrization of the form

$$\mathbf{y}(u, v) = (u, h(u)\cos v, h(u)\sin v).$$

The formulas for k_μ and k_π in (3) above then show that the minimality condition is equivalent to

$$hh'' = 1 + h'^2.$$

Because u does not appear explicitly in this differential equation, there is a standard elementary way to solve it. We merely record that the solution is

$$h(u) = a\cosh\left(\frac{u}{a} + b\right),$$

where $a \neq 0$ and b are constants. Thus M is part of a catenoid.

Case 3. g' is zero at some points, nonzero at others. This cannot happen. For definiteness, suppose that $g'(u) > 0$ for $u < u_0$ but $g'(u_0) = 0$. By Case 2, the profile curve $(g(u), h(u), 0)$ is a catenary for $u < u_0$. The shape of the catenary makes it clear that slope h'/g' cannot approach infinity as $u \to u_0$. ◆

This result shows that *catenoids are the only complete nonplanar surfaces of revolution that are minimal.* (Completeness, discussed in Chapter 8, implies that the surface cannot be part of a larger surface.)

Helicoids and catenoids are called the elementary minimal surfaces. Two others are given in the exercises for this chapter (Exercise 5 in Section 5 and Exercise 15 in Section 6). Soap-film models of an immense variety of minimal surfaces can easily be exhibited by the methods given in [dC], where the term "minimal" is explained.

The expression $\sqrt{g'^2 + h'^2}$, which appears so frequently in the formulas above, is just the speed of the profile curve $\alpha(u) = (g(u), h(u), 0)$. Thus we can radically simplify these formulas by a reparametrization that has unit speed. The surface of revolution is unchanged; it has merely been given a new parametrization, called *canonical.*

7.3 Lemma For a canonical parametrization of a surface of revolution,

$$E = 1, \quad F = 0, \quad G = h^2,$$

and the Gaussian curvature is

$$K = \frac{-h''}{h}.$$

Proof. Since $g'^2 + h'^2 = 1$ for a canonical parametrization, these expressions for E, F, and G follow immediately from those at the start of this section. The formula for K in (2) becomes

$$K = \frac{-g'}{h}\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix} = \frac{-g'^2h'' + g'g''h'}{h}.$$

But this can be simplified. Differentiation of $g'^2 + h'^2 = 1$ gives $g'\, g'' = -h'\, h''$, and when this is substituted above, we get $K = -h''/h$. ◆

The effect of using a canonical parametrization is to shift the emphasis from measurements in the space *outside M* (for example, along the axis of revolution) to measurement *within M.* This important idea will be developed more fully as we proceed.

7.4 Example *Canonical parametrization of the catenoid* ($c = 1$). An arc length function for the catenary $\alpha(u) = (u, \cosh u)$ is $s(u) = \sinh u$. Hence a unit-speed reparametrization is

$$\beta(s) = (g(s), h(s)) = \left(\sinh^{-1} s, \sqrt{1+s^2}\right),$$

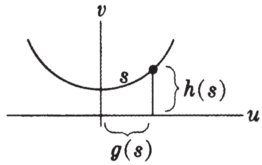

FIG. 5.38

as indicated in Fig. 5.38. The resulting canonical parametrization of the catenoid is given by

$$\bar{\boldsymbol{x}}(s, v) = \left(\sinh^{-1} s, \sqrt{1+s^2}\ \cos v, \sqrt{1+s^2}\ \sin v\right).$$

Hence by the preceding lemma,

$$K(s) = -\frac{h''(s)}{h(s)} = \frac{-1}{(1+s^2)^2}.$$

This formula for K in terms of $\bar{\boldsymbol{x}}$ is consistent with the formula

$$K(u) = \frac{1}{\cosh^4 u}$$

found in Example 6.1 for the parametrization $\mathbf{x}_1$. In fact, since $s(u) = \sinh u$, we have

$$K(s(u)) = \frac{-1}{(1+s^2(u))^2} = \frac{-1}{(1+\sinh^2 u)^2} = \frac{-1}{\cosh^4 u}.$$

The simple formula for K in Lemma 7.3 suggests a way to construct surfaces of revolution with *prescribed* Gaussian curvature. Given a function

$$K = K(u) \text{ on some interval,}$$

first solve the differential equation $h'' + Kh = 0$ for h, subject to initial conditions $h(0) > 0$ and $|h'(0)| < 1$. (The first of these conditions is a convenience; the second is a necessity since we must have $g'^2 + h'^2 = 1$.)

