Table of Integrals, Series, and Products

Seventh Edition

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I.S. Gradshteyn and I.M. Ryzhik

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Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript "11" added to its entry reference number, where the equivalent superscript number for the sixth edition was "10." Similarly, an asterisk on an entry's reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O'Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site `www.az-tec.com/gr/errata`.

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The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x} \ln \tan x\,dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \tag{0.1}$$

By making the natural substitution $u = \cot x - 1$, we obtain

$$\int_0^\infty u^{p-1} \ln(1+u)\,du = \frac{\pi}{p} \operatorname{cosec} p\pi. \tag{0.2}$$

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln\left(\tan^p x + \cot^p x\right) \ln \tan x \, dx = 0.$$

The substitution $u = \tan x$ yields

$$I = \int_0^\infty \frac{\ln\left(u^p + u^{-p}\right) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^\infty \frac{v e^v}{1 + e^{2v}} \ln\left(e^{pv} + e^{-pv}\right) \, dv = \int_{-\infty}^\infty v \, \frac{\ln\left(2 \cosh pv\right)}{2 \cosh v} \, dv.$$

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel's integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer's and MacRobert's functions.
21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x + \cos x}{\sin x - \cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$] are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of $\varphi_2(x)$ in the list] after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions [that is, $\varphi_2(x) = x$]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \dfrac{1}{x}$ are equally complex since $\ln \dfrac{1}{x} = -\ln x$. In the section on "powers and algebraic functions," polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

*For any natural number n, the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

Use of the Tables*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function, and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Gradshteyn and Ryzhik's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α, β, γ, δ, t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions $\operatorname{arcsinh} z$, $\operatorname{arccosh} z$, $\operatorname{arctanh} z$, and $\operatorname{arccoth} z$ are introduced through the definitions

$$\arcsin z = \frac{1}{i} \operatorname{arcsinh}(iz)$$
$$\arccos z = \frac{1}{i} \operatorname{arccosh}(z)$$
$$\arctan z = \frac{1}{i} \operatorname{arctanh}(iz)$$
$$\operatorname{arccot} z = i \operatorname{arccoth}(iz)$$

*Prepared by Alan Jeffrey for the English language edition.

or

$$\begin{aligned}\operatorname{arcsinh} z &= \frac{1}{i}\arcsin(iz)\\ \operatorname{arccosh} z &= i\arccos z\\ \operatorname{arctanh} z &= \frac{1}{i}\arctan(iz)\\ \operatorname{arccoth} z &= \frac{1}{i}\operatorname{arccot}(-iz)\end{aligned}$$

The numerical constants $\boldsymbol{C}$ and $\boldsymbol{G}$ which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant $\boldsymbol{C}$ is defined by the limit

$$\boldsymbol{C} = \lim_{s\to\infty}\left(\sum_{m=1}^{s}\frac{1}{m} - \ln s\right) = 0.577215\ldots.$$

On occasion, other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

$$\gamma = e^{\boldsymbol{C}} = 1.781072\ldots.$$

Catalan's constant $\boldsymbol{G}$ is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

by the expression

$$\boldsymbol{G} = \frac{1}{2}\int_0^1 \mathbf{K}\,dk = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots.$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for } n = 0, 1, \ldots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots.$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \qquad \text{for } n = 0, 1, \ldots$$

The E_n are all integral, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, ...

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \ldots.$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66,$$
$$B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \quad \ldots.$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x) \qquad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n} \left\{ B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right\}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right\} \qquad n = 1, 2, \ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} \left(2^{n-k} - 1\right) B_{n-k} B_n(x) \qquad n = 2, 3, \ldots$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right) \left\{ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right\}$$

the Euler polynomial of order n.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$

$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}$$

$$\operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

$$\begin{aligned}\Pi\left(\varphi, n^2, k\right) &= \int_0^{\varphi} \frac{da}{\left(1-n^2\sin^2 a\right)\sqrt{1-k^2\sin^2 a}} \\ &= \int_0^{\sin\varphi} \frac{dx}{\left(1-n^2x^2\right)\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}\end{aligned} \qquad \left(-\infty < n^2 < \infty\right)$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\operatorname{zn}(u,k)$, frequently written $Z(u)$, is defined by the relation

$$\operatorname{zn}(u,k) = Z(u) = \int_0^u \left\{\operatorname{dn}^2 v - \frac{\boldsymbol{E}}{\boldsymbol{K}}\right\} dv = \boldsymbol{E}(u) - \frac{\boldsymbol{E}}{\boldsymbol{K}}u.$$

