Upon completion of this chapter, you will be able to

- Graphically solve any optimization problem having two design variables
- Plot constraints and identify their feasible/infeasible side
- Identify the feasible region (feasible set) for a problem
- Plot objective function contours through the feasible region
- Graphically locate the optimum solution for a problem and identify active/inactive constraints
- Identify problems that may have multiple, unbounded, or infeasible solutions
- Explain basic concepts and terms associated with optimum design

Optimization problems having only two design variables can be solved by observing how they are graphically represented. All constraint functions are plotted, and a set of feasible designs (the feasible set) for the problem is identified. Objective function contours are then drawn, and the optimum design is determined by visual inspection. In this chapter, we illustrate the graphical solution process and introduce several concepts related to optimum design problems. In Section 3.1, a design optimization problem is formulated and used to describe the solution process. Several more example problems are solved in later sections to illustrate concepts and the procedure.

### 3.1 GRAPHICAL SOLUTION PROCESS

#### 3.1.1 Profit Maximization Problem

**STEP 1: PROJECT/PROBLEM DESCRIPTION**  
A company manufactures two machines, A and B. Using available resources, either 28 A or 14 B can be manufactured daily. The sales
department can sell up to 14 A machines or 24 B machines. The shipping facility can handle no more than 16 machines per day. The company makes a profit of $400 on each A machine and $600 on each B machine. How many A and B machines should the company manufacture every day to maximize its profit?

**STEP 2: DATA AND INFORMATION COLLECTION** Data and information are defined in the project statement.

**STEP 3: DEFINITION OF DESIGN VARIABLES** The following two design variables are identified in the problem statement:

\[ x_1 = \text{number of A machines manufactured each day} \]
\[ x_2 = \text{number of B machines manufactured each day} \]

**STEP 4: OPTIMIZATION CRITERION** The objective is to maximize daily profit, which can be expressed in terms of design variables as

\[ P = 400x_1 + 600x_2 \] (a)

**STEP 5: FORMULATION OF CONSTRAINTS** Design constraints are placed on manufacturing capacity, on sales personnel, and on the shipping and handling facility. The constraint on the shipping and handling facility is quite straightforward:

\[ x_1 + x_2 \leq 16 \] (shipping and handling constraint) (b)

Constraints on manufacturing and sales facilities are a bit tricky. First, consider the manufacturing limitation. It is assumed that if the company is manufacturing \( x_1 \) A machines per day, then the remaining resources and equipment can be proportionately used to manufacture \( x_2 \) B machines, and vice versa. Therefore, noting that \( x_1/28 \) is the fraction of resources used to produce A and \( x_2/14 \) is the fraction used to produce B, the constraint is expressed as

\[ \frac{x_1}{28} + \frac{x_2}{14} \leq 1 \] (manufacturing constraint) (c)

Similarly, the constraint on sales department resources is given as

\[ \frac{x_1}{14} + \frac{x_2}{24} \leq 1 \] (limitation on sales department) (d)

Finally, the design variables must be non-negative as

\[ x_1, x_2 \geq 0 \] (e)

Note that for this problem, the formulation remains valid even when a design variable has zero value. The problem has two design variables and five inequality constraints. All functions of the problem are linear in variables \( x_1 \) and \( x_2 \). Therefore, it is a linear programming problem. Note also that for a meaningful solution, both design variables must have integer values at the optimum point.
3.1.2 Step-by-Step Graphical Solution Procedure

**STEP 1: COORDINATE SYSTEM SET-UP** The first step in the solution process is to set up an origin for the \(x\)-\(y\) coordinate system and scales along the \(x\)- and \(y\)-axes. By looking at the constraint functions, a coordinate system for the profit maximization problem can be set up using a range of 0 to 25 along both the \(x\) and \(y\) axes. In some cases, the scale may need to be adjusted after the problem has been graphed because the original scale may provide too small or too large a graph for the problem.

**STEP 2: INEQUALITY CONSTRAINT BOUNDARY PLOT** To illustrate the graphing of a constraint, let us consider the inequality \(x_1 + x_2 \leq 16\) given in Eq. (b). To represent the constraint graphically, we first need to plot the constraint boundary; that is, the points that satisfy the constraint as an equality \(x_1 + x_2 = 16\). This is a linear function of the variables \(x_1\) and \(x_2\). To plot such a function, we need two points that satisfy the equation \(x_1 + x_2 = 16\). Let these points be calculated as \((16,0)\) and \((0,16)\). Locating them on the graph and joining them by a straight line produces the line \(F \rightarrow J\), as shown in Figure 3.1. Line \(F \rightarrow J\) then represents the boundary of the feasible region for the inequality constraint \(x_1 + x_2 \leq 16\). Points on one side of this line violate the constraint, while those on the other side satisfy it.

**STEP 3: IDENTIFICATION OF THE FEASIBLE REGION FOR AN INEQUALITY** The next task is to determine which side of constraint boundary \(F \rightarrow J\) is feasible for the constraint \(x_1 + x_2 \leq 16\). To accomplish this, we select a point on either side of \(F \rightarrow J\) and evaluate the constraint function there. For example, at point \((0,0)\), the left side of the constraint has a value of 0. Because the value is less than 16, the constraint is satisfied and the region

![Figure 3.1: Constraint boundary for the inequality \(x_1 + x_2 \leq 16\) in the profit maximization problem.](image-url)
below F–J is feasible. We can test the constraint at another point on the opposite side of F–J, say at point (10,10). At this point the constraint is violated because the left side of the constraint function is 20, which is larger than 16. Therefore, the region above F–J is infeasible with respect to the constraint, as shown in Figure 3.2. The infeasible region is “shaded-out,” a convention that is used throughout this text.

Note that if this were an equality constraint \( x_1 + x_2 = 16 \), the feasible region for it would only be the points on line F–J. Although there are infinite points on F–J, the feasible region for the equality constraint is much smaller than that for the same constraint written as an inequality. This shows the importance of properly formulating all the constraints of the problem.

**STEP 4: IDENTIFICATION OF THE FEASIBLE REGION** By following the procedure that is described in step 3, all inequalities are plotted on the graph and the feasible side of each one is identified (if equality constraints were present, they would also be plotted at this stage). Note that the constraints \( x_1 \geq 0 \), \( x_2 \geq 0 \) restrict the feasible region to the first quadrant of the coordinate system. The intersection of feasible regions for all constraints provides the feasible region for the profit maximization problem, indicated as ABCDE in Figure 3.3. Any point in this region or on its boundary provides a feasible solution to the problem.

