4.1 INTRODUCTION

In this chapter and the next we consider the frequency analysis of continuous-time signals and systems—the Fourier series for periodic signals in this chapter, and the Fourier transform for both periodic and aperiodic signals as well as for systems in Chapter 5. In these chapters we consider:

- **Spectral representation**—The frequency representation of periodic and aperiodic signals indicates how their power or energy is allocated to different frequencies. Such a distribution over frequency is called the **spectrum of the signal**. For a periodic signal the spectrum is discrete, as its power is concentrated at frequencies multiples of a so-called **fundamental frequency**, directly related to the period of the signal. On the other hand, the spectrum of an aperiodic signal is a continuous function of frequency. The concept of spectrum is similar to the one used in optics for light, or in material science for metals, each indicating the distribution of power or energy over frequency. The Fourier representation is also useful in finding the frequency response of linear time-invariant systems, which is related to the transfer function obtained with the Laplace transform. The frequency response of a system indicates how an LTI system responds to sinusoids of different frequencies. Such a response characterizes the system and permits easy computation of its steady-state response, and will be equally important in the synthesis of systems.

- **Eigenfunctions and Fourier analysis**—It is important to understand the driving force behind the representation of signals in terms of basic signals when applied to LTI systems. For instance, the convolution integral that gives the output of an LTI system resulted from the representation of its input signal in terms of shifted impulses. Along with this result came the concept of the impulse response of an LTI system. Likewise, the Laplace transform can be seen as the representation of signals in terms of general eigenfunctions. In this chapter and the next we will see that complex...
exponentials or sinusoids are used in the Fourier representation of periodic as well as aperiodic signals by taking advantage of the eigenfunction property of LTI systems. The results of the Fourier series in this chapter will be extended to the Fourier transform in Chapter 5.

- **Steady-state analysis**—Fourier analysis is in the steady state, while Laplace analysis considers both transient and steady state. Thus, if one is interested in transients, as in control theory, Laplace is a meaningful transformation. On the other hand, if one is interested in the frequency analysis, or steady state, as in communications theory, the Fourier transform is the one to use. There will be cases, however, where in control and communications both Laplace and Fourier analysis are considered.

- **Application of Fourier analysis**—The frequency representation of signals and systems is extremely important in signal processing and in communications. It explains filtering, modulation of messages in a communication system, the meaning of bandwidth, and how to design filters. Likewise, the frequency representation turns out to be essential in the sampling of analog signals—the bridge between analog and digital signal processing.

### 4.2 EIGENFUNCTIONS REVISITED

As indicated in Chapter 3, the most important property of stable LTI systems is that when the input is a complex exponential (or a combination of a cosine and a sine) of a certain frequency, the output of the system is the input times a complex constant connected with how the system responds to the frequency at the input. The complex exponential is called an *eigenfunction* of stable LTI systems.

If \( x(t) = e^{j\Omega_0 t}, -\infty < t < \infty \), is the input to a causal and a stable system with impulse response \( h(t) \), the output in the steady state is given by

\[
y(t) = e^{j\Omega_0 t} H(j\Omega_0)
\]

where

\[
H(j\Omega_0) = \int_{0}^{\infty} h(\tau) e^{-j\Omega_0 \tau} d\tau
\]

is the frequency response of the system at \( \Omega_0 \). The signal \( x(t) = e^{j\Omega_0 t} \) is said to be an *eigenfunction* of the LTI system as it appears at both input and output.

This can be seen by finding the output corresponding to \( x(t) = e^{j\Omega_0 t} \) by means of the convolution integral,

\[
y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau = e^{j\Omega_0 t} \int_{0}^{\infty} h(\tau)e^{-j\Omega_0 \tau} d\tau
\]

\[
= e^{j\Omega_0 t} H(j\Omega_0)
\]
where we let \( H(j\Omega_0) \) equal the integral in the second equation. The input signal appears in the output modified by the frequency response of the system \( H(j\Omega_0) \) at the frequency \( \Omega_0 \) of the input. Notice that the convolution integral limits indicate that the input started at \(-\infty\) and that we are considering the output at finite time \( t \)—this means that we are in steady state. The steady-state response of a stable LTI system is attained by either considering that the initial time when the input is applied to the system is \(-\infty\) and we reach a finite time \( t \), or by starting at time 0 and going to \( \infty \).

The above result for one frequency can be easily extended to the case of several frequencies present at the input. If the input signal \( x(t) \) is a linear combination of complex exponentials, with different amplitudes, frequencies, and phases, or

\[
x(t) = \sum_k X_k e^{j\Omega_k t}
\]

where \( X_k \) are complex values, since the output corresponding to \( X_k e^{j\Omega_k t} \) is \( X_k e^{j\Omega_k t} H(j\Omega_k) \) by superposition the response to \( x(t) \) is

\[
y(t) = \sum_k X_k e^{j\Omega_k t} H(j\Omega_k)
\]

\[
= \sum_k X_k |H(j\Omega_k)| e^{j(\Omega_k t + \angle H(j\Omega_k))}
\]

(4.3)

The above is valid for any signal that is a combination of exponentials of arbitrary frequencies. As we will see in this chapter, when \( x(t) \) is periodic it can be represented by the Fourier series, which is a combination of complex exponentials harmonically related (i.e., the frequencies of the exponentials are multiples of the fundamental frequency of the periodic signal). Thus, when a periodic signal is applied to a causal and stable LTI system its output is computed as in Equation (4.3).

The significance of the eigenfunction property is also seen when the input signal is an integral (a sum, after all) of complex exponentials, with continuously varying frequency, as the integrand. That is, if

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} d\Omega
\]

then using superposition and the eigenfunction property of a stable LTI system, with frequency response \( H(j\Omega) \), the output is

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} H(j\Omega) d\Omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)|H(j\Omega)| e^{j(\Omega t + \angle H(j\Omega))} d\Omega
\]

(4.4)
The above representation of \( x(t) \) corresponds to the Fourier representation of aperiodic signals, which will be covered in Chapter 5. Again here, the eigenfunction property of LTI systems provides an efficient way to compute the output. Furthermore, we also find that by letting \( Y(j\Omega) = X(j\Omega)H(j\Omega) \) the above equation gives an expression to compute \( y(t) \) from \( Y(j\Omega) \). The product \( Y(j\Omega) = X(j\Omega)H(j\Omega) \) corresponds to the Fourier transform of the convolution integral \( y(t) = x(t) * h(t) \), and is connected with the convolution property of the Laplace transform. It is important to start noticing these connections, to understand the link between Laplace and Fourier analysis.

**Remarks**

- Notice the difference of notation for the frequency representation of signals and systems used above. If \( x(t) \) is a periodic signal its frequency representation is given by \( \{X_k\} \), and if aperiodic by \( X(j\Omega) \), while for a system with impulse response \( h(t) \) its frequency response is given by \( H(j\Omega) \).
- When considering the eigenfunction property, the stability of the LTI system is necessary to ensure that \( H(j\Omega) \) exists for all frequencies.
- The eigenfunction property applied to a linear circuit gives the same result as the one obtained from phasors in the sinusoidal steady state. That is, if

\[
x(t) = A \cos(\Omega_0 t + \theta) = \frac{Ae^{j\theta}}{2} e^{j\Omega_0 t} + \frac{Ae^{-j\theta}}{2} e^{-j\Omega_0 t}
\]  

is the input of a circuit represented by the transfer function

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]}
\]

then the corresponding steady-state output is given by

\[
y_{ss}(t) = \frac{Ae^{j\theta}}{2} e^{j\Omega_0 t} H(j\Omega_0) + \frac{Ae^{-j\theta}}{2} e^{-j\Omega_0 t} H(-j\Omega_0)
\]

\[
= A|H(j\Omega_0)| \cos(\Omega_0 t + \theta + \angle H(j\Omega_0))
\]

where, very importantly, the frequency of the output coincides with that of the input, and the amplitude and phase of the input are changed by the magnitude and phase of the frequency response of the system for the frequency \( \Omega_0 \). The frequency response is \( H(j\Omega_0) = H(s)|_{s=j\Omega_0} \), and as we will see its magnitude is an even function of frequency, or \( |H(j\Omega)| = |H(-j\Omega)| \), and its phase is an odd function of frequency, or \( \angle H(j\Omega_0) = -\angle H(-j\Omega_0) \). Using these two conditions we obtain Equation (4.6).

The phasor corresponding to the input

\[
x(t) = A \cos(\Omega_0 t + \theta)
\]

is defined as a vector,

\[
X = A \angle \theta
\]

rotating in the polar plane at the frequency of \( \Omega_0 \). The phasor has a magnitude \( A \) and an angle \( \theta \) with respect to the positive real axis. The projection of the phasor onto the real axis, as it rotates at the given
frequency, with time generates a cosine of the indicated frequency, amplitude, and phase. The transfer function is computed at \( s = j\Omega_0 \) or
\[
H(s)|_{s=j\Omega_0} = H(j\Omega_0) = \frac{Y}{X}
\]
(ratio of the phasors corresponding to the output \( Y \) and the input \( X \)). The phasor for the output is thus
\[
Y = H(j\Omega_0)X = |Y|e^{j\angle Y}
\]
Such a phasor is then converted into the sinusoid (which equals Eq. 4.6):
\[
y_{ss}(t) = \Re\{Ye^{j\Omega_0t}\} = |Y|\cos(\Omega_0t + \angle Y)
\]

A very important application of LTI systems is filtering, where one is interested in preserving desired frequency components of a signal and getting rid of less-desirable components. That an LTI system can be used for filtering is seen in Equations (4.3) and (4.4). In the case of a periodic signal, the magnitude \(|H(j\Omega_k)|\) can be set ideally to one for those components we wish to keep and to zero for those we wish to get rid of. Likewise, for an aperiodic signal, the magnitude \(|H(j\Omega)|\) could be set ideally to one for those components we wish to keep and zero for those components we wish to get rid of. Depending on the filtering application, an LTI system with the appropriate characteristics can be designed, obtaining the desired transfer function \( H(s) \).

For a stable LTI with transfer function \( H(s) \) if the input is
\[
x(t) = \Re\{Ae^{j(\Omega_0t + \theta)}\} = A\cos(\Omega_0t + \theta)
\]
the steady-state output is given by
\[
y(t) = \Re\{AH(j\Omega_0)e^{j(\Omega_0t + \theta)}\} = A|H(j\Omega_0)|\cos(\Omega_0t + \theta + \angle H(j\Omega_0))
\]
where
\[
H(j\Omega_0) = H(s)|_{s=j\Omega_0}
\]

**Example 4.1**

Consider the RC circuit shown in Figure 4.1. Let the voltage source be \( v_s(t) = 4 \cos(t + \pi/4) \) volts, the resistor be \( R = 1\Omega \), and the capacitor \( C = 1\text{ F} \). Find the steady-state voltage across the capacitor.

**Solution**

This problem can be approached in two ways.

- **Phasor approach.** From the phasor circuit in Figure 4.1, by voltage division we have the following phasor ratio, where \( V_s \) is the phasor corresponding to the source \( v_s(t) \) and \( V_c \) the phasor
FIGURE 4.1
RC circuit and corresponding phasor circuit.

Corresponding to \( v_c(t) \):

\[
\frac{V_c}{V_s} = \frac{-j}{1-j} = \frac{-j(1+j)}{2} = \frac{\sqrt{2}}{2} \angle -\pi/4
\]

Since \( V_s = 4\angle \pi/4 \), then

\[ V_c = 2\sqrt{2}\angle 0 \]

so that in the steady state,

\[ v_c(t) = 2\sqrt{2} \cos(t) \]

- Eigenfunction approach. Considering the output is the voltage across the capacitor and the input is the voltage source, the transfer function is obtained using voltage division as

\[
H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/s}{1 + 1/s} = \frac{1}{s+1}
\]

so that the system frequency response at the input frequency \( \Omega_0 = 1 \) is

\[ H(j1) = \frac{\sqrt{2}}{2} \angle -\pi/4 \]

According to the eigenfunction property the steady-state response of the capacitor is

\[
v_c(t) = 4|H(j1)| \cos(t + \pi/4 + \angle H(j1)) = 2\sqrt{2} \cos(t)
\]

which coincides with the solution found using phasors.

- Example 4.2

An ideal communication system provides as output the input signal with only a possible delay in the transmission. Such an ideal system does not cause any distortion to the input signal beyond
the delay. Find the frequency response of the ideal communication system, and use it to determine the steady-state response when the delay caused by the system is \( \tau = 3 \text{ sec} \), and the input is \( x(t) = 2 \cos(4t - \pi/4) \).

**Solution**

The impulse response of the ideal system is \( h(t) = \delta(t - \tau) \) where \( \tau \) is the delay of the transmission. In fact, the output according to the convolution integral gives

\[
y(t) = \int_{0}^{\infty} \delta(t - \rho)d\rho = x(t - \tau)
\]

as expected. Let us then find the frequency response of the ideal communication system. According to the eigenvalue property, if the input is \( x(t) = e^{j\Omega_0 t} \), then the output is

\[
y(t) = e^{j\Omega_0 t}H(j\Omega_0)
\]

but also

\[
y(t) = x(t - \tau) = e^{j\Omega_0 (t-\tau)}
\]

so that comparing these equations we have that

\[
H(j\Omega_0) = e^{-j\tau\Omega_0}
\]

For a generic frequency \( 0 \leq \Omega < \infty \), we would get

\[
H(j\Omega) = e^{-j\tau\Omega}
\]

which is a complex function of \( \Omega \), with a unity magnitude \( |H(j\Omega)| = 1 \), and a linear phase \( \angle H(j\Omega) = -\tau\Omega \). This system is called an all-pass system, since it allows all frequency components of the input to go through with a phase change only.

Consider the case when \( \tau = 3 \), and that we input into this system \( x(t) = 2 \cos(4t - \pi/4) \), then \( H(j\Omega) = 1e^{-j3\Omega} \), so that the output in the steady state is

\[
y(t) = 2|H(j4)| \cos(4t - \pi/4 + \angle H(j4))
= 2 \cos(4(t - 3) - \pi/4)
= x(t - 3)
\]

where we used \( H(j4) = 1e^{-j12} \) (i.e., \( |H(j4)| = 1 \) and \( \angle H(j4) = 12 \)).

**Example 4.3**

Although there are better methods to compute the frequency response of a system represented by a differential equation, the eigenfunction property can be easily used for that. Consider the RC
circuit shown in Figure 4.1 where the input is
\[ v_s(t) = 1 + \cos(10,000t) \]
with components of low frequency, \( \Omega = 0 \), and of large frequency, \( \Omega = 10,000 \text{ rad/sec} \). The output \( v_c(t) \) is the voltage across the capacitor in steady state. We wish to find the frequency response of this circuit to verify that it is a low-pass filter (it allows low-frequency components to go through, but filters out high-frequency components).

**Solution**

Using Kirchhoff’s voltage law, this circuit is represented by a first-order differential equation,
\[ v_s(t) = v_c(t) + \frac{dv_c(t)}{dt} \]

Now, if the input is \( v_s(t) = e^{j\Omega t} \), for a generic frequency \( \Omega \), then the output is \( v_c(t) = e^{j\Omega t}H(j\Omega) \). Replacing these in the differential equation, we have
\[ e^{j\Omega t} = e^{j\Omega t}H(j\Omega) + \frac{d(e^{j\Omega t}H(j\Omega))}{dt} \]

\[ = e^{j\Omega t}H(j\Omega) + j\Omega e^{j\Omega t}H(j\Omega) \]

so that
\[ H(j\Omega) = \frac{1}{1 + j\Omega} \]
or the frequency response of the filter for any frequency \( \Omega \). The magnitude of \( H(j\Omega) \) is
\[ |H(j\Omega)| = \frac{1}{\sqrt{1 + \Omega^2}} \]
which is close to one for small values of the frequency, and tends to zero when the frequency values are large—the characteristics of a low-pass filter.