To get a canonical parametrization, we need a function g satisfying the equation $g'^2 + h'^2 = 1$. Evidently,

$$g(u) = \int_0^u \sqrt{1 - h'^2(t)}\, dt$$

will do the job.

We conclude that for any interval around 0 on which the initial conditions

$$h > 0 \text{ and } |h'| < 1$$

both hold, revolving the profile curve ($g(u)$, $h(u)$, 0) around the x axis produces a surface that has, by Lemma 7.3, Gaussian curvature $K = -h''/h$.

A natural use of this scheme is to look for surfaces that have *constant* curvature. Consider first the K positive case.

7.5 Example *Surfaces of revolution with constant positive curvature.* We apply the procedure to the constant function $K = 1/c^2$. The differential equation $h'' + h/c^2 = 0$ has general solution

$$h(u) = a\cos\left(\frac{u}{c} + b\right).$$

The constant b represents only a translation of coordinates so we may as well set $b = 0$. As usual, nothing is lost by requiring $h > 0$; hence $a > 0$. Thus the functions

$$g(u) = \int_0^u \left(1 - \frac{a^2}{c^2}\sin^2\frac{t}{c}\right)^{\frac{1}{2}} dt, \quad h(u) = a\cos\frac{u}{c}$$

give rise to a surface of revolution M_a with constant Gaussian curvature

$$K = 1/c^2.$$

As mentioned above, the conditions $h > 0$ and $|h'| < 1$ determine the largest interval I on which the procedure works. The constant c is fixed, but the constant a is at our disposal, and it distinguishes three cases.

Case 1. $a = c$. Here

$$g(u) = \int_0^u \cos\frac{t}{c}\,dt = c\sin\frac{u}{c}, \quad h(u) = a\cos\frac{u}{c}. \tag{4}$$

Thus the maximum interval I is $-\pi c/2 < u < \pi c/2$, and the profile curve ($g(u)$, $h(u)$) is a semicircle. Revolution about the x axis produces a sphere Σ of radius c—except for its two points on the axis.

Case 2. $0 < a < c$. Here h is positive on the same interval as above and $|h'| < 1$ is always true, so g is well defined. The profile curve has the same length $\pi c/2$, but it now forms a shallower arch, which rests on the x axis at $\pm a^*$, where $a^* = g(\pi c/2) > a$ (Fig. 5.39). As c shrinks down from c to 0, one can check that a^* increases from c to $\pi c/2$. The resulting surface of revolution, round when $a = c$, first becomes football-shaped and then grows ever thinner, becoming, for a small, a needle of length just less than $\pi c/2$.

By contrast with Case 1, the intercepts ($\pm a_*$, 0, 0) cannot be added to M now since this surface is actually pointed at each end (Fig. 5.39).

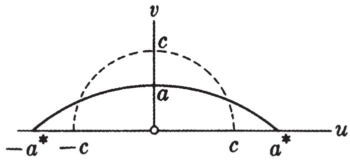

FIG. 5.39

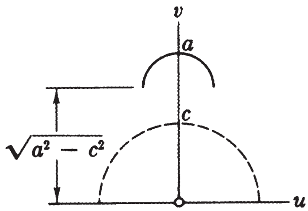

FIG. 5.40

The differential equation $h'' + 1/c^2\, h = 0$ has delicately adjusted the shape of M_a so that its principal curvatures are no longer equal but still give