**STEP 5: PLOTTING OF OBJECTIVE FUNCTION CONTOURS** The next task is to plot the objective function on the graph and locate its optimum points. For the present problem, the objective is to maximize the profit \( P = 400x_1 + 600x_2 \), which involves three variables: \( P \), \( x_1 \), and \( x_2 \). The function needs to be represented on the graph so that the value of \( P \) can be
compared for different feasible designs to locate the best design. However, because there are infinite feasible points, it is not possible to evaluate the objective function at every point. One way of overcoming this impasse is to plot the contours of the objective function.

A contour is a curve on the graph that connects all points having the same objective function value. A collection of points on a contour is also called the level set. If the objective function is to be minimized, the contours are also called isocost curves. To plot a contour through the feasible region, we need to assign it a value. To obtain this value, consider a point in the feasible region and evaluate the profit function there. For example, at point (6,4), \( P = 6 \times 400 + 4 \times 600 = 4800 \). To plot the \( P = 4800 \) contour, we plot the function \( 400x_1 + 600x_2 = 4800 \). This contour is a straight line, as shown in Figure 3.4.

**STEP 6: IDENTIFICATION OF THE OPTIMUM SOLUTION** To locate an optimum point for the objective function, we need at least two contours that pass through the feasible region. We can then observe trends for the values of the objective function at different feasible points to locate the best solution point. Contours for \( P = 2400, 4800, \) and 7200 are plotted in Figure 3.5. We now observe the following trend: As the contours move up toward point D, feasible designs can be found with larger values for \( P \). It is clear from observation that point D has the largest value for \( P \) in the feasible region. We now simply read the coordinates of point D (4, 12) to obtain the optimum design, having a maximum value for the profit function as \( P = 8800 \).

Thus, the best strategy for the company is to manufacture 4 A and 12 B machines to maximize its daily profit. The inequality constraints in Eqs. (b) and (c) are active at the optimum; that is, they are satisfied at equality. These represent limitations on shipping and handling.
FIGURE 3.4 Plot of $P = 4800$ objective function contour for the profit maximization problem.

FIGURE 3.5 Graphical solution to the profit maximization problem: optimum point $D = (4, 12)$; maximum profit, $P = 8800$. 

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facilities, and on manufacturing. The company can think about relaxing these constraints to improve its profit. All other inequalities are strictly satisfied and therefore \textit{inactive}.

Note that in this example the design variables must have integer values. Fortunately, the optimum solution has integer values for the variables. If this had not been the case, we would have used the procedure suggested in Section 2.11.8 or in Chapter 15 to solve this problem. Note also that for this example all functions are linear in design variables. Therefore, all curves in Figures 3.1 through 3.5 are \textit{straight lines}. In general, the functions of a design problem may not be linear, in which case curves must be plotted to identify the feasible region, and contours or \textit{isocost curves} must be drawn to identify the optimum design. To \textit{plot a nonlinear function}, a table of numerical values for \( x_1 \) and \( x_2 \) must be generated. These points must be then plotted on a graph and connected by a smooth curve.

3.2 \textbf{USE OF MATHEMATICA FOR GRAPHICAL OPTIMIZATION}

It turns out that good programs, such as Mathematica and MATLAB\textsuperscript{R}, are available to implement the step-by-step procedure of the previous section and obtain a graphical solution for the problem on the computer screen. Mathematica is an interactive software package with many capabilities; however, we will explain its use to solve a two-variable optimization problem by plotting all functions on the computer screen. Although other commands for plotting functions are available, the most convenient for working with inequality constraints and objective function contours is the \textit{ContourPlot} command. As with most Mathematica commands, this one is followed by what we call subcommands as “arguments” that define the nature of the plot. All Mathematica commands are case-sensitive, so it is important to pay attention to which letters are capitalized.

Mathematica input is organized into what is called a \textit{notebook}. A notebook is divided into \textit{cells}, with each cell containing input that can be executed independently. To explain the graphical optimization capability of Mathematica, we will again use the profit maximization problem. (Note that the commands used here may change in future releases of the program.) We start by entering in the notebook the problem functions as follows (the first two commands are for initialization of the program):

\begin{verbatim}
<<Graphics`Arrow`
Clear[x1,x2];
P=400*x1+600*x2;
g1=x1+x2-16; (*shipping and handling constraint*)
g2=x1/28+x2/14-1; (*manufacturing constraint*)
g3=x1/14+x2/24-1; (*limitation on sales department*)
g4=-x1; (*non-negativity*)
g5=-x2; (*non-negativity*)
\end{verbatim}

This input illustrates some basic features concerning Mathematica format. Note that the \textbf{ENTER} key acts simply as a carriage return, taking the blinking cursor to the next line. Pressing \textbf{SHIFT} and \textbf{ENTER} actually inputs the typed information into Mathematica. When no immediate output from Mathematica is desired, the input line must end with a
semicolon (;). If the semicolon is omitted, Mathematica will simplify the input and display it on the screen or execute an arithmetic expression and display the result. Comments are bracketed as (*Comment*). Note also that all constraints are assumed to be in the standard "≤" form. This helps in identifying the infeasible region for constraints on the screen using the ContourPlot command.

3.2.1 Plotting Functions

The Mathematica command used to plot the contour of a function, say \( g_1=0 \), is entered as follows:

\[
\text{Plot}g_1=\text{ContourPlot}[g_1,\{x_1,0,25\},\{x_2,0,25\},\text{ContourShading}\to\text{False},\text{Contours}\to\{0\},\text{ContourStyle}\to\{(\text{Thickness}[.01])\},\text{Axes}\to\text{True},\text{AxesLabel}\to\{"x_1","x_2"\},\text{PlotLabel}\to\text{"Profit Maximization Problem"},\text{Epilog}\to\{\text{Disk[\{0,16\},.4,.4]\},\text{Text[\{0,16\},\{2,16\}\]},\text{Disk[\{16,0\},.4,.4]\},\text{Text[\{16,0\},\{17,1.5\}\],\text{Text[\"J\",\{17,0\}\],\text{Text[\"x_1+x_2=16\",\{13,9\}\],\text{Arrow[\{13,8.3\},\{10,6\}\],\DefaultFont\to\{"Times",12\},\ImageSize\to72.5\}};\]
\]

\( \text{Plot}g_1 \) is simply an arbitrary name referring to the data points for the function \( g_1 \) determined by the ContourPlot command; it is used in future commands to refer to this particular plot. This ContourPlot command plots a contour defined by the equation \( g_1=0 \) as shown earlier in Figure 3.1. Arguments of the ContourPlot command containing various subcommands are explained as follows (note that the arguments are separated by commas and are enclosed in square brackets ( [ ] )):