For the input
\[ v_s(t) = 1 + \cos(10,000t) = \cos(0t) + \cos(10,000t) \]
(i.e., it has a zero frequency component and a 10,000-rad/sec frequency component) using Euler’s identity, we have that
\[ v_s(t) = 1 + 0.5\left(e^{j10,000t} + e^{-j10,000t}\right) \]
and the steady-state output of the circuit is
\[ v_c(t) = 1H(j0) + 0.5H(j10,000)e^{j10,000t} + 0.5H(-j10,000)e^{-j10,000t} \approx 1 + \frac{1}{10,000} \cos(10,000t - \pi/2) \approx 1 \]
since

\[ H(j0) = 1 \]

\[ H(j10,000) \approx \frac{1}{j \frac{10}{10^4}} = -\frac{j}{10,000} \]

\[ H(-j10,000) \approx \frac{1}{-j \frac{10}{10^4}} = \frac{j}{10,000} \]

Thus, this circuit acts like a low-pass filter by keeping the DC component (with the low frequency \( \Omega = 0 \)) and essentially getting rid of the high-frequency (\( \Omega = 10,000 \)) component of the signal.

Notice that the frequency response can also be obtained by considering the phasor ratio for a generic frequency \( \Omega \), which by voltage division is

\[ \frac{V_c}{V_s} = \frac{1/j \Omega}{1 + 1/j \Omega} = \frac{1}{1 + j \Omega} \]

which for \( \Omega = 0 \) is 1 and for \( \Omega = 10,000 \) is approximately \(-j/10,000\) (i.e., corresponding to \( H(j0) \) and \( H(j10,000) = H^*(j10,000) \)).

**Fourier and Laplace**

French mathematician Jean-Baptiste-Joseph Fourier (1768–1830) was a contemporary of Laplace with whom he shared many scientific and political experiences [2, 7]. Like Laplace, Fourier was from very humble origins but he was not as politically astute. Laplace and Fourier were affected by the political turmoil of the French Revolution and both came in close contact with Napoleon Bonaparte, French general and emperor. Named chair of the mathematics department of the Ecole Normale, Fourier led the most brilliant period of mathematics and science education in France. His main work was “The Mathematical Theory of Heat Conduction” where he proposed the harmonic analysis of periodic signals. In 1807 he received the grand prize from the French Academy of Sciences for this work. This was despite the objections of Laplace, Lagrange, and Legendre, who were the referees and who indicated that the mathematical treatment lacked rigor. Following Galton’s advice of “Never resent criticism, and never answer it,” Fourier disregarded these criticisms and made no change to his 1822 treatise in heat conduction. Although Fourier was an enthusiast for the Revolution and followed Napoleon on some of his campaigns, in the Second Restoration he had to pawn his belongings to survive. Thanks to his friends, he became secretary of the French Academy, the final position he held.

**4.3 COMPLEX EXPONENTIAL FOURIER SERIES**

The Fourier series is a representation of a periodic signal \( x(t) \) in terms of complex exponentials or sinusoids of frequency multiples of the fundamental frequency of \( x(t) \). The advantage of using the Fourier series to represent periodic signals is not only the spectral characterization obtained, but in finding the response for these signals when applied to LTI systems by means of the eigenfunction property.

Mathematically, the Fourier series is an expansion of periodic signals in terms of normalized orthogonal complex exponentials. The concept of orthogonality of functions is similar to the concept of
perpendicularity of vectors: Perpendicular vectors cannot be represented in terms of each other, as orthogonal functions provide mutually exclusive information. The perpendicularity of two vectors can be established using the dot or scalar product of the vectors, and the orthogonality of functions is established by the inner product, or the integration of the product of the function and its conjugate.

Consider a set of complex functions \( \{ \psi_k(t) \} \) defined in an interval \([a, b]\), and such that for any pair of these functions, let’s say \( \psi_\ell(t) \) and \( \psi_m(t) \), \( \ell \neq m \), their inner product is

\[
\int_a^b \psi_\ell(t)\psi_m^*(t)dt = \begin{cases} 
0 & \ell \neq m \\
1 & \ell = m 
\end{cases}
\] (4.8)

Such a set of functions is called orthonormal (i.e., orthogonal and normalized).

A finite-energy signal \( x(t) \) defined in \([a, b]\) can be approximated by a series

\[
\hat{x}(t) = \sum_k a_k \psi_k(t)
\] (4.9)

according to some error criterion. For instance, we could minimize the energy of the error function \( \varepsilon(t) = x(t) - \hat{x}(t) \) or

\[
\int_a^b |\varepsilon(t)|^2 dt = \int_a^b \left| x(t) - \sum_k a_k \psi_k(t) \right|^2 dt
\] (4.10)

The expansion can be finite or infinite, and may not approximate the signal point by point.

Fourier proposed sinusoids as the functions \( \{ \psi_k(t) \} \) to represent periodic signals, and solved the quadratic minimization posed in Equation (4.10) to obtain the coefficients of the representation. For most signals, the resulting Fourier series has an infinite number of terms and coincides with the signal pointwise. We will start with a more general expansion that uses complex exponentials and from it obtain the sinusoidal form. In Chapter 5 we extend the Fourier series to represent aperiodic signals—leading to the Fourier transform that is in turn connected with the Laplace transform.

Recall that a periodic signal \( x(t) \) is such that

- It is defined for \(-\infty < t < \infty\) (i.e., it has an infinite support).
- For any integer \( k \), \( x(t + kT_0) = x(t) \), where \( T_0 \) is the fundamental period of the signal or the smallest positive real number that makes this possible.

The Fourier series representation of a periodic signal \( x(t) \), of period \( T_0 \), is given by an infinite sum of weighted complex exponentials (cosines and sines) with frequencies multiples of the signal’s fundamental frequency \( \Omega_0 = 2\pi/T_0 \text{ rad/sec} \), or

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \Omega_0 = \frac{2\pi}{T_0}
\] (4.11)
where the Fourier coefficients $X_k$ are found according to

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)e^{-j\Omega_0 t}dt \quad (4.12)$$

for $k = 0, \pm 1, \pm 2, \ldots$, and any $t_0$. The form of Equation (4.12) indicates that the information needed for the Fourier series can be obtained from any period of $x(t)$.

**Remarks**

- The Fourier series uses the Fourier basis \{e^{j\Omega_0 t}, k \text{ integer}\} to represent the periodic signal $x(t)$ of period $T_0$. The Fourier basis functions are also periodic of period $T_0$ (i.e., for an integer $m$,

$$e^{j\Omega_0 (t+mT_0)} = e^{j\Omega_0 t}e^{jm2\pi} = e^{j\Omega_0 t}$$

as $e^{jm2\pi} = 1$).

- The Fourier basis functions are orthonormal over a period—that is,

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j\Omega_0 t}[e^{j\Omega_0 t}]^* dt = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases} \quad (4.13)$$

That is, $e^{j\Omega_0 t}$ and $e^{j\ell\Omega_0 t}$ are said to be orthogonal when for $k \neq \ell$ the above integral is zero, and they are normal (or normalized) when for $k = \ell$ the above integral is unity. The functions $e^{j\Omega_0 t}$ and $e^{j\ell\Omega_0 t}$ are orthogonal since

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j\Omega_0 t}[e^{j\ell\Omega_0 t}]^* dt = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j(k-\ell)\Omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} [\cos((k-\ell)\Omega_0 t) + j\sin((k-\ell)\Omega_0 t)] dt$$

$$= 0 \quad k \neq \ell$$

The above integrals are zero given that the integrands are sinusoids and the limits of the integrals cover one or more periods of the integrands. The normality of the Fourier functions is easily shown when for $k = \ell$ the above integral is

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j\ell\Omega_0 t} dt = 1$$
The Fourier coefficients \( \{X_k\} \) are easily obtained using the orthonormality of the Fourier functions: First, we multiply the expression for \( x(t) \) in Equation (4.11) by \( e^{-j\Omega_0 t} \) and then integrate over a period to get

\[
\int_{T_0} x(t)e^{-j\Omega_0 t} dt = \sum_k X_k \int_{T_0} e^{j(k-\ell)\Omega_0 t} dt
\]

\[
= \sum_k X_k T_0 \delta(k - \ell)
\]

\[
= X_\ell T_0
\]

given that when \( k = \ell \), then \( \int_{T_0} e^{i(k-\ell)\Omega_0 t} dt = T_0 \); otherwise it is zero according to the orthogonality of the Fourier exponentials. This then gives us the expression for the Fourier coefficients \( \{X_\ell\} \) in Equation (4.12).

You need to recognize that the \( k \) and \( \ell \) are dummy variables in the Fourier series, and as such the expression for the coefficients is the same regardless of whether we use \( \ell \) or \( k \).

It is important to realize from the given Fourier series equations that for a periodic signal \( x(t) \), of period \( T_0 \), any period

\[
x(t), \ t_0 \leq t \leq t_0 + T_0
\]

provides all the necessary information in the time-domain characterizing \( x(t) \). In an equivalent way the coefficients and their corresponding frequencies \( \{X_k, k\Omega_0\} \) provide all the necessary information about \( x(t) \) in the frequency domain.

### 4.4 LINE SPECTRA

The Fourier series provides a way to determine the frequency components of a periodic signal and the significance of these frequency components. Such information is provided by the power spectrum of the signal. For periodic signals, the power spectrum provides information as to how the power of the signal is distributed over the different frequencies present in the signal. We thus learn not only what frequency components are present in the signal but also the strength of these frequency components. In practice, the power spectrum can be computed and displayed using a spectrum analyzer, which will be described in Chapter 5.

#### 4.4.1 Parseval’s Theorem—Power Distribution over Frequency

Although periodic signals are infinite-energy signals, they have finite power. The Fourier series provides a way to find how much of the signal power is in a certain band of frequencies.

The power \( P_x \) of a periodic signal \( x(t) \), of period \( T_0 \), can be equivalently calculated in either the time or the frequency domain:

\[
P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_k |X_k|^2
\]

(4.14)
The power of a periodic signal $x(t)$ of period $T_0$ is given by

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Replacing the Fourier series of $x(t)$ in the power equation we have that

$$\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \frac{1}{T_0} \int_{T_0} \left( \sum_k \sum_m X_k X_m^* e^{j\Omega_0 kt} e^{-j\Omega_0 mt} dt \right)$$

$$= \sum_k \sum_m X_k X_m^* \frac{1}{T_0} \int_{T_0} e^{j\Omega_0 kt} e^{-j\Omega_0 mt} dt$$

$$= \sum_k |X_k|^2$$

after we apply the orthonormality of the Fourier exponentials. Even though $x(t)$ is real, we let $|x(t)|^2 = x(t)x^*(t)$ in the above equations, permitting us to express them in terms of $X_k$ and its conjugate. The above indicates that the power of $x(t)$ can be computed in either the time or the frequency domain giving exactly the same result.

Moreover, considering the signal to be a sum of harmonically related components or

$$x(t) = \sum_k X_k e^{j\Omega_0 kt} = \sum_k x_k(t)$$

the power of each of these components is given by

$$\frac{1}{T_0} \int_{T_0} |x_k(t)|^2 dt = \frac{1}{T_0} \int_{T_0} |X_k e^{j\Omega_0 kt}|^2 dt = \frac{1}{T_0} \int_{T_0} |X_k|^2 dt = |X_k|^2$$

and the power of $x(t)$ is the sum of the powers of the Fourier series components. This indicates that the power of the signal is distributed over the harmonic frequencies $\{k\Omega_0\}$. A plot of $|X_k|^2$ versus the harmonic frequencies $k\Omega_0$, $k = 0, \pm 1, \pm 2, \ldots$, displays how the power of the signal is distributed over the harmonic frequencies. Given the discrete nature of the harmonic frequencies $\{k\Omega_0\}$ this plot consists of a line at each frequency and as such it is called the power line spectrum (that is, a periodic signal has no power in nonharmonic frequencies). Since $\{X_k\}$ are complex, we define two additional spectra, one that displays the magnitude $|X_k|$ versus $k\Omega_0$, called the magnitude line spectrum, and the phase line spectrum or $\angle X_k$ versus $k\Omega_0$ showing the phase of the coefficients $\{X_k\}$ for $k\Omega_0$. The power line spectrum is simply the magnitude spectrum squared.

A periodic signal $x(t)$, of period $T_0$, is represented in the frequency by its

**Magnitude line spectrum**: $|X_k|$ vs $k\Omega_0$ \hspace{1cm} (4.15)

**Phase line spectrum**: $\angle X_k$ vs $k\Omega_0$ \hspace{1cm} (4.16)
The power line spectrum $|X_k|^2$ versus $k\Omega_0$ of $x(t)$ displays the distribution of the power of the signal over frequency.

4.4.2 Symmetry of Line Spectra

For a real-valued periodic signal $x(t)$, of period $T_0$, represented in the frequency domain by the Fourier coefficients $\{X_k = |X_k|e^{j\angle X_k}\}$ at harmonic frequencies $\{k\Omega_0 = 2\pi k/T_0\}$, we have that

$$X_k = X^*_k$$

or equivalently that

1. $|X_k| = |X^*_k|$ (i.e., magnitude $|X_k|$ is even function of $k\Omega_0$)
2. $\angle X_k = -\angle X^*_k$ (i.e., phase $\angle X_k$ is odd function of $k\Omega_0$)

Thus, for real-valued signals we only need to display for $k \geq 0$ the

- Magnitude line spectrum: Plot of $|X_k|$ versus $k\Omega_0$
- Phase line spectrum: Plot of $\angle X_k$ versus $k\Omega_0$

For a real signal $x(t)$, the Fourier series of its complex conjugate $x^*(t)$ is

$$x^*(t) = \left[ \sum_{\ell} X_\ell e^{j\ell\Omega_0 t} \right]^* = \sum_{\ell} X^*_\ell e^{-j\ell\Omega_0 t} = \sum_{k} X^*_k e^{jk\Omega_0 t}$$

Since $x(t) = x^*(t)$, the above equation is equal to

$$x(t) = \sum_{k} X_k e^{jk\Omega_0 t}$$

Comparing the Fourier series coefficients in the expressions, we have that $X^*_k = X_k$, which means that if $X_k = |X_k|e^{j\angle X_k}$, then

$$|X_k| = |X^*_k|$$
$$\angle X_k = -\angle X^*_k$$

or that the magnitude is an even function of $k$, while the phase is an odd function of $k$. Thus, the line spectra corresponding to real-valued signals is given for only positive harmonic frequencies, with the understanding that for negative values of the harmonic frequencies the magnitude line spectrum is even and the phase line spectrum is odd.
4.5 TRIGONOMETRIC FOURIER SERIES

The trigonometric Fourier series of a real-valued, periodic signal \( x(t) \), of period \( T_0 \), is an equivalent representation that uses sinusoids rather than complex exponentials as the basis functions. It is given by

\[
x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \theta_k)
\]

\[
= c_0 + 2 \sum_{k=1}^{\infty} [c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)] \quad \Omega_0 = \frac{2\pi}{T_0} \quad (4.19)
\]

where \( X_0 = c_0 \) is called the DC component, and \( \{2|X_k| \cos(k\Omega_0 t + \theta_k)\} \) are the \( k \)th harmonics for \( k = 1, 2, \ldots \). The frequencies \( \{k\Omega_0\} \) are said to be harmonically related. The coefficients \( \{c_k, d_k\} \) are obtained from \( x(t) \) as follows:

\[
c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t) \, dt \quad k = 0, 1, \ldots
\]

\[
d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t) \, dt \quad k = 1, 2, \ldots \quad (4.20)
\]

The coefficients \( X_k = |X_k|e^{j\theta_k} \) are connected with the coefficients \( c_k \) and \( d_k \) by

\[
|X_k| = \sqrt{c_k^2 + d_k^2}
\]

\[
\theta_k = -\tan^{-1} \left[ \frac{d_k}{c_k} \right]
\]

The functions \( \{\cos(k\Omega_0 t), \sin(k\Omega_0 t)\} \) are orthonormal.