$$K = k_\mu k_\pi = \frac{1}{c^2}.$$

Case 3. $a > c$. Here the maximum interval is shorter than in Case 1. The formula for $g(u)$ in (4) shows that the endpoints now are $\pm a_*$, where $a_* < c$ is determined by $\sin a_*/c = c/a < 1$. Thus,

$$h(a_*) = a \cos a_*/c = \sqrt{a^2 - c^2}.$$

As a increases from $a = c$, the resulting surface of revolution M_a is at first somewhat like the outer half of a torus. But when a is very large, it becomes a huge circular band (Fig. 5.40), whose very short profile curve is sharply curved (k_μ must be large since $k_\pi \approx 1/a$ and $k_\mu k_\pi = 1/c^2$).

A corresponding analysis for constant negative curvature leads to an infinite family of surfaces of revolution with $K = -1/c^2$ (Exercises 7 and 8). The simplest of these surfaces is

7.6 Example *The bugle surface B.* The profile curve of B (in the xy plane) is characterized by this geometric condition: It starts at the point $(0, c)$ and moves so that its tangent line reaches the x axis after running for distance exactly c. This curve, a *tractrix*, can be described analytically as

$$\alpha(u) = (u, h(u)),\ u > 0,$$

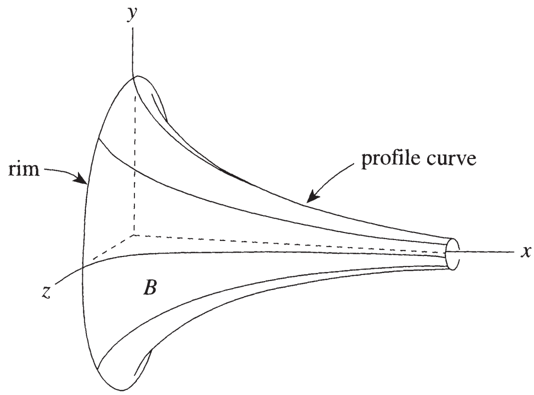

FIG. 5.41

where h is the solution of the differential equation

$$h' = \frac{-h}{\sqrt{c^2 - h^2}}$$

such that $h(u) \to c$ as $u \to 0$. The resulting surface of revolution B is called a *bugle surface* or *tractroid* (Fig. 5.41). Using the differential equation above, we deduce from the earlier formulas (3) that the principal curvatures of B are

$$k_\mu = \frac{-h'}{c}, \quad k_\pi = \frac{1}{ch'}.$$

Thus the bugle surface has constant negative curvature

$$K = -\frac{1}{c^2}.$$

This surface cannot be extended across its rim—not part of B—to form a larger surface in $\mathbf{R}^3$ since $k_\mu(u) \to \infty$ as $u \to 0$. ◆

When this surface was first discovered, it seemed to be the analogue, for K a negative constant, of the sphere; it was thus called a *pseudosphere*. However, as we shall see later on, the true analogue of the sphere is quite a different surface and cannot be found in $\mathbf{R}^3$.

Exercises

1. Find the Gaussian curvature of the surface obtained by revolving the curve $y = e^{-x^2/2}$ around the x axis. Sketch this surface and indicate the regions where $K > 0$ and $K < 0$.

2. (a) Show that when $y = f(x)$ is revolved around the x axis, the Gaussian curvature $K(x)$ has the same sign (–, 0, +) as $-f''(x)$ for all x.
(b) Deduce that for a surface of revolution with arbitrary axis, the Gaussian curvature K is positive on parallels through *convex* intervals on the profile curve (where the curve bulges away from the axis) and negative on parallels through *concave* intervals (where the curve sags toward the axis).

3. Prove that a flat surface of revolution is part of a plane, cone, or cylinder.

4. (*Computer.*)
(a) Write computer commands that, given a profile curve $u \to (g(u), h(u))$, (i) plot the resulting surface of revolution for $a \leqq u \leqq b$, and (ii) return its Gaussian curvature $K(u)$.
(b) Test (a) on the torus and catenoid in Example 7.1.