- \( g_1 \): function to be plotted.
- \( \{x_1,0,25\},\{x_2,0,25\} \): ranges for the variables \( x_1 \) and \( x_2 \); 0 to 25.
- \( \text{ContourShading}\to\text{False} \): indicates that shading will not be used to plot contours, whereas \( \text{ContourShading}\to\text{True} \) would indicate that shading will be used. Note that most subcommands are followed by an arrow (\( \to \)) or (\( \rightarrow \)) and a set of parameters enclosed in braces (\{\}):
- \( \text{Contours}\to\{0\} \): contour values for \( g_1 \); one contour is requested having 0 value.
- \( \text{ContourStyle}\to\{(\text{Thickness}[.01])\} \): defines characteristics of the contour such as thickness and color. Here, the thickness of the contour is specified as ".01". It is given as a fraction of the total width of the graph and needs to be determined by trial and error.
- \( \text{Axes}\to\text{True} \): indicates whether axes should be drawn at the origin; in the present case, where the origin \( (0,0) \) is located at the bottom left corner of the graph, the Axes subcommand is irrelevant except that it allows for the use of the AxesLabel command.
- \( \text{AxesLabel}\to\{"x_1","x_2"\} \): allows one to indicate labels for each axis.
- \( \text{PlotLabel}\to\text{"Profit Maximization Problem"} \): places a label at the top of the graph.
- \( \text{Epilog}\to\{\ldots\} \): allows insertion of additional graphics primitives and text in the figure on the screen figure on the screen; \( \text{Disk}[\{0,16\},.4,.4]\) allows insertion of a dot at the location \( (0,16) \) of radius .4 in both directions; \( \text{Text[\"(0,16)\",(2,16)]\} \) allows "(0,16)" to be placed at the location \( (2,16) \).
3.2 USE OF MATHEMATICA FOR GRAPHICAL OPTIMIZATION

ImageSize → 72.5: indicates that the width of the plot should be 5 inches; the size of the plot can be adjusted by selecting the image and dragging one of the black square control points; the images in Mathematica can be copied and pasted to a word processor file.

DefaultFont → {"Times",12}: specifies the preferred font and size for the text.

3.2.2 Identification and Shading of Infeasible Region for an Inequality

Figure 3.2 is created using a slightly modified ContourPlot command used earlier for Figure 3.1:

Plot1=ContourPlot[g1,{x1,0,25},{x2,0,25}, ContourShading→False, Contours→{0,.65},
ContourStyle→{Thickness[.01]}, {GrayLevel[.8],Thickness[.025]}], Axes→True,
AxesLabel→{"x1","x2"}, PlotLabel→"Profit Maximization Problem",
Epilog→{Disk[{10,10}, {.4,.4}], Text["(10,10)",{11,9}], Disk[{0,0}, {.4,.4}],
Text["(0,0)",{2,.5}], Text["x1+x2=16",{18,7}], Arrow[{18,6.3},{12,4}],
Text["Infeasible",{17,17}], Text["x1+x2>16",{17,15.5}], Text["Feasible",{5,6}],
Text["x1+x2<16",{5,4.5}]), DefaultFont→{"Times",12}, ImageSize→72.5];

Here, two contour lines are specified, the second one having a small positive value. This is indicated by the command: Contours→{0,.65}. The constraint boundary is represented by the contour g1=0. The contour g1=0.65 will pass through the infeasible region, where the positive number 0.65 is determined by trial and error.

To shade the infeasible region, the characteristics of the contour are changed. Each set of brackets {} with the ContourStyle subcommand corresponds to a specific contour. In this case, {Thickness[.01]} provides characteristics for the first contour g1=0, and {GrayLevel[.8],Thickness[.025]} provides characteristics for the second contour g1=0.65. GrayLevel specifies a color for the contour line. A gray level of 0 yields a black line, whereas a gray level of 1 yields a white line. Thus, this ContourPlot command essentially draws one thin, black line and one thick, gray line. This way the infeasible side of an inequality is shaded out.

3.2.3 Identification of Feasible Region

By using the foregoing procedure, all constraint functions for the problem are plotted and their feasible sides are identified. The plot functions for the five constraints g1 through g5 are named Plotg1, Plotg2, Plotg3, Plotg4, and Plotg5. All of these functions are quite similar to the one that was created using the ContourPlot command explained earlier. As an example, the Plotg4 function is given as

Plot4=ContourPlot[g4,{x1,-1,25},{x2,-1,25}, ContourShading→False, Contours→{0,.35},
ContourStyle→{Thickness[.01]}, {GrayLevel[.8],Thickness[.02]}],
DisplayFunction→Identity];

The DisplayFunction→Identity subcommand is added to the ContourPlot command to suppress display of output from each Plotg function; without that, Mathematica
executes each Plotg_i function and displays the results. Next, with the following Show command, the five plots are combined to display the complete feasible set in Figure 3.3:

\[
\text{Show}\{\text{Plotg1, Plotg2, Plotg3, Plotg4, Plotg5}, \text{Axes} \to \text{True}, \text{AxesLabel} \to \{"x1", "x2"}, \\
\text{PlotLabel} \to \text{"Profit Maximization Problem"}, \text{DefaultFont} \to \{"Times", 12}, \text{Epilog} \to \\
\{\text{Text["g1",\{2.5,16.2\}], Text["g2",\{24,4\}], Text["g3",\{2,24\}], Text["g5",\{21,1\}], Text["g4",\{1,10\}], Text["Feasible",\{5,6\}]}\}, \text{DefaultFont} \to \{"Times", 12}, \\
\text{ImageSize} \to 72.5, \text{DisplayFunction} \to \$\text{DisplayFunction}\};
\]

The Text subcommands are included to add text to the graph at various locations. The DisplayFunction \$\text{DisplayFunction} subcommand is added to display the final graph; without that it is not displayed.

### 3.2.4 Plotting of Objective Function Contours

The next task is to plot the objective function contours and locate its optimum point. The objective function contours of values 2400, 4800, 7200, and 8800, shown in Figure 3.4, are drawn by using the ContourPlot command as follows:

\[
\text{PlotP}=\text{ContourPlot}[P, \{x1,0,25\}, \{x2,0,25\}, \text{ContourShading} \to \text{False}, \text{Contours} \to \{4800\}, \\
\text{ContourStyle} \to \{\text{Dashing[\{.03,.04\}], Thickness[.007]}\}, \text{Axes} \to \text{True}, \\
\text{AxesLabel} \to \{"x1","x2"}, \text{PlotLabel} \to \text{"Profit Maximization Problem"}, \\
\text{DefaultFont} \to \{"Times", 12\}, \text{Epilog} \to \{\text{Disk[\{6,4\},\{.4,.4\}], Text["P= 4800",\{9.75,4\}]}, \\
\text{ImageSize} \to 72.5\];
\]

The ContourStyle subcommand provides four sets of characteristics, one for each contour. \text{Dashing[\{a,b\}]} yields a dashed line with "a" as the length of each dash and "b" as the space between dashes. These parameters represent a fraction of the total width of the graph.