Using the relation \( X_k = X_{-k}^* \) obtained in the previous section, we express the exponential Fourier series of a real-valued periodic signal \( x(t) \) as

\[
x(t) = X_0 + \sum_{k=1}^{\infty} [X_ke^{jk\Omega_0 t} + X_{-k}e^{-jk\Omega_0 t}]
\]

\[
= X_0 + \sum_{k=1}^{\infty} \left[ |X_k|e^{j(k\Omega_0 t + \theta_k)} + |X_k|e^{-j(k\Omega_0 t + \theta_k)} \right]
\]

\[
= X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \theta_k)
\]

which is the top equation in Equation (4.19).
Let us then show how the coefficients $c_k$ and $d_k$ can be obtained directly from the signal. Using the relation $X_k = X^*_{-k}$ and the fact that for a complex number $z = a + jb$, then $z + z^* = (a + jb) + (a - jb) = 2a = 2\Re(z)$, we have that

$$x(t) = X_0 + \sum_{k=1}^{\infty} [X_ke^{jk\Omega_0t} + X_{-k}e^{-jk\Omega_0t}]$$

Since $X_k$ is complex (verify this!),

$$2\Re[X_ke^{jk\Omega_0t}] = 2\Re[X_k] \cos(k\Omega_0t) - 2\Im[X_k] \sin(k\Omega_0t)$$

Now, if we let

$$c_k = \Re[X_k] = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0t) \, dt \quad k = 1, 2, \ldots$$

$$d_k = -\Im[X_k] = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0t) \, dt \quad k = 1, 2, \ldots$$

we then have

$$x(t) = X_0 + \sum_{k=1}^{\infty} \left( 2\Re[X_k] \cos(k\Omega_0t) - 2\Im[X_k] \sin(k\Omega_0t) \right)$$

and since the average $X_0 = c_0$ we obtain the second form of the trigonometric Fourier series shown in Equation (4.19). Notice that $d_0 = 0$ and so it is not necessary to define it.

The coefficients $X_k = |X_k|e^{jk\theta_k}$ are connected with the coefficients $c_k$ and $d_k$ by

$$|X_k| = \sqrt{c_k^2 + d_k^2}$$

$$\theta_k = -\tan^{-1} \left( \frac{d_k}{c_k} \right)$$

This can be shown by adding the phasors corresponding to $c_k \cos(k\Omega_0t)$ and $d_k \sin(k\Omega_0t)$ and finding the magnitude and phase of the resulting phasor.
Finally, since the exponential basis \( \{ e^{jk\Omega_0 t} \} = \{ \cos(k\Omega_0 t) + j\sin(k\Omega_0 t) \} \), the sinusoidal bases \( \cos(k\Omega_0 t) \) and \( \sin(k\Omega_0 t) \) just like the exponential basis are periodic, of period \( T_0 \), and orthonormal.

\[ e^{jk\Omega_0 t} = \cos(k\Omega_0 t) + j\sin(k\Omega_0 t) \]

### Example 4.4
Find the Fourier series of a raised-cosine signal \( B \geq A \),

\[ x(t) = B + A \cos(\Omega_0 t + \theta) \]

which is periodic of period \( T_0 \) and fundamental frequency \( \Omega_0 = 2\pi/T_0 \). Call \( y(t) = B + \cos(\Omega_0 t - \pi/2) \). Find its Fourier series coefficients and compare them to those for \( x(t) \). Use symbolic MATLAB to compute the Fourier series of \( y(t) = 1 + \sin(100t) \). Find and plot its magnitude and phase line spectra.

**Solution**
In this case we do not need to compute the Fourier coefficients since \( x(t) \) is already in the trigonometric form. From Equation (4.19) its dc value is \( B \), and \( A \) is the coefficient of the first harmonic in the trigonometric Fourier series, so that \( X_0 = B \), \( \|X_1\| = A/2 \), and \( \angle X_1 = \theta \). Likewise, using Euler’s identity we obtain that

\[ x(t) = B + \frac{A}{2} \left[ e^{j(\Omega_0 t + \theta)} + e^{-j(\Omega_0 t + \theta)} \right] \]

which gives

\[ X_0 = B \]
\[ X_1 = \frac{Ae^{j\theta}}{2} \]
\[ X_{-1} = X_1^* \]

If we let \( \theta = -\pi/2 \) in \( x(t) \), we get

\[ y(t) = B + A \sin(\Omega_0 t) \]

Its Fourier series coefficients are \( Y_0 = B \) and \( Y_1 = Ae^{-j\pi/2}/2 \) so that \( \|Y_1\| = \|Y_{-1}\| = A/2 \) and \( \angle Y_1 = -\angle Y_{-1} = -\pi/2 \). The magnitude and phase line spectra of the raised cosine \( (\theta = 0) \) and of the raised sine \( (\theta = -\pi/2) \) are shown in Figure 4.2. For both \( x(t) \) and \( y(t) \) there are only two frequencies—the dc frequency and \( \Omega_0 \)—and as such the power of the signal is concentrated at those two frequencies as shown in Figure 4.2. The difference between the line spectra of \( x(t) \) and \( y(t) \) is in the phase.
Using symbolic MATLAB integration we can easily find the Fourier series coefficients, and the corresponding magnitude and phase are then plotted using `stem` to obtain the line spectra. Using our MATLAB function `fourierseries` the magnitude and phase of the line spectrum corresponding to the periodic raised sine $y(t) = 1 + \sin(100t)$ is shown in Figure 4.3.

```
function [X, w] = fourierseries(x, T0, N)
    %%%
    % symbolic Fourier Series computation
    % x: periodic signal
    % T0: period
    % N: number of harmonics
    % X,w: Fourier series coefficients at harmonic frequencies
    %%%% syms t
    % computation of N Fourier series coefficients
    for k = 1:N,
        X1(k) = int(x * exp(-j * 2 * pi * (k - 1) * t/T0), t, 0, T0)/T0;
        X(k) = subs(X1(k));
        w(k) = (k-1) * 2 * pi/T0; % harmonic frequencies
    end
```
4.6 Fourier Coefficients from Laplace

The computation of the $X_k$ coefficients (see Eq. 4.12) requires integration that for some signals can be rather complicated. The integration can be avoided whenever we know the Laplace transform of a period of the signal as we will show. In general, the Laplace transform of a period of the signal exists over the whole $s$-plane, given that it is a finite-support signal. In some cases, the dc coefficient cannot be computed with the Laplace transform, but the dc term is easy to compute directly.

Remarks Just because a signal is a sum of sinusoids, which are always periodic, is not enough for it to have a Fourier series. The signal should be periodic. The signal $x(t) = \cos(t) - \sin(\pi t)$ has components with periods $T_1 = 2\pi$ and $T_2 = 2$ so that the ratio $T_1/T_2 = \pi$ is not a rational number. Thus, $x(t)$ is not periodic and no Fourier series for it is possible.

FIGURE 4.3
Line spectra of Fourier series of $y(t) = 1 + \sin(100t)$ (top figure). Notice the even and the odd symmetries of the magnitude and the phase spectra. The phase is $-\pi/2$ at $\Omega = 100$ rad/sec.
For a periodic signal \( x(t) \), of period \( T_0 \), if we know or can easily compute the Laplace transform of a period of \( x(t) \),

\[
x_1(t) = x(t)[u(t_0) - u(t - t_0 - T_0)] \quad \text{for any } t_0
\]

Then the Fourier coefficients of \( x(t) \) are given by

\[
X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]_{s=jk\Omega_0} \quad \Omega_0 = \frac{2\pi}{T_0} \text{ fundamental frequency} \quad (4.21)
\]

This can be seen by comparing the equation for the \( X_k \) coefficients with the Laplace transform of a period \( x_1(t) = x(t)[u(t_0) - u(t - t_0 - T_0)] \) of \( x(t) \). Indeed, we have that

\[
X_k = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t)e^{-jk\Omega_0 t} dt
\]

\[
= \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t)e^{-st} dt \bigg|_{s=jk\Omega_0}
\]

\[
= \frac{1}{T_0} \mathcal{L}[x_1(t)]_{s=jk\Omega_0}
\]

**Example 4.5**

Consider the periodic pulse train \( x(t) \), of period \( T_0 = 1 \), shown in Figure 4.4. Find its Fourier series.

**Solution**

Before finding the Fourier coefficients, we see that this signal has a dc component of 1, and that \( x(t) - 1 \) (zero-average signal) is well represented by cosines, given its even symmetry, and as such

**FIGURE 4.4**
Train of rectangular pulses.
4.6 Fourier Coefficients from Laplace

The Fourier coefficients will be real. Doing this analysis before the computations is important so we know what to expect.

The Fourier coefficients are obtained directly using their integral formulas or from the Laplace transform of a period. Since $T_0 = 1$, the fundamental frequency of $x(t)$ is $\Omega_0 = 2\pi \text{ rad/sec}$. Using the integral expression for the Fourier coefficients we have

$$X_k = \frac{1}{T_0} \int_{-1/4}^{3/4} x(t) e^{-j\Omega_0 kt} \, dt = \int_{-1/4}^{1/4} 2 e^{-j2\pi kt} \, dt$$

$$= \frac{2}{\pi k} \left[ \frac{e^{j\pi k/2} - e^{-j\pi k/2}}{2j} \right] = \frac{\sin(\pi k/2)}{(\pi k/2)}$$

which are real as we predicted. The Fourier series is then

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/2)}{(\pi k/2)} e^{jk2\pi t}$$

To find the Fourier coefficients with the Laplace transform, let the period be $x_1(t) = x(t)$ for $-0.5 \leq t \leq 0.5$. Delaying it by 0.25 we get $x_1(t - 0.25) = 2[u(t) - u(t - 0.5)]$ with a Laplace transform

$$e^{-0.25s}X_1(s) = \frac{2}{s}(1 - e^{-0.5s})$$

so that $X_1(s) = (2/s)[e^{0.25s} - e^{-0.25s}]$, and therefore

$$X_k = \frac{1}{T_0} \mathcal{L} \left[ x_1(t) \right] \bigg|_{s = jk\Omega_0}$$

$$= \frac{2}{jk\Omega_0 T_0} 2j \sin(k\Omega_0/4)$$

and for $\Omega_0 = 2\pi$, $T_0 = 1$, we get

$$X_k = \frac{\sin(\pi k/2)}{\pi k/2} \quad k \neq 0$$

Since the above equation gives zero over zero when $k = 0$ (i.e., it is undefined), the dc value is found from the integral formula as

$$X_0 = \frac{1}{4} \int_{-1/4}^{1/4} 2 \, dt = 1$$

These Fourier coefficients coincide with the ones found before.

The following script is used to find the Fourier coefficients with our function `fourierseries` and to plot the magnitude and phase line spectra.
% Example 4.5---Fourier series of train of pulses

clear all; clf
syms t
T0 = 1; m = heaviside(t) \(-\) heaviside(t \(-\) T0/4) + heaviside(t \(-\) 3 \(*\) T0/4); x = 2 \(*\) m
[X,w] = fourierseries(x,T0,20);
subplot(221); ezplot(x,[0 T0]); grid
subplot(223); stem(w,abs(X))
subplot(224); stem(w,angle(X))

Notice that in this case:

1. The $X_k$ Fourier coefficients of the train of pulses are given in terms of the $\sin(x)/x$ or the sinc function. This function was presented in Chapter 1. Recall that the sinc is
   - Even—that is, $\sin(x)/x = \sin(-x)/(-x)$.
   - The value at $x = 0$ is found by means of L'Hôpital's rule because the numerator and the denominator of sinc are zero for $x = 0$, so
     \[
     \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{d\sin(x)/dx}{dx/dx} = 1
     \]
   - It is bounded, indeed
     \[
     -1 \leq \frac{\sin(x)}{x} \leq 1
     \]

2. Since the dc component of $x(t)$ is 1, once it is subtracted it is clear that the rest of the series can be represented as a sum of cosines:
   \[
   x(t) = 1 + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{\sin(\pi k/2)}{\pi k/2} e^{jk2\pi t}
   \]
   \[
   = 1 + 2 \sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{\pi k/2} \cos(2\pi kt)
   \]
   This can also be seen by considering the trigonometric Fourier series of $x(t)$. Since $x(t) \sin(k\Omega_0 t)$ is odd, as $x(t)$ is even and $\sin(k\Omega_0 t)$ is odd, then the coefficients corresponding to the sines in the expansion will be zero. On the other hand, $x(t) \cos(k\Omega_0 t)$ is even and gives nonzero Fourier coefficients. See Equations (4.20).

3. In general, the Fourier coefficients are complex and as such need to be represented by their magnitudes and phases. In this case, the $X_k$ coefficients are real-valued, and in particular zero when $k\pi/2 = \pm m\pi$, $m$ an integer, or when $k = \pm 2, \pm 4, \ldots$. Since the $X_k$ values are real, the corresponding phase would be zero when $X_k \geq 0$, and $\pm\pi$ when $X_k < 0$. In Figure 4.5 we show a period of the signal, and the magnitude and phase line spectra displayed only for positive values of frequency (with the understanding that the magnitude spectrum is even and the phase is odd functions of the frequency).
4.6 Fourier Coefficients from Laplace

FIGURE 4.5
Period of train of rectangular pulses (top) and its magnitude and phase line spectra (bottom).

4. The $X_k$ coefficients and its squares, related to the power line spectrum, are obtained using the `fourierseries` function (see Figure 4.5):

\[
\begin{array}{cccc}
  k & X_k = X_{-k} & X_k^2 \\
  0 & 1 & 1 \\
  1 & 0.64 & 0.41 \\
  2 & 0 & 0 \\
  3 & -0.21 & 0.041 \\
  4 & 0 & 0 \\
  5 & 0.13 & 0.016 \\
  6 & 0 & 0 \\
  7 & -0.09 & 0.008 \\
\end{array}
\]
Notice that about 11 of them (including the zero values), or the dc value and 5 harmonics, provide a very good approximation of the pulse train, and would occupy a bandwidth of approximately $10\pi$ rad/sec. The power contribution, as indicated by $X_k^2$ after $k = \pm 6$, is relatively small.

**Example 4.6**

Find the Fourier series of the full-wave rectified signal $x(t) = |\cos(\pi t)|$ shown in Figure 4.6. This signal is used in the design of dc sources. The rectification of an ac signal is the first step in this design.

**Solution**

The integral to find the Fourier coefficients is

$$X_k = \int_{-0.5}^{0.5} \cos(\pi t)e^{-j2\pi kt} dt$$

which can be computed by using Euler's identity or any other method. We want to show that this can be avoided by using the Laplace transform.

