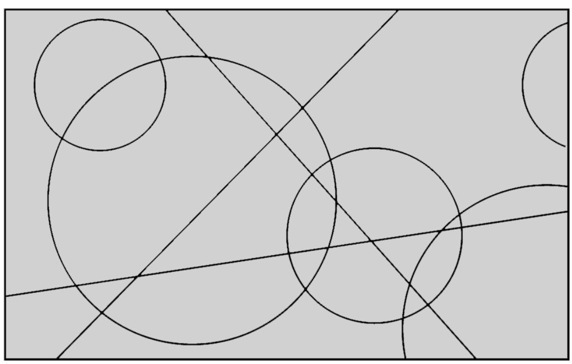
EXPLORATIONS IN TOPOLOGY: MAP COLORING, SURFACES, AND KNOTS



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To the memory of Dona Clyman
and to all my students: you made
this project worthwhile

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Preface

This book will (I hope!) give you a rich experience with low-dimensional topology, get you really hooked on solving mathematical problems, and launch you into an adventure of creating ideas, solutions, and techniques—one that gives you a "research experience" with topology. Of course, I also hope you get excited about this adventure, talk about it with your friends, and have a good time.

Topology is a geometric way of looking at the world and the ideas you will encounter in this book emerged from a long evolutionary process. I hope that this book engages you in this process while it introduces you to the topological point of view. Along the way, you will encounter many of the concrete, simply-stated problems for which much of modern topology was created to solve—problems associated with maps, networks, surfaces, and knots.

The book is more than just an introduction to topological thinking. It was designed to provide you with the opportunity to

- experience solving open-ended problems by guessing, drawing pictures, constructing models, looking for patterns, formulating conjectures, finding counter-examples, making arguments, asking questions, using analogies, and making generalizations;
- observe new mathematical techniques springing from solutions to problems;
- see the connectivity of mathematical ideas—the solution to one problem helps solve other problems; the answer to one problem leads to new questions.

Your participation in all this will help you place mathematical connections into bold relief and will elicit from you gasps of surprise! This participation will have frustrations, but it will also be full of pleasure, drama, and beauty. It will enhance your geometrical and topological intuition, empower you with new approaches to solving problems, and provide you with tools to help you on your next mathematical journey.

Finally, the book is set up so that—as much as possible—*you* are in charge of the development of the mathematical ideas that emerge. You ask the questions that propel the unfolding of your own topological knowledge and understanding and you initiate the process of answering them. The structure of the book maximizes the possibilities for all of this to happen. It poses key questions to begin the discussions and give them shape and direction. It models the problem-solving process and provides you with ways to communicate and explain solutions. And then it lets you loose, as a former student of mine put it, "to write our own book."

Format of the Chapters

Each chapter (excluding chapter 14) poses a big problem and takes a stab at its solution. This takes place in the context of a story whose characters are employees of Acme Maps. The big problem and smaller related problems come up naturally as the characters carry out their work. The jobs Acme takes on expand naturally as the book unfolds. The characters fiddle around with the problems much as real problem-solvers would. They make mistakes. They find themselves in dead ends and work their way out. They create their own definitions and terminology. Sometimes they don't solve the big problem, but they work on it and come up with partial solutions. These solutions are sometimes what a "mainstream" text would call theorems. As you follow the story, paper-and-pencil icons involve you by pointing out explicit tasks for you to carry out. These tasks are called **Your Turn**. A set of **Investigations, Questions, Puzzles, and More** replaces the traditional problem set at the end of the chapter. An investigation is a non-routine, open-ended problem: Look at such-and-such. What do you see? What can you say? Can you explain it? In an investigation you make observations, look for patterns, make a conjecture, and prove the conjecture (or come up with a counterexample and refine the conjecture). When I teach a course that goes with this book I ask students to carry out many of these investigations. They form the heart of the course.

To relate what goes on in the chapter to the rest of the world of mathematics each chapter ends with a section entitled **Notes**. It places the problems, concepts, and results of the chapter in their historical context. It introduces the standard terminology used by other mathematicians. Occasionally it will summarize the results of the chapter and set these results out in the form of theorems. A list of appropriate **References** follows the Notes section.

The format of chapter 14 differs from that of chapters 1-13. This chapter is an annotated list of possible projects that extend the concepts of the book, but which may involve research beyond the materials of the book. It includes an extensive list of helpful resource materials and a guide for carrying out a project and communicating its results.