### 3.2.5 Identification of Optimum Solution

The Show command used to plot the feasible region for the problem in Figure 3.3 can be extended to plot the profit function contours as well. Figure 3.5 contains the graphical representation of the problem, obtained using the following Show command:

\[
\text{Show}\{\text{Plotg1, Plotg2, Plotg3, Plotg4, Plotg5, PlotP}, \text{Axes} \to \text{True}, \text{AxesLabel} \to \{"x1","x2"}, \\
\text{PlotLabel} \to \text{"Profit Maximization Problem"}, \text{DefaultFont} \to \{"Times", 12}, \\
\text{Epilog} \to \{\text{Text["g1",\{2.5,16.2\}], Text["g2",\{24,4\}], Text["g3",\{3,23\}], Text["g5",\{23,1\}], Text["g4",\{1,10\}], Text["P= 2400",\{3.5,2\}], Text["P= 8800",\{17,3.5\}], Text["G",\{1,24.5\}], Text["C",\{10.5,4\}], Text["D",\{3.5,11\}], Text["A",\{1,1\}], Text["B",\{14,-1\}], Text["J",\{16,-1\}], Text["H",\{25,-1\}], Text["E",\{-1,14\}], Text["F",\{-1,16\}], \\
\text{DefaultFont} \to \{"Times", 12\}, \text{ImageSize} \to 72.5, \text{DisplayFunction} \to \$\text{DisplayFunction}\];
\]

Additional Text subcommands have been added to label different objective function contours and different points. The final graph is used to obtain the graphical solution. The Disk subcommand can be added to the Epilog command to put a dot at the optimum point.
3.3 Use of MATLAB for Graphical Optimization

MATLAB has many capabilities for solving engineering problems. For example, it can plot problem functions and graphically solve a two-variable optimization problem. In this section, we explain how to use the program for this purpose; other uses of the program for solving optimization problems are explained in Chapter 7.

There are two modes of input with MATLAB. We can enter commands interactively, one at a time, with results displayed immediately after each one. Alternatively, we can create an input file, called an m-file that is executed in batch mode. The m-file can be created using the text editor in MATLAB. To access this editor, select "File," "New," and "m-file." When saved, this file will have the suffix ".m" (dot m). To submit or run the file, after starting MATLAB, we simply type the name of the file we wish to run in the command window, without the suffix (the current directory in the MATLAB program must be one where the file is located). In this section, we will solve the profit maximization problem of the previous section using MATLAB. It is important to note that with future releases, the commands we will discuss may change.

3.3.1 Plotting of Function Contours

For plotting all of the constraints with MATLAB and identifying the feasible region, it is assumed that all inequality constraints are written in the standard "≤" form. The M-file for the profit maximization problem with explanatory comments is displayed in Table 3.1. Note that the file comments are preceded by the percent sign, %. The comments are ignored during MATLAB execution. For contour plots, the first command in the input file is entered as follows:

\[
[x1,x2]=\text{meshgrid}(-1.0:0.5:25.0, -1.0:0.5:25.0);
\]

This command creates a grid or array of points where all functions to be plotted are evaluated. The command indicates that \( x_1 \) and \( x_2 \) will start at \(-1.0\) and increase in increments of 0.5 up to 25.0. These variables now represent two-dimensional arrays and require special attention in operations using them. "*" (star) and "/" (slash) indicate scalar multiplication and division, respectively, whereas ".*" (dot star) and "./" (dot slash) indicate element-by-element multiplication and division. The ".^" (dot hat) is used to apply an exponent to each element of a vector or a matrix. The semicolon ";" after a command prevents MATLAB from displaying the numerical results immediately (i.e., all of the values for \( x_1 \) and \( x_2 \)).

This use of a semicolon is a convention in MATLAB for most commands. Note that matrix division and multiplication capabilities are not used in the present example, as the variables in the problem functions are only multiplied or divided by a scalar rather than another variable. If, for instance, a term such as \( x_1 \times x_2 \) is present, then the element-by-element operation \( x_1.*x2 \) is necessary. The "contour" command is used for plotting all problem functions on the screen.

The procedure for identifying the infeasible side of an inequality is to plot two contours for the inequality: one of value 0 and the other of a small positive value. The second
TABLE 3.1 MATLAB file for the profit maximization problem

m-file with explanatory comments

%Create a grid from -1 to 25 with an increment of 0.5 for the variables x1 and x2
[x1,x2]=meshgrid(-1:0.5:25.0,-1:0.5:25.0);
%Enter functions for the profit maximization problem
f=400*x1+600*x2;
g1=x1+x2-16;
g2=x1/28+x2/14-1;
g3=x1/14+x2/24-1;
g4=-x1;
g5=-x2;
%Initialization statements; these need not end with a semicolon
cla reset
axis auto %Minimum and maximum values for axes are determined automatically
%Limits for x- and y-axes may also be specified with the command
%axis ([xmin xmax ymin ymax])
xlabel('x1'), ylabel('x2') %Specifies labels for x- and y-axes
title ('Profit Maximization Problem') %Displays a title for the problem
hold on %Retains the current plot and axes properties for all subsequent plots
%Use the "contour" command to plot constraint and cost functions
cv1=[0 .5]; %Specifies two contour values, 0 and .5
const1=contour(x1,x2,g1,cv1,'k'); %Plots two specified contours of g1; k=black color
clabel(const1) %Automatically puts the contour value on the graph
text(1,16,'g1') %Writes g1 at the location (1, 16)
cv2=[0 .03];
const2=contour(x1,x2,g2,cv2,'k');
clabel(const2)
text(23,3,'g2')
const3=contour(x1,x2,g3,cv2,'k');
clabel(const3)
text(1,23,'g3')
cv3=[0 .5];
const4=contour(x1,x2,g4,cv3,'k');
clabel(const4)
text(.25,20,'g4')
const5=contour(x1,x2,g5,cv3,'k');
clabel(const5)
text(19,.5,'g5')
text(1.5,7,'Feasible Region')
fv=[2400, 4800, 7200, 8800]; %Defines 4 contours for the profit function
fs=contour(x1,x2,f,fv,'k-'); %'k-' specifies black dashed lines for profit function contours
clabel(fs)
hold off %Indicates end of this plotting sequence
%Subsequent plots will appear in separate windows

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contour will pass through the problem’s infeasible region. The thickness of the infeasible contour is changed to indicate the infeasible side of the inequality using the graph-editing capability, which is explained in the following subsection.