A period $x_1(t)$ of $x(t)$ can be expressed as

$$x_1(t - 0.5) = \sin(\pi t)u(t) + \sin(\pi(t - 1))u(t - 1)$$

**FIGURE 4.6**

(a) Full-wave rectified signal $x(t)$ and (b) one of its periods $x_1(t)$. 
and using the Laplace transform we have

\[ X_1(s) e^{-0.5s} = \frac{\pi}{s^2 + \pi^2} [1 + e^{-s}] \]

so that

\[ X_1(s) = \frac{\pi}{s^2 + \pi^2} [e^{0.5s} + e^{-0.5s}] \]

The Fourier coefficients are then

\[ X_k = \frac{1}{T_0} X_1(s)|_{s=j\Omega_0 k} \]

where \( T_0 = 1 \) and \( \Omega_0 = 2\pi \), giving

\[ X_k = \frac{\pi}{(j2\pi k)^2 + \pi^2} 2 \cos(2\pi k/2) \]

\[ = \frac{2(-1)^k}{\pi(1 - 4k^2)} \]
since \( \cos(\pi k) = (-1)^k \). The DC value of the full-wave rectified signal is \( X_0 = 2/\pi \). Notice that the Fourier coefficients are real given that the signal is even.

The MATLAB script used in the previous example can be used again with the following modification for the generation of a period of \( x(t) \). The results are shown in Figure 4.7.

```matlab
%% Example 4.6---Fourier series of full-wave rectified signal
%% period generation
T0 = 1;
m = heaviside(t) - heaviside(t - T0); x = abs(cos(pi * t)) * m
```

**Example 4.7**

Computing the derivative of a signal enhances higher harmonics. To illustrate this consider the train of triangular pulses \( y(t) \) (Figure 4.8) with fundamental period \( T_0 = 2 \). Let \( x(t) = dy(t)/dt \). Find its Fourier series and compare \( |X_k| \) with \( |Y_k| \) to determine which of these signals is smoother—that is, which one has lower frequency components.

**Solution**

A period of \( y(t) \), \(-1 \leq t \leq 1\), is given by

\[
y_1(t) = r(t + 1) - 2r(t) + r(t - 1)
\]

with a Laplace transform

\[
Y_1(s) = \frac{1}{s^2} \left[ e^s - 2 + e^{-s} \right]
\]
so that the Fourier coefficients are given by \( T_0 = 2, \Omega_0 = \pi \):

\[
Y_k = \frac{1}{T_0} Y_1(s)|_{s=j\Omega_0 k} = \frac{1}{2(j\pi k)^2} [2 \cos(\pi k) - 2] = \frac{1 - \cos(\pi k)}{\pi^2 k^2} = \frac{1 - (-1)^k}{\pi^2 k^2} \quad k \neq 0
\]

This is also equal to

\[
Y_k = 0.5 \left[ \frac{\sin(\pi k/2)}{(\pi k/2)} \right]^2
\]

using the identity \( 1 - \cos(\pi k) = 2 \sin^2(\pi k/2) \). By observing \( y(t) \) we deduce that its DC value is \( Y_0 = 0.5 \).

Let us then consider the periodic signal \( x(t) = dy(t)/dt \) (shown in Fig. 4.8(b)) with a dc value \( X_0 = 0 \). For \(-1 \leq t \leq 1\), its period is \( x_1(t) = u(t + 1) - 2u(t) + u(t - 1) \) and

\[
X_1(s) = \frac{1}{s} \left[ e^s - 2 + e^{-s} \right]
\]

which gives the Fourier series coefficients \( T_0 = 2, \Omega \) (the period and the fundamental frequency are equal to the ones for \( y(t) \))

\[
X_k = \frac{\sin^2(k\pi/2)}{k\pi/2} j
\]

since \( X_k = \frac{1}{T} X_1(s)|_{s=j\pi k} \).

**FIGURE 4.9**

Magnitude and phase line spectra of (a) triangular signal \( y(t) \) (top left) and (b) its derivative \( x(t) \) (top right). Ignoring the dc values, the \(|Y_k|\) decay faster to zero than the \(|X_k|\), thus \( y(t) \) is smoother than \( x(t) \).
For $k \neq 0$ we have $|Y_k| = |X_k|/(\pi k)$, so that as $k$ increases the frequency components of $y(t)$ decrease in magnitude faster than the corresponding ones of $x(t)$. Thus, $y(t)$ is smoother than $x(t)$. The magnitude line spectrum $|Y_k|$, ignoring its average, goes faster to zero than the magnitude line spectrum $|X_k|$, as seen in Figure 4.9.

Notice that in this case $y(t)$ is even and its Fourier coefficients $Y_k$ are real, while $x(t)$ is odd and its Fourier coefficients $X_k$ are purely imaginary. If we subtract the average of $y(t)$, the signal $y(t)$ can be clearly approximated as a series of cosines, thus the need for real coefficients in its complex exponential Fourier series. The signal $x(t)$ is zero-average and as such it can be clearly approximated by a series of sines requiring its Fourier coefficients $X_k$ to be imaginary.

Example 4.8
Integration of a periodic signal, provided it has zero mean, gives a smoother signal. To see this, find and compare the magnitude line spectra of a sawtooth signal $x(t)$, of period $T_0 = 2$, and its integral

$$y(t) = \int x(t)dt$$

shown Figure 4.10.

Solution
Before doing any calculations it is important to realize that the integral would not exist if the dc is not zero. Using the following script we can compute the Fourier series coefficients of $x(t)$ and $y(t)$.

A period of $x(t)$ is

$$x_1(t) = tw(t) + (t - 2)w(t - 1) \quad 0 \leq t \leq 2$$

where $w(t) = u(t) - u(t - 1)$ is a rectangular window.
4.7 Convergence of the Fourier Series

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FIGURE 4.11
(a) Periods of the sawtooth signal \( x(t) \) and (b) its integral \( y(t) \) and their magnitude and phase line spectra.

The signal \( y(t) \) is smoother than \( x(t) \); \( y(t) \) is a continuous function of time, while \( x(t) \) is discontinuous. This is indicated as well by the magnitude line spectra of the two signals. Ignoring the dc components, the \( |Y_k| \) of \( y(t) \) decay a lot faster to zero than the \( |X_k| \) of \( x(t) \) (See Figure 4.11). As we will see in Section 4.10, computing the derivative of a periodic signal is equivalent to multiplying its Fourier series coefficients by \( j\Omega_0 k \), which emphasizes the higher harmonics. If the periodic signal is zero-mean so that its integral exists, the Fourier coefficients of the integral can be found by dividing them by \( j\Omega_0 k \) so that now the low harmonics are emphasized.

4.7 CONVERGENCE OF THE FOURIER SERIES

It can be said, without overstating it, that any periodic signal of practical interest has a Fourier series. Only very strange signals would not have a converging Fourier series. Establishing convergence is necessary because the Fourier series has an infinite number of terms. To establish some general
conditions under which the series converges, we need to classify signals with respect to their smoothness.

A signal \( x(t) \) is said to be **piecewise smooth** if it has a finite number of discontinuities, while a **smooth** signal has a derivative that changes continuously. Thus, smooth signals can be considered special cases of piecewise smooth signals.

The Fourier series of a piecewise smooth (continuous or discontinuous) periodic signal \( x(t) \) converges for all values of \( t \). The mathematician Dirichlet showed that for the Fourier series to converge to the periodic signal \( x(t) \), the signal should satisfy the following sufficient (not necessary) conditions over a period:

1. Be absolutely integrable.
2. Have a finite number of maxima, minima, and discontinuities.

The infinite series equals \( x(t) \) at every continuity point and equals the average

\[
0.5[x(t + 0+) + x(t + 0-)]
\]

of the right limit \( x(t + 0+) \) and the left limit \( x(t + 0- \) at every discontinuity point. If \( x(t) \) is continuous everywhere, then the series converges absolutely and uniformly.

Although the Fourier series converges to the arithmetic average at discontinuities, it can be observed that there is some ringing before and after the discontinuity points. This is called the **Gibb’s phenomenon**. To understand this phenomenon it is necessary to explain how the Fourier series can be seen as an approximation to the actual signal, and how when a signal has discontinuities the convergence is not uniform around them. It will become clear that the smoother the signal \( x(t) \) is, the easier it is to approximate it with a Fourier series with a finite number of terms.

When the signal is continuous everywhere, the convergence is such that at each point \( t \) the series approximates the actual value \( x(t) \) as we increase the number of terms in the approximation. However, that is not the case when discontinuities occur in the signal. This is despite the fact that a minimum mean-square approximation seems to indicate that the approximation could give a zero error. Let

\[
x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t} \tag{4.24}
\]

be the \( N \)-th order approximation of a periodic signal \( x(t) \), of fundamental frequency \( \Omega_0 \), that minimizes the average quadratic error over a period

\[
E_N = \frac{1}{T_0} \int_{T_0} |x(t) - x_N(t)|^2 dt \tag{4.25}
\]
with respect to the Fourier coefficients $X_k$. To minimize $E_N$ with respect to the coefficients $X_k$ we set its derivative with respect to $X_k$ to zero. Let $\varepsilon(t) = x(t) - x_N(t)$, so that

$$
\frac{dE_N}{dX_k} = \frac{1}{T_0} \int_{T_0} 2\varepsilon(t) \frac{de^*(t)}{dX_k} dt
$$

$$
= -\frac{1}{T_0} \int_{T_0} 2[x(t) - x_N(t)]e^{-j\Omega_0 t} dt
$$

$$
= 0
$$

which after replacing $x_N(t)$ and using the orthogonality of the Fourier exponentials gives

$$X_k = \frac{1}{T_0} \int_{T_0} x(t)e^{-j\Omega_0 t} dt$$

(4.26)

corresponding to the Fourier coefficients of $x(t)$ for $-N \leq k \leq N$. As $N \to \infty$ the average error $E_N \to 0$.

The only issue left is how $x_N(t)$ converges to $x(t)$. As indicated before, if $x(t)$ is smooth $x_N(t)$ approximates $x(t)$ at every point, but if there are discontinuities the approximation is in an average fashion. The Gibb's phenomenon indicates that around discontinuities there will be ringing, regardless of the order $N$ of the approximation, even though the average quadratic error $E_N$ goes to zero as $N$ increases. This phenomenon will be explained in Chapter 5 as the effect of using a rectangular window to obtain a finite-frequency representation of a periodic signal.

**Example 4.9**

To illustrate the Gibb's phenomenon consider the approximation of a train of pulses $x(t)$ with zero mean and period $T_0 = 1$ (see the dashed signal in Figure 4.12) with a Fourier series $x_N(t)$ with $N = 1, \ldots, 20$.

**Solution**

We compute analytically the Fourier coefficients of $x(t)$ and use them to obtain an approximation $x_N(t)$ of $x(t)$ having a zero DC component and up to 20 harmonics. The dashed-line plot in Figure 4.12 is $x(t)$ and the solid-line plot is $x_N(t)$ when $N = 20$. The discontinuities of the pulse train cause the Gibb's phenomenon. Even if we increase the number of harmonics there is an overshoot in the approximation around the discontinuities.
FIGURE 4.12
Approximate Fourier series \( x_N(t) \) of the pulse train \( x(t) \) (discontinuous) using the DC component and 20 harmonics. The approximate \( x_N(t) \) displays the Gibb’s phenomenon around the discontinuities.

When you execute the above script, it pauses to display the approximation for an increasing number of terms in the approximation. At each of these values ringing around the discontinuities the Gibb’s phenomenon is displayed.
**Example 4.10**

Consider the mean-square error optimization to obtain an approximation of the periodic signal \( x(t) \) shown in Figure 4.4 from Example 4.5. We wish to obtain an approximate \( x_2(t) = \alpha + 2\beta \cos(\Omega_0 t) \), given that it is clear that \( x(t) \) has an average, and that once we subtract it from the signal the resulting signal is approximated by a cosine function. Minimize the mean-square error

\[
E_2 = \frac{1}{T_0} \int_{t_0}^{T_0} |x(t) - x_2(t)|^2 dt
\]

with respect to \( \alpha \) and \( \beta \) to find these values.

**Solution**

To minimize \( E_2 \) we set to zero its derivatives with respect to \( \alpha \) and \( \beta \) to get

\[
\frac{dE_2}{d\alpha} = -\frac{1}{T_0} \int_{t_0}^{T_0} 2[x(t) - \alpha - 2\beta \cos(\Omega_0 t)] dt = -\frac{1}{T_0} \int_{t_0}^{T_0} 2[x(t) - \alpha] dt = 0
\]

\[
\frac{dE_2}{d\beta} = -\frac{1}{T_0} \int_{t_0}^{T_0} 2[x(t) - \alpha - 2\beta \cos(\Omega_0 t)] \cos(\Omega_0 t) dt = 0
\]

which, after getting rid of \( \frac{2}{T_0} \) of both sides of the above equations and applying the orthogonality of the Fourier basis, gives

\[
\alpha = \frac{1}{T_0} \int_{t_0}^{T_0} x(t) dt
\]

\[
\beta = \frac{1}{T_0} \int_{t_0}^{T_0} x(t) \cos(\Omega_0 t) dt
\]

For the signal in Figure 4.4 we obtain

\[
\alpha = 1
\]

\[
\beta = \frac{2}{\pi}
\]

giving as approximation the signal

\[
x_2(t) = 1 + \frac{4}{\pi} \cos(2\pi t)
\]

which at \( t = 0 \) gives \( x_2(0) = 2.27 \) instead of the expected 2; \( x_2(0.25) = 1 \) (because of the discontinuity at this point, this value is the average of 2 and 0, the values, respectively, before and after the discontinuity) instead of 2 and \( x_2(0.5) = -0.27 \) instead of the expected 0. □
Example 4.11

Consider the train of pulses in Example 4.5. Determine how many Fourier coefficients are necessary to get a representation containing 97% of the power of the periodic signal.

Solution

The desired 97% of the power of $x(t)$ is

$$0.97 \frac{1}{T_0} \int_{T_0} x^2(t) dt = 0.97 \int_{-0.25}^{0.25} 4 dt = 1.94$$

and so we need to find an integer $N$ such that

$$\sum_{k=-N}^{N} |X_k|^2 = \sum_{k=-N}^{N} \left| \frac{\sin(\pi k/2)}{(\pi k/2)} \right|^2 = 1.94$$

The value of $N$ is found by trial and error, adding consecutive values of the magnitude squared of Fourier coefficients. Using MATLAB, it is found that for $N = 5$ (dc and 5 harmonics) the Fourier series approximation has a power of 1.93. Thus, 11 Fourier coefficients give a very good approximation to the periodic train of pulses, with about 97% of the signal power.

4.8 TIME AND FREQUENCY SHIFTING

Time shifting and frequency shifting are duals of each other.

- **Time-shifting:** A periodic signal $x(t)$, of period $T_0$, remains periodic of the same period when shifted in time. If $X_k$ are the Fourier coefficients of $x(t)$, the Fourier coefficients for $x(t - t_0)$ are

$$\left\{ X_k e^{-jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k - k\Omega_0 t_0)} \right\}$$

That is, only a change in phase is caused by the time shift. The magnitude spectrum remains the same.