This is related to the theta functions by the relationship

$$\operatorname{zn}(u,k) = \frac{\partial}{\partial u}\ln\Theta(u)$$

giving

(i). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}}\frac{\vartheta_1'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}$$

(ii). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}}\frac{\vartheta_2'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u}$$

(iii). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}}\frac{\vartheta_3'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_3\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

(iv). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}}\frac{\vartheta_4'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts, the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions, the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d\left(\ln z!\right)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -\boldsymbol{C} + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n}\, dt \qquad (n = 0, 1, \ldots, \quad \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\operatorname{Ei}(z)$ through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^{\infty} e^{-t} t^{-1}\, dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \qquad [z > 1].$$

The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \left\{ e^{-z} - z\, E_{n-1}(z) \right\} \qquad [n > 1]$$

and

$$E_n'(z) = -E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots \right\} \qquad \left[|\arg z| < \frac{3\pi}{2} \right]$$

while for large n,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n\left(6x^2 - 8nx + n^2\right)}{(x+n)^6} + R(n,x) \right\},$$

where

$$-0.36 n^{-4} \le R(n,x) \le \left(1 + \frac{1}{x+n-1} \right) n^{-4} \qquad [x > 0].$$

The sine and cosine integrals $\operatorname{si}(x)$ and $\operatorname{ci}(x)$ are related to the functions $\operatorname{Si}(x)$ and $\operatorname{Ci}(x)$ by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\operatorname{Ci}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t}\, dt.$$

The hyperbolic sine and cosine integrals $\operatorname{shi}(x)$ and $\operatorname{chi}(x)$ are defined by the relations

$$\operatorname{shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt$$

and

$$\operatorname{chi}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t}\, dt.$$

Some authors write

$$\operatorname{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t}\, dt$$

so that

$$\operatorname{Cin}(x) = -\operatorname{Ci}(x) + \ln x + \boldsymbol{C}.$$

The error function $\operatorname{erf}(x)$ is defined by the relation

$$\operatorname{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and the complementary error function $\operatorname{erfc}(x)$ is related to the error function $\operatorname{erfc}(x)$ and to $\Phi(x)$ by the expression

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Gradshteyn and Ryzhik as

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2\, dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2\, dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2}\, dt, \qquad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2}\, dt,$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\, dt, \qquad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\, dt.$$

These are related by the expressions

$$S(x) = S_1\left(x\sqrt{\frac{2}{\pi}}\right) = S_2\left(x^2\right)$$

and

$$C(x) = C_1\left(x\sqrt{\frac{2}{\pi}}\right) = C_2\left(x^2\right)$$

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $He_n(x)$ by the relations

$$He_n(x) = 2^{-n/2}\, H_n\left(\frac{x}{\sqrt{2}}\right)$$

and

$$H_n(x) = 2^{n/2}\, He_n\left(x\sqrt{2}\right).$$

These functions satisfy the differential equations

$$\frac{d^2\,H_n}{dx^2} - 2x\frac{d\,H_n}{dx} + 2n\,H_n = 0$$

and

$$\frac{d^2\,He_n}{dx^2} - x\frac{d\,He_n}{dx} + n\,He_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2x\,H_n - 2n\,H_{n-1}$$

and

$$He_{n+1} = x\,He_n - n\,He_{n-1}\,.$$

The first six orthogonal polynomials He_n are

$He_0 = 1, \quad He_1 = x, \quad He_2 = x^2 - 1, \quad He_3 = x^3 - 3x, \quad He_4 = x^4 - 6x^2 + 3, \quad He_5 = x^5 - 10x^3 + 15x.$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n}\Gamma(n+1)\,J_n(z).$$