In this way all constraint functions are plotted and the problem’s feasible region is identified. By observing the trend of the objective function contours, the optimum point for the problem is identified.

3.3.2 Editing of Graph

Once the graph has been created using the commands just described, we can edit it before printing it or copying it to a text editor. In particular, we may need to modify the appearance of the constraints’ infeasible contours and edit any text. To do this, first select “Current Object Properties...” under the “Edit” tab on the graph window. Then double-click on any item in the graph to edit its properties. For instance, we can increase the thickness of the infeasible contours to shade out the infeasible region. In addition, text may be added, deleted, or moved as desired. Note that if MATLAB is re-run, any changes made directly to the graph are lost. For this reason, it is a good idea to save the graph as a "fig" file, which may be recalled with MATLAB.

Another way to shade out the infeasible region is to plot several closely spaced contours in it using the following commands:

```matlab
cv1=[0:0.01:0.5]; %[Starting contour: Increment: Final contour]
const1=contour(x1,x2,g1,cv1,'g'); % g = green color
```

There are two ways to transfer the graph to a text document. First, select “Copy Figure” under the “Edit” tab so that the figure can be pasted as a bitmap into a document. Alternatively, select “Export...” under the “File” tab. The figure is exported as the specified file type and can be inserted into another document through the “Insert” command. The final MATLAB graph for the profit maximization problem is shown in Figure 3.6.

3.4 DESIGN PROBLEM WITH MULTIPLE SOLUTIONS

A situation can arise in which a constraint is parallel to the cost function. If the constraint is active at the optimum, there are multiple solutions to the problem. To illustrate this situation, consider the following design problem:

Minimize

\[ f(x) = -x_1 - 0.5x_2 \]

subject to

\[ 2x_1 + 3x_2 \leq 12, \quad 2x_1 + x_2 \leq 8, \quad -x_1 \leq 0, \quad -x_2 \leq 0 \]

In this problem, the second constraint is parallel to the cost function. Therefore, there is a possibility of multiple optimum designs. Figure 3.7 provides a graphical solution to the problem. It is seen that any point on the line B–C gives an optimum design, giving the problem infinite optimum solutions.
FIGURE 3.6 This shows a graphical representation of the profit maximization problem with MATLAB.

FIGURE 3.7 Example problem with multiple solutions.
3.5 PROBLEM WITH UNBOUNDDED SOLUTIONS

Some design problems may not have a bounded solution. This situation can arise if we forget a constraint or incorrectly formulate the problem. To illustrate such a situation, consider the following design problem:

Minimize \( f(x) = -x_1 + 2x_2 \)  
subject to  
\[-2x_1 + x_2 \leq 0, \quad -2x_1 + 3x_2 \leq 6, \quad -x_1 \leq 0, \quad -x_2 \leq 0 \]

The feasible set for the problem is shown in Figure 3.8 with several cost function contours. It is seen that the feasible set is unbounded. Therefore, there is no finite optimum solution, and we must re-examine the way the problem was formulated to correct the situation. Figure 3.8 shows that the problem is underconstrained.

3.6 INFEASIBLE PROBLEM

If we are not careful in formulating it, a design problem may not have a solution, which happens when there are conflicting requirements or inconsistent constraint equations. There may also be no solution when we put too many constraints on the system; that is, the
constraints are so restrictive that no feasible solution is possible. These are called infeasible problems. To illustrate them, consider the following:

\[
\begin{align*}
\text{Minimize} & \quad f(x) = x_1 + 2x_2 \\
\text{subject to} & \quad 3x_1 + 2x_2 \leq 6, \quad 2x_1 + 3x_2 \geq 12, \quad x_1, x_2 \leq 5, \quad x_1, x_2 \geq 0
\end{align*}
\]

Constraints for the problem are plotted in Figure 3.9 and their infeasible side is shaded-out. It is evident that there is no region within the design space that satisfies all constraints; that is, there is no feasible region for the problem. Thus, the problem is infeasible. Basically, the first two constraints impose conflicting requirements. The first requires the feasible design to be below the line A–G, whereas the second requires it to be above the line C–F. Since the two lines do not intersect in the first quadrant, the problem has no feasible region.

3.7 GRAPHICAL SOLUTION FOR THE MINIMUM-WEIGHT TUBULAR COLUMN

The design problem formulated in Section 2.7 will now be solved by the graphical method using the following data: \( P = 10 \) MN, \( E = 207 \) GPa, \( \rho = 7833 \) kg/m\(^3\), \( l = 5.0 \) m, and \( \sigma_a = 248 \) MPa. Using these data, formulation 1 for the problem is defined as “Find mean radius \( R \) (m) and thickness \( t \) (m) to minimize the mass function”:

\[
f(R, t) = 2\rho\pi R t = 2(7833)(5)\pi R t = 2.4608 \times 10^5 R t, \text{ kg}
\]

subject to the four inequality constraints
Note that the explicit bound constraints discussed in Section 2.7 are simply replaced by the non-negativity constraints \( g_3 \) and \( g_4 \). The constraints for the problem are plotted in Figure 3.10, and the feasible region is indicated. Cost function contours for \( f = 1000 \text{ kg} \), \( 1500 \text{ kg} \), and \( 1579 \text{ kg} \) are also shown. In this example the cost function contours run parallel to the stress constraint \( g_1 \). Since \( g_1 \) is active at the optimum, the problem has infinite optimum designs, that is, the entire curve A–B in Figure 3.10. We can read the coordinates of any point on the curve A–B as an optimum solution. In particular, point A, where constraints \( g_1 \) and \( g_2 \) intersect, is also an optimum point where \( R^* = 0.1575 \text{ m} \) and \( t^* = 0.0405 \text{ m} \).

The superscript \(^*\) on a variable indicates its optimum value, a notation that will be used throughout this text.