- **Frequency-shifting:** When a periodic signal $x(t)$, of period $T_0$, modulates a complex exponential $e^{j\Omega_1 t}$:
  - The modulated signal $x(t)e^{j\Omega_1 t}$ is periodic of period $T_0$ if $\Omega_1 = M\Omega_0$ for an integer $M \geq 1$.
  - The Fourier coefficients $X_k$ are shifted to frequencies $k\Omega_0 + \Omega_1$.
  - The modulated signal is real-valued by multiplying $x(t)$ by $\cos(\Omega_1 t)$.

If we delay or advance in time a periodic signal, the resulting signal is periodic of the same period. Only a change in the phase of the coefficients occurs to accommodate for the shift. Indeed, if

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}$$
we then have that

\[ x(t - t_0) = \sum_k X_k e^{jk\Omega_0(t - t_0)} = \sum_k \left[ X_k e^{-jk\Omega_0 t_0} \right] e^{jk\Omega_0 t} \]

\[ x(t + t_0) = \sum_k X_k e^{jk\Omega_0(t + t_0)} = \sum_k \left[ X_k e^{jk\Omega_0 t_0} \right] e^{jk\Omega_0 t} \]

so that the Fourier coefficients \( \{X_k\} \) corresponding to \( x(t) \) are changed to \( \{X_k e^{\pm jk\Omega_0 t_0}\} \) for \( x(t \mp t_0) \). In both cases, they have the same magnitude \( |X_k| \) but different phases.

In a dual way, if we multiply the above periodic signal \( x(t) \) by a complex exponential of frequency \( \Omega_1, e^{j\Omega_1 t} \), we obtain a so-called modulated signal \( y(t) \) and its spectrum is shifted in frequency by \( \Omega_1 \) with respect to the spectrum of the periodic signal \( x(t) \). In fact,

\[ y(t) = x(t)e^{j\Omega_1 t} = \sum_k X_k e^{j(k\Omega_0 + \Omega_1) t} \]

indicating that the harmonic frequencies are shifted by \( \Omega_1 \). The signal \( y(t) \) is not necessarily periodic. Since \( T_0 \) is the period of \( x(t) \), then

\[ y(t + T_0) = x(t + T_0)e^{j\Omega_1(t + T_0)} \]

and for it to be equal to \( y(t) \), then \( \Omega_1 T_0 = 2\pi M \), for an integer \( M \neq 0 \) or

\[ \Omega_1 = M\Omega_0 \quad M >> 1 \]

which goes along with the condition that the modulating frequency \( \Omega_1 \) is chosen much larger than \( \Omega_0 \). The modulated signal is then given by

\[ y(t) = \sum_k X_k e^{j(k\Omega_0 + \Omega_1) t} = \sum_k X_k e^{j(k\Omega_0 + M) t} = \sum_\ell X_{\ell - M} e^{j\Omega_0 \ell t} \]

so that the Fourier coefficients are shifted to new frequencies \( \Omega_0(k + M) \).

To keep the modulated signal real-valued, one multiplies the periodic signal \( x(t) \) by a cosine of frequency \( \Omega_1 = M\Omega_0 \) for \( M >> 1 \) to obtain a modulated signal

\[ y_1(t) = x(t) \cos(\Omega_1 t) = \sum_k 0.5X_k \left[ e^{j[k\Omega_0 + \Omega_1] t} + e^{j[k\Omega_0 - \Omega_1] t} \right] \]

so that the harmonic components are now centered around \( \pm \Omega_1 \).
Example 4.12

To illustrate the modulation property using MATLAB consider modulating a sinusoid \( \cos(20\pi t) \) with a train of square pulses

\[
x_1(t) = 0.5[1 + \text{sign}(\sin(\pi t))]
\]

and with a sinusoid

\[
x_2(t) = \cos(\pi t)
\]

Use our function `fourierseries` to find the Fourier series of the modulated signals and plot their magnitude line spectra.

Solution

The function \( \text{sign} \) is defined as

\[
\text{sign}(x(t)) = \begin{cases} 
-1 & x(t) < 0 \\
1 & x(t) \geq 0 
\end{cases}
\]  

That is, it determines the sign of the signal. Thus, \( 0.5[1 + \text{sign}(\sin(\pi t))] = u(t) - u(t - 1) \) equals 1 for \( 0 \leq t \leq 1 \), and 0 for \( 1 < t \leq 2 \), which corresponds to a period of a train of square pulses.

The following script allows us to compute the Fourier coefficients of the two modulated signals.

```plaintext
%% Example 4.12---Modulation

syms t
T0 = 2;
m = heaviside(t) - heaviside(t - T0/2);
m1 = heaviside(t) - heaviside(t - T0);
x = m * cos(20 * pi * t);
x1 = m1 * cos(pi * t) * cos(20 * pi * t);
[X, w] = fourierseries(x, T0, 60);
[X1, w1] = fourierseries(x1, T0, 60);
```

The modulated signals and their corresponding magnitude line spectra are shown in Figure 4.13. The Fourier coefficients of the modulated signals are now clustered around the frequency \( 20\pi \).
4.9 Response of LTI Systems to Periodic Signals

The most important property of LTI systems is the eigenfunction property.

**Eigenfunction property**: In steady state, the response to a complex exponential (or a sinusoid) of a certain frequency is the same complex exponential (or sinusoid), but its amplitude and phase are affected by the frequency response of the system at that frequency.

Suppose that the impulse response of an LTI system is \( h(t) \) and that \( H(s) = \mathcal{L}[h(t)] \) is the corresponding transfer function. If the input to this system is a periodic signal \( x(t) \), of period \( T_0 \), with Fourier series

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \Omega_0 = \frac{2\pi}{T_0}
\]

then according to the eigenfunction property the output in the steady state is

\[
y_{ss}(t) = \sum_{k=-\infty}^{\infty} [X_k H(jk\Omega_0)] e^{jk\Omega_0 t}
\]
If we call $Y_k = X_k H(jk\Omega_0)$ we have a Fourier series representation of $y_{ss}(t)$ with $Y_k$ as its Fourier coefficients.

### 4.9.1 Sinusoidal Steady State

If the input of a stable and causal LTI system, with impulse response $h(t)$, is $x(t) = Ae^{j\Omega_0 t}$, the output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = Ae^{j\Omega_0 t} \int_{0}^{\infty} h(\tau)e^{-j\Omega_0 \tau}d\tau$$

$$= Ae^{j\Omega_0 t}H(j\Omega_0) = A|H(j\Omega_0)|e^{j\Omega_0 t + \angle H(j\Omega_0)}$$

The limits of the first integral indicate that the system is causal (the $h(\tau) = 0$ for $\tau < 0$) and that the input $x(t-\tau)$ is applied from $-\infty$ (when $\tau = \infty$) to $t$ (when $\tau = 0$); thus $y(t)$ is the steady-state response of the system. If the input is a sinusoid—for example,

$$x_1(t) = Re[x(t) = Ae^{j\Omega_0 t}] = A \cos(\Omega_0 t)$$

then the corresponding steady-state response is

$$y_1(t) = Re[A|H(j\Omega_0)|e^{j\Omega_0 t + \angle H(j\Omega_0)}]$$

$$= A|H(j\Omega_0)| \cos(\Omega_0 t + \angle H(j\Omega_0)).$$

As in the eigenfunction property, the frequency of the output coincides with the frequency of the input, however, the magnitude and the phase of the input signal is changed by the response of the system at the input frequency.

The following script simulates the convolution of a sinusoid $x(t)$ of frequency $\Omega = 20\pi$, amplitude 10, and random phase with the impulse response $h(t)$ (a modulated decaying exponential) of an LTI system. The convolution integral is approximated using the MATLAB function `conv`.

```matlab
%% Simulation of Convolution
clear all; clf
Ts = 0.01; Tend = 2; t = 0:Ts:Tend;
x = 10*cos(20*pi*t + pi*(rand(1,1) - 0.5)); % input signal
h = 20*exp(-10.*t).*cos(40*pi*t); % impulse response
y = Ts*conv(x, h);
```
4.9 Response of LTI Systems to Periodic Signals

\[
M = \text{length}(x);
\]

\[
\text{figure}(1)
\]

\[
x1 = [\text{zeros}(1, 5) \ x(1:M)];
\]

\[
z = y(1); \ y1 = [\text{zeros}(1, 5) \ z \ \text{zeros}(1, M - 1)];
\]

\[
t0 = -5 \ast \text{Ts}:\text{Ts}:\text{Tend};
\]

\[
\text{for} \ k = 0:M - 6,
\]

\[
\text{pause}(0.05)
\]

\[
h0 = \text{fliplr}(h);
\]

\[
h1 = [h0(M - k - 5:M) \ \text{zeros}(1, M - k - 1)];
\]

\[
\text{subplot}(211)
\]

\[
\text{plot}(t0, h1, 'r')
\]

\[
\text{hold on}
\]

\[
\text{plot}(t0, x1, 'k')
\]

\[
\text{title('Convolution of x(t) and h(t)')}
\]

\[
\text{ylabel('x(\tau), h(t-\tau)')}; \ \text{grid}; \ \text{axis([min(t0) max(t0) 1.1*min(x) 1.1*max(x)])}
\]

\[
\text{hold off}
\]

\[
\text{subplot}(212)
\]

\[
\text{plot}(t0, y1, 'b')
\]

\[
\text{ylabel('y(t) = (x \ast h)(t)'); \ grid; \ axis([min(t0) max(t0) 0.1*min(x) 0.1*max(x)])}
\]

\[
\text{z} = [z \ y(k + 2)];
\]

\[
y1 = [\text{zeros}(1, 5) \ z \ \text{zeros}(1, M - \text{length}(z))];
\]

\[
\text{end}
\]

Figure 4.14 displays the last step of the convolution integral simulation. Notice that the steady state is attained in a very short time (around \( t = 0.5 \text{ sec} \)). The transient changes every time that the script is executed due to the random phase.

\textbf{FIGURE 4.14}

Convolution simulation: (a) input \( x(t) \) (solid line) and \( h(t-\tau) \) (dashed line), and (b) output \( y(t) \): transient and steady-state response.
If the input $x(t)$ of a causal and stable LTI system, with impulse response $h(t)$, is periodic of period $T_0$ and has the Fourier series

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k) \quad \Omega_0 = \frac{2\pi}{T_0} \quad (4.34)$$

the steady-state response of the system is

$$y(t) = X_0|H(j0)| \cos(\angle H(j0)) + 2 \sum_{k=1}^{\infty} |X_k||H(jk\Omega_0)| \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0)) \quad (4.35)$$

where

$$H(jk\Omega_0) = \int_{0}^{\infty} h(\tau)e^{-j\Omega_0 \tau} \, d\tau \quad (4.36)$$

is the frequency response of the system at $k\Omega_0$.

**Remarks**

- If the input signal $x(t)$ is a combination of sinusoids of frequencies that are not harmonically related, the signal is not periodic, but the eigenfunction property still holds. For instance, if

$$x(t) = \sum_{k} A_k \cos(\Omega_k t + \theta_k)$$

and the frequency response of the LTI system is $H(j\Omega)$, the steady-state response is

$$y(t) = \sum_{k} A_k|H(j\Omega_k)| \cos(\Omega_k t + \theta_k + \angle H(j\Omega_k))$$

- It is important to realize that if the LTI system is represented by a differential equation and the input is a sinusoid, or combination of sinusoids, it is not necessary to use the Laplace transform to obtain the complete response and then let $t \to \infty$ to find the sinusoidal steady-state response. The Laplace transform is only needed to find the transfer function of the system, which can then be used in Equation (4.35) to find the sinusoidal steady state.

**4.9.2 Filtering of Periodic Signals**

According to Equation (4.35) if we know the frequency response of the system (Eq. 4.36), at the harmonic frequencies of the periodic input, $H(jk\Omega_0)$, we have that in the steady state the output of the system $y(t)$ is as follows:

- Periodic of the same period as the input.
- Its Fourier coefficients are those of the input $X_k$ multiplied by the frequency response at the harmonic frequencies, $H(jk\Omega_0)$. 
Example 4.13

To illustrate the filtering of a periodic signal, consider a zero-mean pulse train

\[ x(t) = \sum_{k=-\infty, \neq 0}^{\infty} \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t} \]

as the driving source of an RC circuit that realizes a low-pass filter (i.e., a system that tries to keep the low-frequency harmonics and get rid of the high-frequency harmonics of the input). The transfer function of the RC low-pass filter is

\[ H(s) = \frac{1}{1 + s/100} \]

Solution

The following script computes the frequency response of the filter at the harmonic frequencies \( H(jk\Omega_0) \) (see Figure 4.15).

```matlab
%%%%% Example 4.13
%%%%% Freq response of H(s)=1/(s/scale+1) -- low-pass filter
w0 = 2 * pi; % fundamental frequency of input
M = 20; k = 0:M - 1; w1 = k. * w0; % harmonic frequencies
H = 1./(1 + j * w1/100); Hm = abs(H); Ha = angle(H); % frequency response
subplot(211)
stem(w1, Hm, 'filled'); grid; ylabel('—H(j\omega)—')
axis([0 max(w1) 0 1.3])
subplot(212)
stem(w1, Ha, 'filled'); grid
axis([0 max(w1) -1 0])
ylabel('\angle H(j\omega)'); xlabel('w (rad/sec)')
```

The response due to the pulse train can be found by finding the response to each of its Fourier series components and adding them. Approximating \( x(t) \) using \( N = 20 \) harmonics by

\[ x_N(t) = \sum_{k=-20, \neq 0}^{20} \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t} \]

Then the output voltage across the capacitor is given in the steady state,

\[ y_{ss}(t) = \sum_{k=-20, \neq 0}^{20} H(j2k\pi) \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t} \]
Because the magnitude response of the low-pass filter changes very little in the range of frequencies of the input, the output signal is very much like the input (see Figure 4.15). The following script is used to find the response.

```matlab
% low-pass filtering
% FS coefficients of input
X(1) = 0; % mean value
for k = 2:M - 1,
    X(k) = sin((k - 1) * pi/2)/((k - 1) * pi/2);
end

% periodic signal
Ts = 0.001; t1 = 0:Ts:1 - Ts; L = length(t1);
x1 = [ones(1, L /4) zeros(1, L /2) ones(1, L /4)]; x1 = x1 - 0.5; x = [x1 x1];

% output of filter
for k = 2:M - 1,
    y = y + X(k) * Hm(k) * cos(w0 * (k - 1) * t + Ha(k));
    hold on
end
plot(t, y); axis([0 max(t) -.6 .6]); hold off
grid
```

![Graph showing the magnitude and phase response of the low-pass RC filter](image)

**FIGURE 4.15**
(a) Magnitude and phase response of the low-pass RC filter $H(s)$ at harmonic frequencies, and (b) response due to a train of pulses.
4.10 OTHER PROPERTIES OF THE FOURIER SERIES

In this section we present additional properties of the Fourier series that will help us with its computation and with our understanding of the relation between time and frequency. We are in particular interested in showing that even and odd signals have special representations, and that it is possible to find the Fourier series of the sum, product, derivative, and integral of periodic signals without the integration required by the definition of the series.

4.10.1 Reflection and Even and Odd Periodic Signals

If the Fourier series of \( x(t) \), periodic with fundamental frequency \( \Omega_0 \), is

\[
x(t) = \sum_k X_k e^{jk\Omega_0 t}
\]

then the one for its reflected version \( x(-t) \) is

\[
x(-t) = \sum_m X_m e^{-jm\Omega_0 t} = \sum_k X_{-k} e^{jk\Omega_0 t}
\]

so that the Fourier coefficients of \( x(-t) \) are \( X_{-k} \) (remember that \( m \) and \( k \) are just dummy variables). This can be used to simplify the computation of Fourier series of even and odd signals.