In the book by Gray, Mathews, and MacRobert, the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi\,Y_n(z) + (\ln 2 - \mathbf{C})\,J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $Hs_\nu(z)$ and $Hi_\nu(z)$, and some authors write $G_\nu(z) = \left(\frac{1}{2}\right)\pi i\,H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n+1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z)\,O_0(t) + 2\sum_{k=1}^{\infty} J_k(z)\,O_k(t),$$

giving

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{[n/2]}\frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \ldots,$$

where $\left[\frac{1}{2}n\right]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1)\,O_{n+1}(t) + (n+1)\,O_{n-1}(t) - \frac{2\left(n^2-1\right)}{t}\,O_n(t) = \frac{2n}{t}\sin^2\frac{n\pi}{2}.$$

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\mathrm{Ai}(z) = \frac{1}{3}\sqrt{z}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$$

$$\mathrm{Ai}(-z) = \frac{1}{3}\sqrt{z}\left\{J_{1/3}\left(\frac{2}{3}z^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}$$

and by

$$\mathrm{Bi}(z) = \sqrt{\frac{z}{3}}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\},$$

$$\mathrm{Bi}(-z) = \sqrt{\frac{z}{3}}\left\{J_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - J_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}.$$

Parabolic Cylinder Functions and Whittaker Functions

The differential equation

$$\frac{d^2y}{dz^2} + \left(az^2 + bz + c\right)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left(\frac{1}{4}z^2 + a\right)y = 0 \text{ and } \frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0,$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0$$

are sometimes written $U(a,z)$ and $V(a,z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a,z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a,z) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + a\right)\left\{D_{-a-\frac{1}{2}}(-z) + (\sin\pi a)\,D_{-a-\frac{1}{2}}(z)\right\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is

$$\frac{d^2y}{dz^2} + \left(a - 2k^2\cos 2z\right)y = 0 \qquad \text{with } k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ, λ and h^2, and b and $c = 2\sqrt{q}$, respectively. The periodic solutions $\mathrm{se}_n(z,q)$ and $\mathrm{ce}_n(z,q)$ and the modified periodic solutions $\mathrm{Se}_n(z,q)$ and $\mathrm{Ce}_n(z,q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions.* National Bureau of Standards, Columbia University Press, New York, 1951.

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$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Bessel functions of an imaginary argument	8.406, 8.43
$I_x(p,q)$	Normalized incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger function	8.58
$\mathrm{k}_\nu(x)$	Bateman's function	9.210 3
$\mathbf{K}(\mathbf{k}) = \boldsymbol{K}, \quad \mathbf{K}(\mathbf{k}') = \boldsymbol{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Bessel functions of imaginary argument	8.407, 8.43
$\mathrm{kei}(z), \quad \mathrm{ker}(z)$	Thomson functions	8.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\mathrm{li}(x)$	Logarithm integral	8.24
$M_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$O_n(x)$	Neumann's polynomials	8.59
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_\nu(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$	Riemann's differential equation	9.160
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	8.96
$Q_\nu^\mu(z), \quad Q_\nu^\mu(x)$	Associated Legendre functions of the second kind	8.7, 8.8
$Q_\nu(z), \quad Q_\nu(x)$	Legendre functions of the second kind	8.82, 8.83
$S(x)$	Fresnel sine integral	8.25
$S_n(x)$	Schläfli's polynomials	8.59
$s_{\mu,\nu}(z), \quad S_{\mu,\nu}(z)$	Lommel functions	8.57
$\mathrm{se}_{2n+1}(z,q), \quad \mathrm{se}_{2n+2}(z,q)$	Periodic Mathieu functions	8.61
$\mathrm{Se}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+2}(z,q)$	Mathieu functions of an imaginary argument	8.63
$\mathrm{shi}(x)$	Hyperbolic sine integral	8.22
$\mathrm{si}(x)$	Sine integral	8.23
$\mathrm{sn}\, u$	Sine amplitude	8.14
$T_n(x)$	Chebyshev polynomial of the 1^{st} kind	8.94
$U_n(x)$	Chebyshev polynomials of the 2^{nd} kind	8.94

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Notation	Name of the function and the number of the formula containing its definition	
$U_\nu(w,z), \quad V_\nu(w,z)$	Lommel functions of two variables	8.578
$W_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$Y_\nu(z)$	Neumann functions	8.403, 8.41
$Z_\nu(z)$	Bessel functions	8.401
$\mathfrak{Z}_\nu(z)$	Bessel functions	