**FIGURE 3.10** A graphical solution to the problem of designing a minimum-weight tubular column.
3.8 GRAPHICAL SOLUTION FOR A BEAM DESIGN PROBLEM

**STEP 1: PROJECT/PROBLEM DESCRIPTION**  A beam of rectangular cross-section is subjected to a bending moment $M$ (N·m) and a maximum shear force $V$ (N). The bending stress in the beam is calculated as $\sigma = \frac{6M}{bd^2}$ (Pa), and average shear stress is calculated as $\tau = \frac{3V}{2bd}$ (Pa), where $b$ is the width and $d$ is the depth of the beam. The allowable stresses in bending and shear are 10 MPa and 2 MPa, respectively. It is also desirable that the depth of the beam not exceed twice its width and that the cross-sectional area of the beam be minimized. In this section, we formulate and solve the problem using the graphical method.

**STEP 2: DATA AND INFORMATION COLLECTION**  Let bending moment $M = 40$ kN·m and the shear force $V = 150$ kN. All other data and necessary equations are given in the project statement. We shall formulate the problem using a consistent set of units, N and mm.

**STEP 3: DEFINITION OF DESIGN VARIABLES**  The two design variables are

- $d =$ depth of beam, mm
- $b =$ width of beam, mm

**STEP 4: OPTIMIZATION CRITERION**  The cost function for the problem is the cross-sectional area, which is expressed as

$$f(b, d) = bd$$  \hspace{1cm} (a)

**STEP 5: FORMULATION OF CONSTRAINTS**  Constraints for the problem consist of bending stress, shear stress, and depth-to-width ratio. Bending and shear stresses are calculated as

$$\sigma = \frac{6M}{bd^2} = \frac{6(40)(1000)(1000)}{bd^2}, \text{ N/mm}^2$$  \hspace{1cm} (b)

$$\tau = \frac{3V}{2bd} = \frac{3(150)(1000)}{2bd}, \text{ N/mm}^2$$  \hspace{1cm} (c)

Allowable bending stress $\sigma_a$ and allowable shear stress $\tau_a$ are given as

$$\sigma_a = 10 \text{ MPa} = 10 \times 10^6 \text{ N/m}^2 = 10 \text{ N/mm}^2$$  \hspace{1cm} (d)

$$\tau_a = 2 \text{ MPa} = 2 \times 10^6 \text{ N/m}^2 = 2 \text{ N/mm}^2$$  \hspace{1cm} (e)

Using Eqs. (b) through (e), we obtain the bending and shear stress constraints as

$$g_1 = \frac{6(40)(1000)(1000)}{bd^2} - 10 \leq 0 \text{ (bending stress)}$$  \hspace{1cm} (f)

$$g_2 = \frac{3(150)(1000)}{2bd} - 2 \leq 0 \text{ (shear stress)}$$  \hspace{1cm} (g)

I. THE BASIC CONCEPTS
The constraint that requires that the depth be no more than twice the width can be expressed as

\[ g_3 = d - 2b \leq 0 \]  

Finally, both design variables should be non-negative:

\[ g_4 = -b \leq 0; \quad g_5 = -d \leq 0 \]

In reality, \( b \) and \( d \) cannot both have zero value, so we should use some minimum value as a lower bound on them (i.e., \( b \geq b_{\min} \) and \( d \geq d_{\min} \)).

**Graphical Solution**

Using MATLAB, the constraints for the problem are plotted in Figure 3.11, and the feasible region is identified. Note that the cost function is parallel to the constraint \( g_2 \) (both functions have the same form: \( bd = \text{constant} \)). Therefore, any point along the curve A–B represents an optimum solution, so there are infinite optimum designs. This is a desirable situation since a wide choice of optimum solutions is available to meet a designer’s needs.

The optimum cross-sectional area is 112,500 mm\(^2\). Point B corresponds to an optimum design of \( b = 237 \) mm and \( d = 474 \) mm. Point A corresponds to \( b = 527.3 \) mm and \( d = 213.3 \) mm. These points represent the two extreme optimum solutions; all other solutions lie between these two points on the curve A–B.

**EXERCISES FOR CHAPTER 3**

Solve the following problems using the graphical method.

3.1 Minimize \( f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2 \)

subject to

\[ x_1 + x_2 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

I. THE BASIC CONCEPTS
3.2 Maximize \( F(x_1, x_2) = x_1 + 2x_2 \) 
subject to \( 2x_1 + x_2 \leq 4 \)  
\( x_1, x_2 \geq 0 \)

3.3 Minimize \( f(x_1, x_2) = x_1 + 3x_2 \) 
subject to \( x_1 + 4x_2 \geq 48 \)  
\( 5x_1 + x_2 \geq 50 \)  
\( x_1, x_2 \geq 0 \)

3.4 Maximize \( F(x_1, x_2) = x_1 + x_2 + 2x_3 \) 
subject to \( 1 \leq x_1 \leq 4 \)  
\( 3x_2 - 2x_3 = 6 \)  
\( -1 \leq x_3 \leq 2 \)  
\( x_2 \geq 0 \)

3.5 Maximize \( F(x_1, x_2) = 4x_1x_2 \) 
subject to \( x_1 + x_2 \leq 20 \)  
\( x_2 - x_1 \leq 10 \)  
\( x_1, x_2 \geq 0 \)

3.6 Minimize \( f(x_1, x_2) = 5x_1 + 10x_2 \) 
subject to \( 10x_1 + 5x_2 \leq 50 \)  
\( 5x_1 - 5x_2 \geq -20 \)  
\( x_1, x_2 \geq 0 \)

3.7 Minimize \( f(x_1, x_2) = 3x_1 + x_2 \) 
subject to \( 2x_1 + 4x_2 \leq 21 \)  
\( 5x_1 + 3x_2 \leq 18 \)  
\( x_1, x_2 \geq 0 \)

3.8 Minimize \( f(x_1, x_2) = x_1^2 - 2x_2^2 - 4x_1 \) 
subject to \( x_1 + x_2 \leq 6 \)  
\( x_2 \leq 3 \)  
\( x_1, x_2 \geq 0 \)

3.9 Minimize \( f(x_1, x_2) = x_1x_2 \) 
subject to \( x_1 + x_2^2 \leq 0 \)  
\( x_1^2 + x_2^2 \leq 9 \)

3.10 Minimize \( f(x_1, x_2) = 3x_1 + 6x_2 \) 
subject to \( -3x_1 + 3x_2 \leq 2 \)  
\( 4x_1 + 2x_2 \leq 4 \)  
\( -x_1 + 3x_2 \geq 1 \)

Develop an appropriate graphical representation for the following problems and determine the minimum and the maximum points for the objective function.