5. If $r = \sqrt{x^2 + y^2}$ is the usual polar coordinate function on the xy plane, and f is a differentiable function, show the M: $z = f(r)$ is a surface of revolution and that its Gaussian curvature K is given by

$$K(r) = \frac{f'(r)f''(r)}{r\left(1 + f'(r)^2\right)^2}.$$

6. Find the Gaussian curvature of that surface M: $z = e^{-r^2/2}$. Sketch this surface, indicating the regions where $K > 0$ and $K < 0$.

7. (*Surfaces of revolution with negative curvature* $K = -1/c^2$) As in the corresponding positive case, there is a family of such surfaces, separated into two subfamilies by a special surface. Essentially all these surfaces are given, using canonical parametrization, by solutions of $h'' - 1/c^2\, h = 0$ as follows:
(a) If $0 < a < c$, let M_a be the surface given by $h(u) = a \sinh u/c$, $u > 0$. Show that its profile curve $(g(u), h(u))$ leaves the origin with slope $a/\sqrt{c^2 - a^2}$ and rises to a maximum height of $\sqrt{c^2 - a^2}$.
(b) If $a = c$, let $\bar{B}$ be the surface given by $h(u) = ce^{u/c}$, $u < 0$. Show that its mirror image B, given by $h(u) = ce^{-u/c}$, $u > 0$, is the bugle surface in Example 7.6.

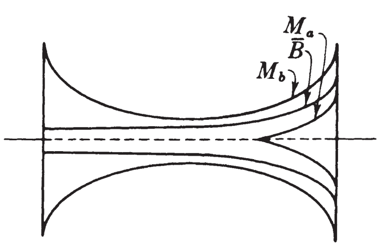

FIG. 5.42

(c) If $b > c$, let M_b be the surface given by $h(u) = b \cosh u/c$. Show that as $|u|$ increases from 0, its profile curve rises symmetrically from height b to height $\sqrt{c^2 + b^2}$.

Sample profile curves of all three types are shown in Fig. 5.42, where M_a and M_b have been translated along the axis of revolution. Explicit formulas for the profile curves in (a) and (b) involve elliptic integrals (see [G]).

8. (a) Taking $c = 1$ for simplicity, show that the tractrix has a parametrization (g, h) with

$$g(u) = e^{-u},\ h(u) = -\sqrt{1 - e^{-2u}} + \operatorname{arctanh}\sqrt{1 - e^{-2u}}.$$

(b) (*Computer graphics.*) Plot a view of the resulting bugle surface similar to that in Fig. 5.41.

9. In a *twisted* surface of revolution, as points rotate around the axis they also move evenly in the axis direction. Explicitly, if the original surface has a usual parametrization in terms of functions $g(u)$ and $h(u)$, then the twisted surface has parametrization

$$\mathbf{x}(u, v) = (g(u) + pv,\ h(u)\cos v,\ h(u)\sin v).$$

where p is a constant.

(a) Find a parametrization of the twisted bugle surface D (*Dini's surface*) with data as in the preceding exercise and $p = 1/5$.

(b) (*Computer.*) Plot the surface D in (b) for $0.01 \leqq u \leqq 2$ and $0 \leqq v \leqq 6\pi$. (Impose smoothness and view the surface from a point with $x < 0$.)

(c) Show that D has constant negative curvature.

5.8 Summary

The shape operator S of a surface M in $\mathbf{R}^3$ measures the rate of change of a unit normal U in any direction on M and thus describes the way the shape of M is changing in that direction. If we imagine U as the "first derivative" of M, then S is the "second derivative." But the shape operator is an algebraic object consisting of linear operators on the tangent planes of M. And it is by an algebraic analysis of S that we have been led to the main geometric invariants of a surface in $\mathbf{R}^3$: its principal curvatures and directions, and its Gaussian and mean curvatures.