Geometric Measure Theory

Geometric measure theory could be described as differential geometry, generalized through measure theory to deal with maps and surfaces that are not necessarily smooth, and applied to the calculus of variations. It dates from the 1960 foundational paper of Herbert Federer and Wendell Fleming on “Normal and Integral Currents,” recognized by the 1986 AMS Steele Prize for a paper of fundamental or lasting importance, and earlier and contemporaneous work of L. C. Young [1, 2], E. De Giorgi [1, 3, 4], and E. R. Reifenberg [1–3] (see Figure 1.0.1). This chapter provides a rough outline of the purpose and basic concepts of geometric measure theory. Later chapters take up these topics more carefully.

1.1 Archetypical Problem  Given a boundary in $\mathbb{R}^n$, find the surface of least area with that boundary. See Figure 1.1.1. Progress on this problem depends crucially on first finding a good space of surfaces to work in.

1.2 Surfaces as Mappings  Classically, one considered only two-dimensional surfaces, defined as mappings of the disc. See Figure 1.2.1. Excellent references include J. C. C. Nitsche’s *Lectures on Minimal Surfaces* [2], now available in English, R. Osserman’s updated *Survey of Minimal Surfaces*, and H. B. Lawson’s *Lectures on Minimal Submanifolds*. It was not until about 1930 that J. Douglas and T. Rado surmounted substantial inherent difficulties to prove that every smooth Jordan curve bounds a disc of least mapping area. Almost no progress was made for higher-dimensional surfaces (until, in a surprising turnaround, B. White [1] showed that for higher-dimensional surfaces the geometric measure theory solution actually solves the mapping problem too).
Figure 1.0.1. Wendell Fleming, Fred Almgren, and Ennio De Giorgi, three of the founders of geometric measure theory, at the Scuola Normale Superiore, Pisa, summer, 1965; and Fleming today. Photographs courtesy of Fleming.
Along with its successes and advantages, the definition of a surface as a mapping has certain drawbacks (see Morgan [24]):

1. There is an inevitable *a priori* restriction on the types of singularities that can occur;
2. There is an *a priori* restriction on the topological complexity; and
3. The natural topology lacks compactness properties.

The importance of compactness properties appears in the direct method described in the next section.

### 1.3 The Direct Method

The direct method for finding a surface of least area with a given boundary has three steps.

1. Take a sequence of surfaces with areas decreasing to the infimum.
2. Extract a convergent subsequence.
3. Show that the limit surface is the desired surface of least area.

Figures 1.3.1–1.3.4 show how this method breaks down for lack of compactness in the space of surfaces as mappings, even when the given boundary is the unit circle.
circle. By sending out thin tentacles toward every rational point, the sequence could include all of $\mathbb{R}^3$ in its closure!

1.4 Rectifiable Currents An alternative to surfaces as mappings is provided by rectifiable currents, the $m$-dimensional, oriented surfaces of geometric measure theory. The relevant functions $f : \mathbb{R}^m \to \mathbb{R}^n$ need not be smooth but merely Lipschitz; that is,

$$|f(x) - f(y)| \leq C|x - y|,$$

for some “Lipschitz constant” $C$.

Fortunately, there is a good $m$-dimensional measure on $\mathbb{R}^n$, called Hausdorff measure, $\mathcal{H}^m$. Hausdorff measure agrees with the classical mapping area of an embedded manifold, but it is defined for all subsets of $\mathbb{R}^n$.

A Borel subset $B$ of $\mathbb{R}^n$ is called $(\mathcal{H}^m, m)$ rectifiable if $B$ is a countable union of Lipschitz images of bounded subsets of $\mathbb{R}^m$, with $\mathcal{H}^m(B) < \infty$. (As usual,
we will ignore sets of $\mathcal{H}^m$ measure 0.) That definition sounds rather general, and it includes just about any “$m$-dimensional surface” I can imagine. Nevertheless, these sets will support a kind of differential geometry: for example, it turns out that a rectifiable set $B$ has a canonical tangent plane at almost every point.

Finally, a \textit{rectifiable current} is an oriented rectifiable set with integer multiplicities, finite area, and compact support. By general measure theory, one can integrate a smooth differential form $\varphi$ over such an oriented rectifiable set $S$, and hence view $S$ as a \textit{current}; that is, a linear functional on differential forms,

$$\varphi \mapsto \int_S \varphi.$$ 

This perspective yields a new natural topology on the space of surfaces, dual to an appropriate topology on differential forms. This topology has useful compactness
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properties, given by the fundamental compactness theorem in Section 1.5. Viewing rectifiable sets as currents also provides a boundary operator \( \partial \) from \( m \)-dimensional rectifiable currents to \((m - 1)\)-dimensional currents, defined by

\[
(\partial S)(\varphi) = S(d\varphi),
\]

where \( d\varphi \) is the exterior derivative of \( \varphi \). By Stokes’s theorem, this definition coincides with the usual notion of boundary for smooth, compact, oriented manifolds with boundary. In general, the current \( \partial S \) is not rectifiable, even if \( S \) is rectifiable.

1.5 The Compactness Theorem Let \( c \) be a positive constant. Then the set of all \( m \)-dimensional rectifiable currents \( T \) in a fixed large closed ball in \( \mathbb{R}^n \), such that the boundary \( \partial T \) is also rectifiable and such that the area of both \( T \) and \( \partial T \) is bounded by \( c \), is compact in an appropriate weak topology.

1.6 Advantages of Rectifiable Currents Notice that rectifiable currents have none of the three drawbacks mentioned in Section 1.2. There is certainly no restriction on singularities or topological complexity. Moreover, the compactness theorem provides the ideal compactness properties. In fact, the direct method described in Section 1.3 succeeds in the context of rectifiable currents. In the figures of Section 1.3, the amount of area in the tentacles goes to 0. Therefore, they disappear in the limit in the new topology. What remains is the disc, the desired solution.

All of these results hold in all dimensions and codimensions.

1.7 The Regularity of Area-Minimizing Rectifiable Currents One serious suspicion hangs over this new space of surfaces: The solutions they provide to the problem of least area, the so-called area-minimizing rectifiable currents, may be generalized objects without any geometric significance. The following interior regularity results allay such concerns. (We give more precise statements in Chapter 8.)

1. A two-dimensional area-minimizing rectifiable current in \( \mathbb{R}^3 \) is a smooth embedded manifold.

2. For \( m \leq 6 \), an \( m \)-dimensional area-minimizing rectifiable current in \( \mathbb{R}^{m+1} \) is a smooth embedded manifold.

Thus, in low dimensions the area-minimizing hypersurfaces provided by geometric measure theory actually turn out to be smooth embedded manifolds. However, in higher dimensions, singularities occur, for geometric and not merely
technical reasons (see Section 10.7). Despite marked progress, understanding such singularities remains a tremendous challenge.

1.8 More General Ambient Spaces  Basic geometric measure theory extends from $\mathbb{R}^n$ to Riemannian manifolds via $C^1$ embeddings in $\mathbb{R}^n$ or Lipschitz charts. Ambrosio and Kirchheim among others have been developing an intrinsic approach to geometric measure theory in certain metric spaces.