Notation

Symbol	Meaning
$\lfloor x \rfloor$	The integral part of the real number x (also denoted by $[x]$)
$\int_a^{(b+)} \quad \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a, proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve C
PV $\int$	Principal value integral
$\overline{z} = x - iy$	The complex conjugate of $z = x + iy$
$n!$	$= 1 \cdot 2 \cdot 3 \ldots n, \qquad 0! = 1$
$(2n+1)!!$	$= 1 \cdot 3 \ldots (2n+1)$. (double factorial notation)
$(2n)!!$	$= 2 \cdot 4 \ldots (2n)$. (double factorial notation)
$0!! = 1$ and $(-1)!! = 1$	(cf. 3.372 for $n = 0$)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$)
$\binom{p}{n}$	$= \frac{p(p-1)\ldots(p-n+1)}{1 \cdot 2 \ldots n} = \frac{p!}{n!(p-n)!}, \quad \binom{p}{0} = 1, \quad \binom{p}{n} = \frac{p!}{n!(p-n)!}$ $[n = 1, 2, \ldots, p \geq n]$
$\binom{x}{n}$	$= x(x-1)\ldots(x-n+1)/n!\ [n = 0, 1, \ldots]$
$(a)_n$	$= a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Pochhammer symbol)
$\sum_{k=m}^{n} u_k$	$= u_m + u_{m+1} + \ldots + u_n$. If $n < m$, we define $\sum_{k=m}^{n} u_k = 0$
$\sum_n{}', \quad \sum_{m,n}{}'$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$\sum, \quad \prod$	An empty $\sum$ has value 0, and an empty $\prod$ has value 1
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Symbol	Meaning
$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$	Kronecker delta
τ	Theta function parameter (cf. 8.18)
$\times$ and $\wedge$	Vector product (cf. 10.11)
$\cdot$	Scalar product (cf. 10.11)
∇ or "del"	Vector operator (cf. 10.21)
∇^2	Laplacian (cf. 10.31)
$\sim$	Asymptotically equal to
$\arg z$	The argument of the complex number $z = x + iy$
curl or rot	Vector operator (cf. 10.21)
div	Vector operator (divergence) (cf. 10.21)
$\mathcal{F}$	Fourier transform (cf. 17.21)
$\mathcal{F}_c$	Fourier cosine transform (cf. 17.31)
$\mathcal{F}_s$	Fourier sine transform (cf. 17.31)
grad	Vector operator (gradient) (cf. 10.21)
h_i and g_{ij}	Metric coefficients (cf. 10.51)
$^{\mathrm{H}}$	Hermitian transpose of a vector or matrix (cf. 13.123)
$\mathrm{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	Heaviside step function
$\operatorname{Im} z \equiv y$	The imaginary part of the complex number $z = x + iy$
k	The letter k (when not used as an index of summation) denotes a number in the interval [0, 1]. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1-k^2}$ is denoted by k'.
$\mathcal{L}$	Laplace transform (cf. 17.11)
$\mathcal{M}$	Mellin transform (cf. 17.41)
$\mathbb{N}$	The natural numbers $(0, 1, 2, \dots)$
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0. If there exists an $M > 0$ such that $\lvert g(z)\rvert \leq M\lvert f(z)\rvert$ in some sufficiently small neighborhood of the point z_0, we write $g(z) = O(f(z))$.

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Symbol	Meaning
q	The nome, a theta function parameter (cf. 8.18)
$\mathbb{R}$	The real numbers
$R(x)$	A rational function
$\operatorname{Re} z \equiv x$	The real part of the complex number $z = x + iy$
S_n^m	Stirling number of the first kind (cd. 9.74)
$\mathfrak{S}_n^m$	Stirling number of the second kind (cd. 9.74)
$\operatorname{sign} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$	The sign (signum) of the real number x
$^{\mathrm{T}}$	Transpose of a vector or matrix (cf. 13.115)
$\mathbb{Z}$	The integers $(0, \pm 1, \pm 2, \dots)$
Z_b	Bilateral z transform (cf. 18.1)
Z_u	Unilateral z transform (cf. 18.1)

Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

ET I 118 (1) *a*

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms.*

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the *a* indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.