3.11 \( f(x, y) = 2x^2 + y^2 - 2xy - 3x - 2y \) 
subject to \( y - x \leq 0 \)  
\( x^2 + y^2 - 1 = 0 \)

3.12 \( f(x, y) = 4x^2 + 3y^2 - 5xy - 8x \) 
subject to \( x + y = 4 \)

I. THE BASIC CONCEPTS
3.13 \[ f(x, y) = 9x^2 + 13y^2 + 18xy - 4 \]
subject to \[ x^2 + y^2 + 2x = 16 \]

3.14 \[ f(x, y) = 2x + 3y - x^3 - 2y^2 \]
subject to \[ x + 3y \leq 6 \]
\[ 5x + 2y \leq 10 \]
\[ x, y \geq 0 \]

3.15 \[ f(r, t) = (r - 8)^2 + (t - 8)^2 \]
subject to \[ 12 \geq r + t \]
\[ t \leq 5 \]
\[ r, t \geq 0 \]

3.16 \[ f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2 \]
subject to \[ x_1 + x_2 \leq 3 \]

3.17 \[ f(x, y) = 9x^2 + 13y^2 + 18xy - 4 \]
subject to \[ x^2 + y^2 + 2x \geq 16 \]

3.18 \[ f(r, t) = (r - 4)^2 + (t - 4)^2 \]
subject to \[ 10 - r - t \geq 0 \]
\[ 5 \geq r \]
\[ r, t \geq 0 \]

3.19 \[ f(x, y) = -x^2 + 2y \]
subject to \[ -x^2 + 6x + 3y \leq 27 \]
\[ 18x - y^2 \geq 180 \]
\[ x, y \geq 0 \]

3.20 \[ f(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 2)^2 \]
subject to \[ 10 \geq x_1 + 2x_2 \]
\[ 0 \leq x_1 \leq 3 \]
\[ x_2 \geq 0 \]

3.21 Solve the rectangular beam problem of Exercise 2.17 graphically for the following data:
\[ M = 80 \text{kN} \cdot \text{m}, \ V = 150 \text{kN}, \ \sigma_a = 8 \text{MPa}, \ \text{and} \ \tau_a = 3 \text{MPa}. \]

3.22 Solve the cantilever beam problem of Exercise 2.23 graphically for the following data:
\[ P = 10 \text{kN}; \ l = 5.0 \text{m}; \ \text{modulus of elasticity}, \ E = 210 \text{Gpa}; \ \text{allowable bending stress}, \ \sigma_a = 250 \text{MPa}; \ \text{allowable shear stress}, \ \tau_a = 90 \text{MPa}; \ \text{mass density}, \ \rho = 7850 \text{kg/m}^3; \ R_o \leq 20.0 \text{cm}; \ R_i \leq 20.0 \text{cm}. \]

3.23 For the minimum-mass tubular column design problem formulated in Section 2.7, consider the following data: \[ P = 50 \text{kN}; \ l = 5.0 \text{m}; \ \text{modulus of elasticity}, \ E = 210 \text{Gpa}; \ \text{allowable stress}, \ \sigma_a = 250 \text{MPa}; \ \text{mass density}, \ \rho = 7850 \text{kg/m}^3. \]
Treating mean radius \( R \) and wall thickness \( t \) as design variables, solve the design problem graphically, imposing an additional constraint \( R/t \leq 50 \). This constraint is needed to avoid local crippling of the column. Also impose the member size constraints as
\[ 0.01 \leq R \leq 1.0 \text{ m}; \ 5 \leq t \leq 200 \text{ mm} \]

3.24 For Exercise 3.23, treat outer radius \( R_o \) and inner radius \( R_i \) as design variables, and solve the design problem graphically. Impose the same constraints as in Exercise 3.23.

3.25 Formulate the minimum-mass column design problem of Section 2.7 using a hollow square cross-section with outside dimension \( w \) and thickness \( t \) as design variables. Solve the problem graphically using the constraints and the data given in Exercise 3.23.

I. THE BASIC CONCEPTS
3.26 Consider the symmetric (members are identical) case of the two-bar truss problem discussed in Section 2.5 with the following data: \( W = 10 \text{kN}; \theta = 30^\circ; \) height \( h = 1.0 \text{ m}; \) span \( s = 1.5 \text{ m}; \) allowable stress, \( \sigma_a = 250 \text{ MPa}; \) modulus of elasticity, \( E = 210 \text{ GPa}. \)

Formulate the minimum-mass design problem with constraints on member stresses and bounds on design variables. Solve the problem graphically using circular tubes as members.

3.27 Formulate and solve the problem of Exercise 2.1 graphically.

3.28 In the design of the closed-end, thin-walled cylindrical pressure vessel shown in Figure E3.28, the design objective is to select the mean radius \( R \) and wall thickness \( t \) to minimize the total mass. The vessel should contain at least 25.0 m³ of gas at an internal pressure of 3.5 MPa. It is required that the circumferential stress in the pressure vessel not exceed 210 MPa and the circumferential strain not exceed \((1.0 \times 10^{-3})\). The circumferential stress and strain are calculated from the equations

\[
\sigma_c = \frac{PR}{t}, \quad \varepsilon_c = \frac{PR(2-\nu)}{2Et}
\]

where \( \rho = \text{mass density (7850 kg/m}^3\)), \( \sigma_c = \text{circumferential stress (Pa)}, \) \( \varepsilon_c = \text{circumferential strain}, \) \( P = \text{internal pressure (Pa)}, \) \( E = \text{Young’s modulus (210 GPa)}, \) and \( \nu = \text{Poisson’s ratio (0.3)}. \)

Formulate the optimum design problem, and solve it graphically.

3.29 Consider the symmetric three-bar truss design problem formulated in Section 2.10. Formulate and solve the problem graphically for the following data: \( l = 1.0 \text{ m}; P = 100 \text{ kN}; \theta = 30^\circ; \) mass density, \( \rho = 2800 \text{ kg/m}^3\); modulus of elasticity, \( E = 70 \text{ GPa}; \) allowable stress, \( \sigma_a = 140 \text{ MPa}; \Delta_u = 0.5 \text{ cm}; \Delta_a = 0.5 \text{ cm}; \omega_o = 50 \text{ Hz}; \beta = 1.0; A_1, A_2 = 2 \text{ cm}^2. \)

3.30 Consider the cabinet design problem in Section 2.6. Use the equality constraints to eliminate three design variables from the problem. Restate the problem in terms of the remaining three variables, transcribing it into the standard form.