For an even signal \( x(t) \), we have that \( x(t) = x(-t) \), and as such \( X_k = X_{-k} \) and therefore \( x(t) \) is naturally represented in terms of cosines and a dc term. Indeed, its Fourier series is

\[
x(t) = X_0 + \sum_{k=-\infty}^{-1} X_k e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} X_k e^{jk\Omega_0 t}
\]

\[
= X_0 + \sum_{k=1}^{\infty} X_k [e^{jk\Omega_0 t} + e^{-jk\Omega_0 t}]
\]

\[
= X_0 + 2 \sum_{k=1}^{\infty} X_k \cos(k\Omega_0 t)
\]

indicating that \( X_k \) are real-valued. This is also seen from

\[
X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jk\Omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)[\cos(k\Omega_0 t) - j\sin(k\Omega_0)] dt
\]

\[
= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos(k\Omega_0 t) dt
\]

because \( x(t) \sin(k\Omega_0 t) \) is odd and their integral is zero. It will be similar for an odd function for which \( x(t) = -x(-t) \), or \( X_k = -X_{-k} \), in which case the Fourier series has a zero dc value and sine harmonics.
The $X_k$ are purely imaginary. Indeed, for an odd $x(t)$,

$$X_k = \frac{1}{T_0} \int_{-T_0}^{T_0} x(t) e^{-j k \Omega_0 t} dt = \frac{1}{T_0} \int_{-T_0}^{T_0} x(t) [\cos(k \Omega_0 t) - j \sin(k \Omega_0 t)] dt$$

$$= -j \frac{1}{T_0} \int_{-T_0}^{T_0} x(t) \sin(k \Omega_0 t) dt$$

since $x(t) \cos(k \Omega_0 t)$ is odd. The Fourier series of an odd function can thus be written as

$$x(t) = 2 \sum_{k=1}^{\infty} (jX_k) \sin(k \Omega_0 t)$$

(4.39)

According to the even and odd decomposition, any periodic signal $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ is the even and $x_o(t)$ is the odd component of $x(t)$. Finding the Fourier coefficients of $x_e(t)$, which will be real, and those of $x_o(t)$, which will be purely imaginary, we would then have $X_k = X_{ek} + X_{ok}$ since

$$x_e(t) = 0.5[x(t) + x(-t)] \quad \Rightarrow \quad X_{ek} = 0.5[X_k + X_{-k}]$$

$$x_o(t) = 0.5[x(t) - x(-t)] \quad \Rightarrow \quad X_{ok} = 0.5[X_k - X_{-k}]$$

(4.40)

Reflection: If the Fourier coefficients of a periodic signal $x(t)$ are $\{X_k\}$ then those of $x(-t)$, the time-reversed signal with the same period as $x(t)$, are $\{X_{-k}\}$.

Even periodic signal $x(t)$: Its Fourier coefficients $X_k$ are real, and its trigonometric Fourier series is

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} X_k \cos(k \Omega_0 t)$$

(4.41)

Odd periodic signal $x(t)$: Its Fourier coefficients $X_k$ are imaginary, and its trigonometric Fourier series is

$$x(t) = 2 \sum_{k=1}^{\infty} jX_k \sin(k \Omega_0 t)$$

(4.42)

For any periodic signal $x(t) = x_e(t) + x_o(t)$ where $x_e(t)$ and $x_o(t)$ are the even and odd component of $x(t)$, then

$$X_k = X_{ek} + X_{ok}$$

(4.43)

where $\{X_{ek}\}$ are the Fourier coefficients of $x_e(t)$ and $\{X_{ok}\}$ are the Fourier coefficients of $x_o(t)$.

Example 4.14

Consider the periodic signals $x(t)$ and $y(t)$ shown in Figure 4.16. Determine their Fourier coefficients by using the symmetry conditions and the even–odd decomposition.
4.10 Other Properties of the Fourier Series

Solution

The given signal \( x(t) \) is neither even nor odd, but the advance signal \( x(t + 0.5) \) is even with a period of \( T_0 = 2, \Omega_0 = \pi \). Then between \(-1\) and \(1\) the shifted period is

\[
x_1(t + 0.5) = 2[u(t + 0.5) - u(t - 0.5)]
\]

so that its Laplace transform is

\[
X_1(s)e^{0.5s} = \frac{2}{s}[e^{0.5s} - e^{-0.5s}]
\]

which gives the Fourier coefficients

\[
X_k = \frac{1}{2j\pi k} \left[ e^{j\pi k/2} - e^{-j\pi k/2} \right] e^{-j\pi k/2}
\]

\[
= \frac{1}{0.5\pi k} \sin(0.5\pi k) e^{-j\pi k/2}
\]

after replacing \( s \) by \( jk\Omega_0 = jk\pi \) and dividing by the period \( T_0 = 2 \). These coefficients are complex as corresponding to a signal that is neither even nor odd. The dc coefficient is \( X_0 = 1 \).

The given signal \( y(t) \) is neither even nor odd, and cannot be made even or odd by shifting. The even and odd components of a period of \( y(t) \) are shown in Figure 4.17. The even and odd components of a period \( y_1(t) \) between \(-1\) and \(1\) are

\[
y_{1e}(t) = [u(t + 1) - u(t - 1)] + [r(t + 1) - 2r(t) + r(t - 1)]
\]

rectangular pulse triangle

\[
y_{1o}(t) = t[u(t + 1) - u(t - 1)] = [(t + 1)u(t + 1) - u(t + 1)] - [(t - 1)u(t - 1) + u(t - 1)]
\]

\[
= r(t + 1) - r(t - 1) - u(t + 1) - u(t - 1)
\]
Thus, the mean value of \( y_e(t) \) is the area under \( y_1e(t) \) divided by 2 or 1.5, and for \( k \neq 0 \),

\[
Y_{ek} = \frac{1}{T_0} Y_{1e}(s) \bigg|_{s=jk\Omega_0} = \frac{1}{2} \left[ \frac{1}{s} (e^s - e^{-s}) + \frac{1}{s^2} (e^s - 2 + e^{-s}) \right]_{s=jk\pi} = \frac{\sin(k\pi)}{\pi k} + \frac{1 - \cos(k\pi)}{(k\pi)^2} = 0 + \frac{1 - \cos(k\pi)}{(k\pi)^2} = \frac{1 - (-1)^k}{(k\pi)^2}
\]

The mean value of \( y_o(t) \) is zero, and for \( k \neq 0 \),

\[
Y_{ok} = \frac{1}{T_0} Y_{1o}(s) \bigg|_{s=jk\Omega_0} = \frac{1}{2} \left[ \frac{e^s - e^{-s}}{s^2} - \frac{e^s + e^{-s}}{s} \right]_{s=jk\pi} = \frac{-j \sin(k\pi)}{(k\pi)^2} + j \frac{\cos(k\pi)}{k\pi} = 0 + j \frac{\cos(k\pi)}{k\pi} = j \frac{(-1)^k}{k\pi}
\]

Finally, the Fourier series coefficients of \( y(t) \) are

\[
Y_k = \begin{cases} 
Y_{e0} + Y_{o0} = 1.5 + 0 = 1.5 & k = 0 \\
Y_{ek} + Y_{ok} = (1 - (-1)^k)/(k\pi)^2 + j(-1)^k/(k\pi) & k \neq 0
\end{cases}
\]

### 4.10.2 Linearity of Fourier Series—Addition of Periodic Signals

- **Same fundamental frequency**: If \( x(t) \) and \( y(t) \) are periodic signals with the same fundamental frequency \( \Omega_0 \), then the Fourier series coefficients of \( z(t) = \alpha x(t) + \beta y(t) \) for constants \( \alpha \) and \( \beta \) are

\[
Z_k = \alpha X_k + \beta Y_k \quad (4.44)
\]

where \( X_k \) and \( Y_k \) are the Fourier coefficients of \( x(t) \) and \( y(t) \).

- **Different fundamental frequencies**: If \( x(t) \) is periodic of period \( T_1 \), and \( y(t) \) is periodic of period \( T_2 \) such that \( T_2/T_1 = N/M \), for nondivisible integers \( N \) and \( M \), then \( z(t) = \alpha x(t) + \beta y(t) \) is periodic of period
If \( x(t) \) and \( y(t) \) are periodic signals of the same period \( T_0 \), the Fourier coefficients of \( z(t) = \alpha x(t) + \beta y(t) \) (also periodic of period \( T_0 \)) are then \( Z_k = \alpha X_k + \beta Y_k \) where \( X_k \) and \( Y_k \) are the Fourier coefficients of \( x(t) \) and \( y(t) \), respectively.

In general, if \( x(t) \) is periodic of period \( T_1 \), and \( y(t) \) is periodic of period \( T_2 \), their sum \( z(t) = \alpha x(t) + \beta y(t) \) is periodic if the ratio \( T_2/T_1 \) is a rational number (i.e., \( T_2/T_1 = N/M \) for some nondivisible integers \( N \) and \( M \)). If so, the period of \( z(t) \) is \( T_0 = MT_2 = NT_1 \). The fundamental frequency of \( z(t) \) would be \( \Omega_0 = \Omega_1/N = \Omega_2/M \) for \( \Omega_1 \) the fundamental frequency of \( x(t) \) and \( \Omega_2 \) the fundamental frequency of \( y(t) \). The Fourier series of \( z(t) \) is then

\[
z(t) = \alpha x(t) + \beta y(t) = \alpha \sum_k X_k e^{i\Omega_1 kt} + \beta \sum_m Y_m e^{i\Omega_2 mt}
\]

\[
= \alpha \sum_k X_k e^{i\Omega_0 k/NT_1 t} + \beta \sum_m Y_m e^{i\Omega_0 mT_1 t}
\]

\[
= \alpha \sum_{n=0,\pm N,\pm 2N,...} X_n e^{i\Omega_0 nt} + \beta \sum_{\ell=0,\pm M,\pm 2M,...} Y_{\ell/M} e^{i\Omega_0 \ell t}
\]

Thus, the coefficients are

\[
Z_k = \alpha X_k/N + \beta Y_k/M
\]

for integers \( k \) such that \( k/N \) and \( k/M \) are integers.

**Example 4.15**

Consider the sum \( z(t) \) of a periodic signal \( x(t) \) of period \( T_1 = 2 \), with a periodic signal \( y(t) \) with period \( T_2 = 0.2 \). Find the Fourier coefficients \( Z_k \) of \( z(t) \) in terms of the Fourier coefficients \( X_k \) and \( Y_k \) of \( x(t) \) and \( y(t) \).

**Solution**

The ratio \( T_2/T_1 = 1/10 = N/M \) is rational, so \( z(t) \) is periodic of period \( T_0 = T_1 = 10T_2 = 2 \). The fundamental frequency of \( z(t) \) is \( \Omega_0 = \Omega_1 = \pi \), and \( \Omega_2 = 10\Omega_0 = 10\pi \) is the fundamental frequency of \( y(t) \). Thus, the Fourier coefficients of \( z(t) \) are

\[
Z_k = \begin{cases} 
X_k + Y_{k/10} & \text{when } k = 0, \pm 10, \pm 20, \ldots \\
X_k & \text{otherwise}
\end{cases}
\]
4.10.3 Multiplication of Periodic Signals

If \( x(t) \) and \( y(t) \) are periodic signals of same period \( T_0 \), then their product

\[
z(t) = x(t)y(t)
\]

is also periodic of period \( T_0 \), and with Fourier coefficients that are the convolution sum of the Fourier coefficients of \( x(t) \) and \( y(t) \):

\[
Z_k = \sum_{\ell} X_\ell Y_{k-\ell} \quad (4.47)
\]

If \( x(t) \) and \( y(t) \) are periodic with the same period \( T_0 \), then \( z(t) = x(t)y(t) \) is also periodic of period \( T_0 \), since \( z(t + kT_0) = x(t + kT_0)y(t + kT_0) = x(t)y(t) = z(t) \). Furthermore,

\[
x(t)y(t) = \sum_k X_k e^{j\Omega_0 t} \sum_\ell Y_\ell e^{j\ell\Omega_0 t} = \sum_k \sum_\ell X_k Y_\ell e^{j(k+\ell)\Omega_0 t}
\]

\[
= \sum_m \left[ \sum_k X_k Y_{m-k} \right] e^{jm\Omega_0 t} = z(t)
\]

where we let \( m = k + \ell \). The coefficients of the Fourier series of \( z(t) \) are then

\[
Z_m = \sum_k X_k Y_{m-k}
\]

or the convolution sum of the sequences \( X_k \) and \( Y_k \), to be formally defined in Chapter 8.

Example 4.16

Consider the train of rectangular pulses \( x(t) \) shown in Figure 4.4. Let \( z(t) = 0.25x^2(t) \). Use the Fourier series of \( z(t) \) to show that

\[
X_k = \alpha \sum_m X_m X_{k-m}
\]

for some constant \( \alpha \). Determine \( \alpha \).

Solution

The signal \( 0.5x(t) \) is a train of pulses of unit amplitude, so that \( z(t) = (0.5x(t))^2 = 0.5x(t) \). Thus, \( Z_k = 0.5X_k \), but also as a product of \( 0.5x(t) \) with itself we have that

\[
Z_k = \sum_m [0.5X_m][0.5X_{k-m}]
\]
and thus
\[ 0.5X_k = 0.25 \sum_m X_m X_{k-m} \implies X_k = \frac{1}{2} \sum_m X_m X_{k-m} \]
(4.48)

so that \( \alpha = 0.5 \).

The Fourier series of \( z(t) = 0.5x(t) \) according to the results in Example 4.5 is

\[ z(t) = 0.5x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/2)}{\pi k} e^{jk2\pi t} \]

If we define

\[ S(k) = 0.5X_k = \frac{\sin(k\pi/2)}{k\pi} \implies X_k = 2S(k) \]

we have from Equation (4.48) the interesting result

\[ S(k) = \sum_{m=-\infty}^{\infty} S(m)S(k-m) \]

or the convolution sum of the discrete sinc function \( S(k) \) with itself is \( S(k) \).

### 4.10.4 Derivatives and Integrals of Periodic Signals

- **Derivative**: The derivative \( dx(t)/dt \) of a periodic signal \( x(t) \), of period \( T_0 \), is periodic of the same period \( T_0 \). If \( \{X_k\} \) are the coefficients of the Fourier series of \( x(t) \), the Fourier coefficients of \( dx(t)/dt \) are

\[ jk\Omega_0 X_k \]

where \( \Omega_0 \) is the fundamental frequency of \( x(t) \).

- **Integral**: For a zero-mean, periodic signal \( y(t) \), of period \( T_0 \), the integral

\[ z(t) = \int_{-\infty}^{t} y(\tau)d\tau \]

is periodic of the same period as \( y(t) \), with Fourier coefficients

\[ Z_k = \frac{Y_k}{jk\Omega_0} \quad k \neq 0 \]

\[ Z_0 = -\sum_{m \neq 0} Y_m \frac{1}{jm\Omega_0} \quad \Omega_0 = \frac{2\pi}{T_0} \]

These properties come naturally from the Fourier series representation of the periodic signal. Once we find the Fourier series of a periodic signal, we can differentiate it or integrate it (only when the dc
value is zero). The derivative of a periodic signal is obtained by computing the derivative of each of the terms of its Fourier series—that is, if

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}$$

then

$$\frac{dx(t)}{dt} = \sum_k X_k \frac{de^{jk\Omega_0 t}}{dt} = \sum_k [jk\Omega_0 X_k] e^{jk\Omega_0 t}$$

indicating that if the Fourier coefficients of $x(t)$ are $X_k$, the Fourier coefficients of $dx(t)/dt$ are $jk\Omega_0 X_k$.