Overview of the Chapters

- **Chapter 1.** This is an introduction to coloring maps on an island and on the sphere. Particular attention is paid to maps requiring two colors.
- **Chapter 2.** This is an introduction to networks (graphs)—to taking particularly "nice" trips on networks, and identifying those networks where such trips are possible.
- **Chapter 3.** This is an introduction to collecting data about maps and observing relationships among these data.
- **Chapter 4.** This continuation of chapter 3 applies relationships among map data to map coloring.
- **Chapter 5.** This final chapter on coloring maps on the sphere is an exposition of the first "proof" of the four color theorem.
- **Chapter 6.** This chapter considers the torus, a new surface on which to draw maps and networks and to solve (or not) problems which came up for the sphere.
- **Chapter 7.** This chapter looks at twisted strips, cuts them, and adds the Möbius strip to the repertoire of surfaces.
- **Chapter 8.** This chapter takes the idea of a "pattern" for a surface introduced in chapters 6 and 7 and uses it to create new, unusual surfaces (Klein bottle, crosscap) and to observe relationships among them.
- **Chapter 9.** This chapter introduces the Euler number for a surface and describes a technique for creating new surfaces out of old.
- **Chapter 10.** The study of surfaces becomes algebraic as symbol strings replace patterns and rules are developed for arriving at equivalent, recognizable symbols. The consequence: a classification theorem for surfaces.
- **Chapter 11.** This chapter tidies up the classification of surfaces by looking at boundaries (lakes). It also considers the "existence" of surfaces and offers the possibility of "assembling" them in four-space.
- **Chapter 12.** This chapter relates the number of colors needed to color a map on a surface (not the sphere) to its Euler number.
- **Chapter 13.** This introduction to knot theory and one of its invariants is motivated by problems in chemistry and biology.
- **Chapter 14.** As mentioned above, this chapter is devoted to projects.

Success in the Course this Book is based on

To get the goodies a course using this book has to offer, we all have to dig in and take risks—by offering a solution that may turn out to be incorrect, by making a guess that might be wrong, or by using an approach to a problem that might not pan out. We will laugh at our mistakes. We will listen to others' ideas and possibly build on them. We will learn to accept a solution only when **we** understand it, when it feels good in our tummies. It'll be a hoot!

Instructor's Manual

This supplement describes how the author uses this book with a class. It highlights each chapter's important aspects and points out investigations important for the development of the main ideas and it includes solutions to many of problems and hints for others. The manual contains large-size versions of diagrams and patterns that appear in the text so that the instructor could have copies made for use in class.

Acknowledgments

More than thirty years ago, I taught an experimental, low-dimensional topology course with an incredible group of talented and enthusiastic students. I wrote up notes as the course progressed. The course was so successful that I have taught it several times since, revising and supplementing my original notes each time. The notes eventually became this book.

Several individuals played parts in the development and evolution of the course and book. Initial credit goes to my college teachers Albert Tucker and Ralph Fox who, when I was a mathematical tad, first fed me these topological ideas and got me hooked. Over the years, Philip Straffin and David T. Gay taught the course using the notes and Noah Snavely and Adam Spiegler assisted me in teaching it; I appreciate their pedagogical insights and enthusiasm for the project. I thank the The University of Arizona's Department of Mathematics for supporting curriculum development and for valuing innovation in teaching. My colleague Olga Yparaki was particularly excited with the story format and encouraged me to continue developing it. I am especially grateful to Susan Lowell, good friend and author, for suggestions on making the story's characters come alive.

I have borrowed freely from several authors whose books have given me mathematical ideas, problems, and approaches. Most of these appear as references in the text. One author that deserves special mention is Martin Gardner. He has the uncanny ability to come up with problems that are accessible to students, that grab them, and that lead them nicely and naturally to deeper mathematical involvement.

Of course, the lion's share of thanks for this project's development goes to the students who took the course, who struggled with the problems and projects and taught me what worked and what got them excited. It was their enthusiasm for these ideas and for discovery that kept me on the project all these years.

I also need to thank Deborah Yoklic, who typed the first version of the manuscript thirty years ago and who has been an enthusiastic supporter of the project ever since. To those who reviewed the manuscript, I am grateful for the many helpful criticisms and encouraging comments during the phases of its development.

Finally, let me thank my editor Tom Singer at Academic Press who believed in the project from the beginning, Michael Troy at Graphic World Publishing Services who kept me on task and let me know what was happening during production, Fritz Simon who turned my scribbles into effective illustrations and helped me make the manuscript "look like a book," and Tulley Straub who helped render three-dimensional objects realistically. It was a pleasure and a privilege to work with all of you.