To obtain the integral property we assume $y(t)$ is a zero-mean signal so that its integral $z(t)$ is finite. If for some integer $M$, $MT_0 \leq t < (M + 1)T_0$, then

$$z(t) = \int_{-\infty}^{t} y(\tau) d\tau = \int_{-\infty}^{MT_0} y(\tau) d\tau + \int_{MT_0}^{t} y(\tau) d\tau$$

$$= 0 + \int_{MT_0}^{t} y(\tau) d\tau$$

Replacing $y(t)$ by its Fourier series gives

$$z(t) = \int_{MT_0}^{t} y(\tau) d\tau = \int_{MT_0}^{t} \sum_{k \neq 0} Y_k e^{jk\Omega_0 \tau} d\tau$$

$$= \sum_{k \neq 0} Y_k \int_{MT_0}^{t} e^{jk\Omega_0 \tau} d\tau = \sum_{k \neq 0} Y_k \frac{1}{jk\Omega_0} \left[ e^{jk\Omega_0 t} - 1 \right]$$

$$= -\sum_{k \neq 0} Y_k \frac{1}{jk\Omega_0} + \sum_{k \neq 0} Y_k \frac{1}{jk\Omega_0} e^{jk\Omega_0 t}$$

where the first term corresponds to the average $Z_0$ and $Z_k = Y_k/(jk\Omega_0)$, $k \neq 0$, are the rest of the Fourier coefficients of $z(t)$.

**Remarks** It should be now clear why the derivative of a periodic signal $x(t)$ enhances its higher harmonics. Indeed, the Fourier coefficients of the derivative $dx(t)/dt$ are those of $x(t)$, $X_k$, multiplied by $j\Omega_0 k$, which increases with $k$. Likewise, the integration of a zero-mean periodic signal $x(t)$ does the opposite—that is, it makes the signal smoother, as we multiply $X_k$ by decreasing terms $1/(jk\Omega_0)$ as $k$ increases.
**Example 4.17**

Let \( g(t) \) be the derivative of a triangular train of pulses \( f(t) \), of period \( T_0 = 1 \). The period of \( f(t) \), \( 0 \leq t \leq 1 \), is

\[
 f_1(t) = 2r(t) - 4r(t - 0.5) + 2r(t - 1)
\]

Use the Fourier series of \( g(t) \) to find the Fourier series of \( f(t) \).

**Solution**

According to the derivative property we have that

\[
 F_k = \frac{G_k}{jk\Omega_0} \quad k \neq 0
\]

are the Fourier coefficients of \( f(t) \). The signal \( g(t) = df(t)/dt \) has a corresponding period \( g_1(t) = df_1(t)/dt = 2u(t) - 4u(t - 0.5) + 2u(t - 1) \). The Fourier series coefficients of \( g(t) \) are

\[
 G_k = \frac{2e^{-0.5s}}{s} \left( e^{0.5s} - 2 + e^{-0.5s} \right) \bigg|_{s=j2\pi k} = 2(-1)^k \cos(\pi k) - 1 \quad j \neq 0
\]

which are used to obtain the coefficients \( F_k \) for \( k \neq 0 \). The dc component of \( f(t) \) is found to be 0.5 from its plot as \( g(t) \) does not provide it.

**Example 4.18**

Consider the reverse of Example 4.17. That is, given the periodic signal \( g(t) \) of period \( T_0 = 1 \) and Fourier coefficients

\[
 G_k = 2(-1)^k \cos(\pi k) - 1 \quad k \neq 0
\]

and \( G_0 = 0 \). Find the integral

\[
 z(t) = \int_{-\infty}^{t} g(\tau) d\tau
\]

**Solution**

As shown above, \( z(t) \) is also periodic of the same period as \( g(t) \) (i.e., \( T_0 = 1 \)). The Fourier coefficients of \( z(t) \) are

\[
 Z_k = \frac{G_k}{j\Omega_0 k} = (-1)^k \frac{4(\cos(\pi k) - 1)}{(j2\pi k)^2} = (-1)^{k+1} \frac{\cos(\pi k) - 1}{\pi^2 k^2} \quad k \neq 0
\]
and the average term is

\[ Z_0 = - \sum_{m \neq 0} G_m \frac{1}{j2m\pi} = \sum_{m \neq 0} (-1)^m \frac{\cos(\pi m) - 1}{(\pi m)^2} \]

\[ = 0.5 \sum_{m=-\infty, m \neq 0}^{\infty} (-1)^{m+1} \left[ \frac{\sin(\pi m/2)}{(\pi m/2)} \right]^2 \]

where we used \( 1 - \cos(\pi m) = 2 \sin^2(\pi m/2) \). We used the following script to obtain the average, and to approximate the triangular signal using 100 harmonics (see Figure 4.18). The mean is obtained as 0.498.

%%% % Example 4.18  %%%%
clear all
clf; w0 = 2 * pi; N = 100; % parameters of periodic signal
% computation of mean value
DC = 0;
for m = 1:N,
    DC = DC + 2 * (-1)ˆ(m) * (cos(pi * m) - 1)/(pi * m)^2;
end
% computation of Fourier series coefficients
Ts = 0.001; t = 0:Ts:2 - Ts;
for k = 1:N,
    X(k) = (-1)ˆ(k+1) * (cos(pi * k) - 1)/(pi * k)^2;
end
X = [DC X]; % Fourier series coefficients
xa = X(1)*ones(1,length(t));
figure(1)
for $k = 2:N,$
\[
    x_{a} = x_{a} + 2 \ast \text{abs}(X(k)) \ast \cos(w0 \ast (k - 1) \ast t + \text{angle}(X(k))); \ % \text{approximate signal}
\]
end

4.11 WHAT HAVE WE ACCOMPLISHED? WHERE DO WE GO FROM HERE?

Periodic signals are not to be found in practice, so where did Fourier get the intuition to come up with a representation for them? As you will see, the fact that periodic signals are not found in practice does not mean that they are not useful. The Fourier representation of periodic signals will be fundamental in finding a representation for nonperiodic signals.

A very important concept you have learned in this chapter is that the inverse relation between time and frequency provides complementary information for the signal. The frequency domain constitutes the other side of the coin in representing signals. As mentioned before, it is the eigenfunction property of linear time-invariant systems that holds the theory together. It will provide the fundamental principle for filtering. You should have started to experience *déjà vu* in terms of the properties of the Fourier series; some look like a version of the ones in the Laplace transform. This is due to the connection existing between these transforms. You should have also noticed the usefulness of the Laplace transform in finding the Fourier coefficients, avoiding integration whenever possible. Table 4.1 provides the basic properties of the Fourier series for continuous–time periodic signals.

Chapter 5 will extend some of the results obtained in this chapter, thus unifying the treatment of periodic and nonperiodic signals and the concept of spectrum. Also the frequency representation of

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systems will be introduced and exemplified by its application in filtering. Modulation is the basic tool in communications and can be easily explained in the frequency domain.

**PROBLEMS**

4.1. Eigenfunctions and LTI systems

The eigenfunction property is only valid for LTI systems. Consider the cases of nonlinear and of time-varying systems.

(a) A system represented by the following input–output equation is nonlinear:

\[ y(t) = x^2(t) \]

Let \( x(t) = e^{j\pi t/4} \). Find the corresponding system output \( y(t) \). Does the eigenfunction property hold? Explain.

(b) Consider a time-varying system

\[ y(t) = x(t)[u(t) - u(t - 1)] \]

Let \( x(t) = e^{j\pi t/4} \). Find the corresponding system output \( y(t) \). Does the eigenfunction property hold? Explain.

4.2. Eigenfunctions and LTI systems

The output of an LTI system is

\[ y(t) = \int_0^t h(\tau)x(t - \tau)d\tau \]

where the input \( x(t) \) and the impulse response \( h(t) \) of the system are assumed to be causal. Let \( x(t) = 2 \cos(2\pi t)u(t) \). Compute the output \( y(t) \) in the steady state and determine if the eigenfunction property holds.

4.3. Eigenfunctions and frequency response of LTI systems

The input–output equation for an analog averager is

\[ y(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)d\tau \]

Let \( x(t) = e^{j\Omega_0 t} \). Since the system is LTI, then the output should be

\[ y(t) = e^{j\Omega_0 t}H(j\Omega_0) \]

(a) Find \( y(t) \) for the given input and then compare it with the above equation to find \( H(j\Omega_0) \), the response of the averager at frequency \( \Omega_0 \).

(b) Find \( H(s) \) and verify the frequency response value \( H(j\Omega_0) \) obtained above.

4.4. Generality of eigenfunctions

The eigenfunction property holds for any input signal, periodic or not, that can be expressed in sinusoidal form.

(a) Consider the input \( x(t) = \cos(t) + \cos(2\pi t), -\infty < t < \infty \), into an LTI system. Is \( x(t) \) periodic? If so, indicate its period.
(b) Suppose that the system is represented by a first-order differential equation,

\[ y'(t) + 5y(t) = x(t) \]

where \( y(t) \) is the output of the system and the given \( x(t) \) is the input of the system. Find the steady-state response \( y(t) \) due to \( x(t) \) using the eigenfunction property.

### 4.5. Steady state of LTI systems

The transfer function of an LTI system is

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{s + 1}{s^2 + 3s + 2} \]

If the input to this system is \( x(t) = 1 + \cos(t + \pi/4) \), \(-\infty < t < \infty\), what is the output \( y(t) \) in the steady state?

### 4.6. Eigenfunction property of LTI systems and Laplace

The transfer function of an LTI system is given by

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2} \]

and its input is

\[ x(t) = 4u(t) \]

(a) Use the eigenfunction property of LTI systems to find the steady-state response \( y(t) \) of this system.

(b) Verify your result in (a) by means of the Laplace transform.

### 4.7. Different ways to compute the Fourier coefficients—MATLAB

We would like to find the Fourier series of a sawtooth periodic signal \( x(t) \) of period \( T_0 = 1 \). The period of \( x(t) \) is

\[ x_1(t) = r(t)[u(t) - u(t - 1)] \]

(a) Carefully plot \( x(t) \) and compute the Fourier coefficients \( X_k \) using the integral definition.

(b) An easier way to do this is to use the Laplace transform of \( x_1(t) \). Find \( X_k \) this way.

(c) Use MATLAB to plot the signal \( x(t) \) and its magnitude and phase line spectra.

(d) Obtain a trigonometric Fourier series \( \hat{x}(t) \) consisting of the DC term and 40 harmonics to approximate \( x(t) \). Use MATLAB to find the values of \( \hat{x}(t) \) for \( t = 0 \) to 10 in steps of 0.001. How does it compare with \( x(t) \)?

### 4.8. Addition of periodic signals—MATLAB

Consider a sawtooth signal \( x(t) \) with period \( T_0 = 2 \) and period

\[ x_1(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \]

(a) Find the Fourier coefficients \( X_k \) using the Laplace transform. Consider the cases when \( k \) is odd and even \((k \neq 0)\). You need to compute \( X_0 \) directly from the signal.

(b) Let \( y(t) = x(-t) \). Find the Fourier coefficients \( Y_k \).

(c) The sum \( z(t) = x(t) + y(t) \) is a triangular function. Find the Fourier coefficients \( Z_k \) and compare them to \( X_k + Y_k \).

(d) Use MATLAB to plot \( x(t) \), \( y(t) \), and \( z(t) \) and their corresponding magnitude line spectra. Find an approximate of \( z(t) \) using the dc value and 10 harmonics and plot it.
4.9. Fourier series coefficients via Laplace—MATLAB
The computation of the Fourier series coefficients is simplified by the relation between the formula for these coefficients and the Laplace transform of a period of the periodic signal.

(a) A periodic signal \(x(t)\), of period \(T_0 = 2\) sec, has as period with the signal \(x_1(t) = u(t) - u(t - 1)\), so that \(x(t)\) can be represented as

\[
x(t) = \sum_{m=-\infty}^{\infty} x_1(t - mT_0)
\]

Expand this sum, and use the information for \(x_1(t)\) and \(T_0\) to carefully plot the periodic signal \(x(t)\).

(b) Find the Laplace transform of \(x_1(t)\), and let \(s = jk\Omega_0\), where \(\Omega_0 = 2\pi/T_0\) is the fundamental frequency, to obtain the Fourier coefficients of \(x(t)\).

(c) Use MATLAB to plot the magnitude line spectrum of \(x(t)\). Find an approximate of \(x(t)\) using the dc and 40 harmonics. Plot it.

4.10. Half- and full-wave rectifying and Fourier—MATLAB
Rectifying a sinusoid provides a way to create a dc source. In this problem we consider the Fourier series of the full- and half-wave rectified signals. The full-wave rectified signal \(x_f(t)\) has a period \(T_0 = 1\) and its period from 0 to 1 is

\[
x_1(t) = \sin(\pi t) \quad 0 \leq t \leq 1
\]

while the period for the half-wave rectifier signal \(x_h(t)\) is

\[
x_2(t) = \begin{cases} 
\sin(\pi t) & 0 \leq t \leq 1 \\
0 & 1 < t \leq 2
\end{cases}
\]

with period \(T_1 = 2\).

(a) Obtain the Fourier coefficients for both of these periodic signals.

(b) Use the even and odd decomposition of \(x_f(t)\) to obtain its Fourier coefficients. This computation of the Fourier coefficients of \(x_f(t)\) avoids some difficulties when you attempt to plot its magnitude line spectrum. Use MATLAB and your analytic results here to plot the magnitude line spectrum of the half-wave signal and use the dc and 40 harmonics to obtain an approximation of the half-wave signal.

4.11. Smoothness and Fourier series—MATLAB
The smoothness of a period determines the way the magnitude line spectrum decays. Consider the following periodic signals \(x(t)\) and \(y(t)\), both of period \(T_0 = 2\) sec, and with a period from 0 to \(T_0\) equal to

\[
x_1(t) = u(t) - u(t - 1) \\
y_1(t) = r(t) - 2r(t - 1) + r(t - 2)
\]

Find the Fourier series coefficients of \(x(t)\) and \(y(t)\) and use MATLAB to plot their magnitude line spectrum for \(k = 0, \pm 1, \pm 2, \ldots, \pm 20\). Determine which of these spectra decays faster and how it relates to the smoothness of the period. (To see this relate \(|X_k|\) to the corresponding \(|Y_k|\).)

4.12. Time support and frequency content—MATLAB
The support of a period of a periodic signal relates inversely to the support of the line spectrum. Consider two periodic signals: \(x(t)\) of period \(T_0 = 2\) and \(y(t)\) of period \(T_1 = 1\), and with periods

\[
x_1(t) = u(t) - u(t - 1) \quad 0 \leq t \leq 2 \\
y_1(t) = u(t) - u(t - 0.5) \quad 0 \leq t \leq 1
\]
(a) Find the Fourier series coefficients for \(x(t)\) and \(y(t)\).

(b) Use MATLAB to plot the magnitude line spectra of the two signals from 0 to 40\(\pi\) rad/sec. Plot them on the same figure so you can determine which has a broader support. Indicate which signal is smoother and explain how it relates to its line spectrum.

4.13. Derivatives and Fourier Series

Given the Fourier series representation for a periodic signal,

\[x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_0 t}\]

we can compute derivatives of it, just like for any other signal.

(a) Consider the periodic train of pulses shown in Figure 4.19. Compute its derivative

\[y(t) = \frac{dx(t)}{dt}\]

and carefully plot it. Find the Fourier series of \(y(t)\).

(b) Use the Fourier series representation of \(x(t)\) and find its derivative to obtain the Fourier series of \(y(t)\). How does it compare to the Fourier series obtained above?

4.14. Fourier series of sampling delta

The periodic signal

\[\delta_{T_s}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s)\]

will be very useful in the sampling of continuous-time signals.

(a) Find the Fourier series of this signal—that is,

\[\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \Delta_k e^{jk\Omega_s t}\]

find the Fourier coefficients \(\Delta_k\).

(b) Plot the magnitude line spectrum of this signal.

(c) Plot \(\delta_{T_s}(t)\) and its corresponding line spectrum \(\Delta_k\) as functions of time and frequency. Are they both periodic? How are their periods related? Explain.

**FIGURE 4.19**

Problem 4.13: train of rectangular pulses.
4.15. **Figuring out Fourier’s idea**

Fourier proposed to represent a periodic signal as a sum of sinusoids, perhaps an infinite number of them. For instance, consider the representation of a periodic signal \( x(t) \) as a sum of cosines of different frequencies

\[
x(t) = \sum_{k=0}^{\infty} A_k \cos(\Omega_k t + \theta_k)
\]

(a) If \( x(t) \) is periodic of period \( T_0 \), what should the frequencies \( \Omega_k \) be?
(b) Suppose \( x(t) = 2 + \cos(2\pi t) - 3 \cos(6\pi t + \pi/4) \). Is this signal periodic? If so, what is its period \( T_0 \)? Determine its trigonometric Fourier series as given above by specifying the values of \( A_k \) and \( \theta_k \) for all values of \( k = 0, 1, \ldots \).
(c) Let the signal \( x_1(t) = 2 + \cos(2\pi t) - 3 \cos(20t + \pi/4) \) (this signal is almost like \( x(t) \) given above, except that the frequency \( 6\pi \) rad/sec of the second cosine has been approximated by \( 20 \) rad/sec). Is this signal periodic? Can you determine its Fourier series as given above by specifying the values of \( A_k \) and \( \theta_k \) for all values of \( k = 0, 1, \ldots \)? Explain.

4.16. **DC output from a full-wave rectified signal—MATLAB**

Consider a full-wave rectifier that has as output a periodic signal \( x(t) \) of period \( T_0 = 1 \) and a period of it is given as

\[
x_1(t) = \begin{cases} 
\cos(\pi t) & -0.5 \leq t \leq 0.5 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Obtain the Fourier coefficients \( X_k \).
(b) Suppose we pass \( x(t) \) through an ideal filter of transfer function \( H(s) \). Determine the values of this filter at harmonic frequencies \( \pm 2\pi k = 0, \pm 1, \pm 2, \ldots \), so that its output is a constant (i.e., we have a dc source).
(c) Use MATLAB to plot the signal \( x(t) \) and its magnitude line spectrum.

4.17. **Fourier series of sum of periodic signals**

Suppose you have the Fourier series of two periodic signals \( x(t) \) and \( y(t) \) of periods \( T_1 \) and \( T_2 \), respectively. Let \( X_k \) and \( Y_k \) be the Fourier series coefficients corresponding to \( x(t) \) and \( y(t) \).
(a) If \( T_1 = T_2 \), what would be the Fourier series coefficients of \( z(t) = x(t) + y(t) \) in terms of \( X_k \) and \( Y_k \)?
(b) If \( T_1 = 2T_2 \), determine the Fourier series coefficients of \( w(t) = x(t) + y(t) \) in terms of \( X_k \) and \( Y_k \).

4.18. **Manipulation of periodic signals**

Let the following be the Fourier series of a periodic signal \( x(t) \) of period \( T_0 \) (fundamental frequency \( \Omega_0 = 2\pi/T_0 \)):

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_0 kt}
\]

Consider the following functions of \( x(t) \), and determine if they are periodic and what are their periods if so:
- \( y(t) = 2x(t) - 3 \)
- \( z(t) = x(t - 2) + x(t) \)
- \( w(t) = x(2t) \)
Express the Fourier series coefficients \( Y_k, Z_k, \) and \( W_k \) in terms of \( X_k \).
4.19. Using properties to find the Fourier series

Use the Fourier series of a square train of pulses (done in this chapter) to compute the Fourier series of the triangular signal \( x(t) \) with a period,

\[
x_1(t) = r(t) - 2r(t - 1) + r(t - 2)
\]

(a) Find the derivative of \( x(t) \) or \( y(t) = dx(t)/dt \) and carefully plot it. Plot also \( z(t) = y(t) + 1 \). Use the Fourier series of the square train of pulses to compute the Fourier series coefficients of \( y(t) \) and \( z(t) \).

(b) Obtain the trigonometric Fourier series of \( y(t) \) and \( z(t) \) and explain why they are represented by sines and why \( z(t) \) has a nonzero mean.

(c) Obtain the Fourier series coefficients of \( x(t) \) from those of \( y(t) \).

(d) Obtain the sinusoidal form of \( x(t) \) and explain why the cosine representation is more appropriate for this signal than a sine representation.

4.20. Applying Parseval’s result—MATLAB

We wish to approximate the triangular signal \( x(t) \) in Problem 4.19 by its Fourier series with a finite number of terms, let’s say \( 2N \). This approximation should have 95% of the average power of the triangular signal. Use MATLAB to find the value of \( N \).

4.21. Fourier series of multiplication of periodic signals

Consider the Fourier series of two periodic signals,

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_0 kt}
\]

\[
y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j\Omega_1 kt}
\]

(a) Let \( \Omega_1 = \Omega_0 \). Is \( z(t) = x(t)y(t) \) periodic? If so, what is its period and its Fourier series coefficients?

(b) If \( \Omega_1 = 2\Omega_0 \). Is \( w(t) = x(t)y(t) \) periodic? If so, what is its period and its Fourier series coefficients?

4.22. Integration of periodic signals

Consider now the integral of the Fourier series of the pulse signal \( p(t) = x(t) - 1 \) of period \( T_0 = 1 \), where \( x(t) \) is given in Figure 4.20.

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**FIGURE 4.20**

Problem 4.23: train of rectangular pulses.
(a) Given that an integral of \( p(t) \) is the area under the curve, find and plot the function

\[
s(t) = \int_{-\infty}^{t} p(t) \, dt \quad t \leq 1
\]

Indicate the values of \( s(t) \) for \( t = 0, 0.25, 0.5, 0.75, \) and \( 1 \).

(b) Find the Fourier series of \( p(t) \) and \( s(t) \) and relate their Fourier series coefficients.

(c) Suppose you want to compute the integral

\[
\frac{T_0}{2} \int_{-T_0/2}^{T_0/2} p(t) \, dt
\]

using the Fourier series of \( p(t) \). What is the integral equal to?

(d) You can also compute the integral from the plot of \( p(t) \):

\[
\frac{T_0}{2} \int_{-T_0/2}^{T_0/2} p(t) \, dt
\]

What is it? Does it coincide with the result obtained using the Fourier series? Explain.

4.23. Full-wave rectifying and DC sources

Let \( x(t) = \sin^2(2\pi t) \), a periodic signal of period \( T_0 = 1 \), and \( y(t) = |\sin(2\pi t)| \), which is also periodic of period \( T_1 = 0.5 \).

(a) A trigonometric identity gives that

\[
x(t) = \frac{1}{2} [1 - \cos(4\pi t)]
\]

Use this result to find its complex exponential Fourier series.

(b) Use the Laplace transform to find the Fourier series of \( y(t) \).

(c) Are \( x(t) \) and \( y(t) \) identical? Explain.

(d) Indicate how you would use an ideal low-pass filter to get a DC source of unit value from \( x(t) \) and \( y(t) \). Indicate the bandwidth and the magnitude of the filters. Compare these two signals in terms of advantages or disadvantages in generating the desired DC source.

4.24. Windowing and music sounds—MATLAB

In the computer generation of musical sounds, pure tones need to be windowed to make them more interesting. Windowing mimics the way a musician would approach the generation of a certain sound. Increasing the richness of the harmonic frequencies is the result of the windowing, as we will see in this problem. Consider the generation of a musical note with frequencies around \( f_A = 880 \) Hz. Assume our “musician” while playing this note uses three strokes corresponding to a window \( w_1(t) = r(t) - r(t - T_1) - r(t - T_2) + r(t - T_0) \), so that the resulting sound would be the multiplication, or windowing, of a pure sinusoid \( \cos(2\pi f_A t) \) by a periodic signal \( w(t) \), with \( w_1(t) \) a period that repeats every \( T_0 = 5T \) where \( T \) is the period of the sinusoid. Let \( T_1 = T_0/4 \) and \( T_2 = 3T_0/4 \).

(a) Analytically determine the Fourier series of the window \( w(t) \) and plot its line spectrum using MATLAB.

(b) Use the modulation or the convolution properties of the Fourier series to obtain the coefficients of the product \( s(t) = \cos(2\pi f_A t)w(t) \). Use MATLAB to plot the line spectrum of this periodic signal and again determine how many harmonic frequencies you would need to obtain a good approximation to \( s(t) \).
4.26. Square error approximation of periodic signals—MATLAB

(c) The line spectrum of the pure tone \( p(t) = \cos(2\pi f_A t) \) only displays one harmonic, the one corresponding to the \( f_A = 880 \text{ Hz} \) frequency. How many more harmonics does \( s(t) \) have? To listen to the richness in harmonics use the MATLAB function sound to play the sinusoid \( p(t) \) and \( s(t) \) (use \( F_s = 2 \times 880 \text{ Hz} \) to play both).

(d) Consider a combination of notes in a certain scale; for instance, let

\[
p(t) = \sin(2\pi \times 440t) + \sin(2\pi \times 550t) + \sin(2\pi \times 660t)
\]

Use the same windowing \( w(t) \), and let \( s(t) = p(t)w(t) \). Use MATLAB to plot \( p(t) \) and \( s(t) \) and to compute and plot their corresponding line spectra. Use sound to play \( p(nT_s) \) and \( s(nT_s) \) using \( F_s = 1000 \).

4.25. Computation of \( \pi \)—MATLAB

As you know, \( \pi \) is an irrational number that can only be approximated by a number with a finite number of decimals. How to compute this value recursively is a problem of theoretical interest. In this problem we show that the Fourier series can provide that formulation.

(a) Consider a train of rectangular pulses \( x(t) \), with a period

\[
x_1(t) = 2[u(t + 0.25) - u(t - 0.25)] - 1 \quad -0.5 \leq t \leq 0.5
\]

and period \( T_0 = 1 \). Plot the periodic signal and find its trigonometric Fourier series.

(b) Use the above Fourier series to find an infinite sum for \( \pi \).

(c) If \( \pi_N \) is an approximation of the infinite sum with \( N \) coefficients, and \( \pi \) is the value given by MATLAB, find the value of \( N \) so that \( \pi_N \) is 95% of the value of \( \pi \) given by MATLAB.

4.26. Square error approximation of periodic signals—MATLAB

To understand the Fourier series consider a more general problem, where a periodic signal \( x(t) \), of period \( T_0 \), is approximated as a finite sum of terms,

\[
\hat{x}(t) = \sum_{k=-N}^{N} \hat{X}_k \phi_k(t)
\]

where \( \{\phi_k(t)\} \) are orthonormal functions. To pose the problem as an optimization problem, consider the square error

\[
\epsilon = \int_{T_0} |x(t) - \hat{x}(t)|^2 dt
\]

and look for the coefficients \( \{ \hat{X}(k) \} \) that minimize \( \epsilon \).

(a) Assume that \( x(t) \) as well as \( \hat{x}(t) \) are real valued, and that \( x(t) \) is even so that the Fourier series coefficients \( X_k \) are real. Show that the error can be expressed as

\[
\epsilon = \int_{T_0} x^2(t) dt - 2 \sum_{k=-N}^{N} \hat{X}_k \int_{T_0} x(t) \phi_k(t) dt + \sum_{\ell=-N}^{N} |\hat{X}_\ell|^2 T_0
\]

(b) Compute the derivative of \( \epsilon \) with respect to \( \hat{X}_n \) and set it to zero to minimize the error. Find \( \hat{X}_n \).

(c) In the Fourier series the \( \{\phi_k(t)\} \) are the complex exponentials and the \( \{\hat{X}_n\} \) coincide with the Fourier series coefficients. To illustrate the above procedure consider the case of the pulse signal \( x(t) \), of period \( T_0 = 1 \), and a period

\[
x_1(t) = 2[u(t + 0.25) - u(t - 0.25)]
\]
Use MATLAB to compute and plot the approximation \( \hat{x}(t) \) and the error \( \epsilon \) for increasing values of \( N \) from 1 to 100.

(d) Concentrate your plot of \( \hat{x}(t) \) around the one of the discontinuities, and observe the Gibb’s phenomenon. Does it disappear when \( N \) is very large. Plot \( \hat{x}(t) \) around the discontinuity for \( N = 1000 \).

4.27. Walsh functions—MATLAB

As seen in Problem 4.26, the Fourier series is one of a possible class of representations in terms of orthonormal functions. Consider the case of the Walsh functions, which are a set of rectangular pulse signals that are orthonormal in a finite time interval \([0, 1]\). These functions are such that: (1) they take only 1 and \(-1\) values, (2) \( \phi_k(t) = 1 \) for all \( k \), and (3) they are ordered according to the number of sign changes.

(a) Consider obtaining the functions \( \{\phi_k\}_{k=0,\ldots,5} \). The Walsh functions are clearly normal since when squared they are unity for \( t \in [0, 1] \). Let \( \phi_0(t) = 1 \) for \( t \in [0, 1] \) and zero elsewhere. Obtain \( \phi_1(t) \) with one change of sign and that is orthogonal to \( \phi_0(t) \). Find then \( \phi_2(t) \), which has two changes of sign and is orthogonal to both \( \phi_0(t) \) and \( \phi_1(t) \). Continue this process. Carefully plot the \( \{\phi_i(t)\}, i = 0, \ldots, 5 \). Use the MATLAB function \texttt{stairs} to plot these Walsh functions.

(b) Consider the Walsh functions obtained above as sequences of 1s and \(-1\)s of length 8, and carefully write these six sequences. Observe the symmetry of the sequences corresponding to \( \{\phi_i(t), i = 0, 1, 3, 5\} \), and determine the circular shift needed to find the sequence corresponding to \( \phi_2(t) \) from the sequence from \( \phi_1(t) \), and \( \phi_4(t) \) from \( \phi_3(t) \). Write a MATLAB script that generates a matrix \( \Phi \) with entries as the sequences. Find the product \((1/8)\Phi\Phi^T\), and explain how this result connects with the orthonormality of the Walsh functions.

(c) We wish to approximate a ramp function \( x(t) = r(t), \ 0 \leq t \leq 1 \), using \( \{\phi_k\}_{k=0,\ldots,5} \). This could be written as

\[
\mathbf{r} = \Phi \mathbf{a}
\]

where \( \mathbf{r} \) is a vector of \( x(nT) = r(nT) \) where \( T = 1/8 \), \( \mathbf{a} \) are the coefficients of the expansion, and \( \Phi \) is the Walsh matrix found above. Determine the vector \( \mathbf{a} \) and use it to obtain an approximation of \( x(t) \). Plot \( x(t) \) and the approximation \( \hat{x}(t) \) (use \texttt{stairs} for this signal).