3.31 Graphically solve the insulated spherical tank design problem formulated in Section 2.3 for the following data: \( r = 3.0 \text{ m}, \) \( c_1 = $10,000, c_2 = $1000, c_3 = $1, c_4 = $0.1, \Delta T = 5. \)

3.32 Solve the cylindrical tank design problem given in Section 2.8 graphically for the following data: \( c = $1500/\text{m}^2, V = 3000 \text{ m}^3. \)

3.33 Consider the minimum-mass tubular column problem formulated in Section 2.7. Find the optimum solution for it using the graphical method for the data: load, \( P = 100 \text{ kN}; \) length, \( l = 5.0 \text{ m}; \) Young’s modulus, \( E = 210 \text{ GPa}; \) allowable stress, \( \sigma_a = 250 \text{ MPa}; \) mass density, \( \rho = 7850 \text{ kg/m}^3; R \leq 0.4 \text{ m}; t \leq 0.1 \text{ m}; R, t \geq 0. \)

**FIGURE E3.28** Graphic of a cylindrical pressure vessel.
Design a hollow torsion rod, shown in Figure E3.34, to satisfy the following requirements (created by J. M. Trummel):

1. The calculated shear stress $\tau$ shall not exceed the allowable shear stress $\tau_a$ under the normal operating torque $T_o$ (N · m).
2. The calculated angle of twist, $\theta$, shall not exceed the allowable twist, $\theta_a$ (radians).
3. The member shall not buckle under a short duration torque of $T_{max}$ (N · m).

Requirements for the rod and material properties are given in Tables E3.34 (select a material for one rod). Use the following design variables: $x_1 = \text{outside diameter of the rod}$; $x_2 = \text{ratio of inside/outside diameter, } d_i/d_o$.

Using graphical optimization, determine the inside and outside diameters for a minimum-mass rod to meet the preceding design requirements. Compare the hollow rod

![Graphic of a hollow torsion rod.](FIGURE E3.34)

### Table E3.34(a) Rod requirements

<table>
<thead>
<tr>
<th>Torsion rod no.</th>
<th>Length $l$ (m)</th>
<th>Normal torque $T_o$ (kN · m)</th>
<th>Maximum $T_{max}$ (kN · m)</th>
<th>Allowable twist $\theta_a$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.50</td>
<td>10.0</td>
<td>20.0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0.75</td>
<td>15.0</td>
<td>25.0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>20.0</td>
<td>30.0</td>
<td>2</td>
</tr>
</tbody>
</table>

### Table E3.34(b) Materials and properties for the torsion rod

<table>
<thead>
<tr>
<th>Material</th>
<th>Density, $\rho$ (kg/m$^3$)</th>
<th>Allowable shear stress, $\tau_a$ (MPa)</th>
<th>Elastic modulus, $E$ (GPa)</th>
<th>Shear modulus, $G$ (GPa)</th>
<th>Poisson ratio ($\nu$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 4140 alloy steel</td>
<td>7850</td>
<td>275</td>
<td>210</td>
<td>80</td>
<td>0.30</td>
</tr>
<tr>
<td>2. Aluminum alloy 24 ST4</td>
<td>2750</td>
<td>165</td>
<td>75</td>
<td>28</td>
<td>0.32</td>
</tr>
<tr>
<td>3. Magnesium alloy A261</td>
<td>1800</td>
<td>90</td>
<td>45</td>
<td>16</td>
<td>0.35</td>
</tr>
<tr>
<td>4. Beryllium</td>
<td>1850</td>
<td>110</td>
<td>300</td>
<td>147</td>
<td>0.02</td>
</tr>
<tr>
<td>5. Titanium</td>
<td>4500</td>
<td>165</td>
<td>110</td>
<td>42</td>
<td>0.30</td>
</tr>
</tbody>
</table>

I. THE BASIC CONCEPTS
with an equivalent solid rod \((d_i/d_o = 0)\). Use a consistent set of units (e.g., Newtons and millimeters) and let the minimum and maximum values for design variables be given as

\[
0.02 \leq d_o \leq 0.5 \text{ m}, \quad 0.60 \leq \frac{d_i}{d_o} \leq 0.999
\]

**Useful expressions**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>(M = \frac{\pi}{4} \rho (d_o^4 - d_i^4)), kg</td>
</tr>
<tr>
<td>Calculated shear stress</td>
<td>(\tau = \frac{c}{J} T_o), Pa</td>
</tr>
<tr>
<td>Calculated angle of twist</td>
<td>(\theta = \frac{l}{GJ} T_o), radians</td>
</tr>
<tr>
<td>Critical buckling torque</td>
<td>(T_{cr} = \frac{\pi d_i^2 E}{12 \sqrt{2(1-\nu^2)}} \left(1 - \frac{d_i}{d_o}\right)^{25}, \text{ N} \cdot \text{ m})</td>
</tr>
</tbody>
</table>

**Notation**

- \(M\): mass (kg)
- \(d_o\): outside diameter (m)
- \(d_i\): inside diameter (m)
- \(\rho\): mass density of material (kg/m³)
- \(l\): length (m)
- \(T_o\): normal operating torque (N · m)
- \(c\): distance from rod axis to extreme fiber (m)
- \(J\): polar moment of inertia (m⁴)
- \(\theta\): angle of twist (radians)
- \(G\): modulus of rigidity (Pa)
- \(T_{cr}\): critical buckling torque (N · m)
- \(E\): modulus of elasticity (Pa)
- \(\nu\): Poisson’s ratio

*3.35* Formulate and solve Exercise 3.34 using the outside diameter \(d_o\) and the inside diameter \(d_i\) as design variables.

*3.36* Formulate and solve Exercise 3.34 using the mean radius \(R\) and wall thickness \(t\) as design variables. Let the bounds on design variables be given as \(5 \leq R \leq 20\) cm and \(0.2 \leq t \leq 4\) cm.

3.37 Formulate the problem in Exercise 2.3 and solve it using the graphical method.

3.38 Formulate the problem in Exercise 2.4 and solve it using the graphical method.

3.39 Solve Exercise 3.23 for a column pinned at both ends. The buckling load for such a column is given as \(\pi^2 EI/l^2\). Use the graphical method.

3.40 Solve Exercise 3.23 for a column fixed at both ends. The buckling load for such a column is given as \(4\pi^2 EI/l^2\). Use the graphical method.

3.41 Solve Exercise 3.23 for a column fixed at one end and pinned at the other. The buckling load for such a column is given as \(2\pi^2 EI/l^2\). Use the graphical method.
3.42 Solve Exercise 3.24 for a column pinned at both ends. The buckling load for such a column is given as $\pi^2 EI/l^2$. Use the graphical method.

3.43 Solve Exercise 3.24 for a column fixed at both ends. The buckling load for such a column is given as $4\pi^2 EI/l^2$. Use the graphical method.

3.44 Solve Exercise 3.24 for a column fixed at one end and pinned at the other. The buckling load for such a column is given as $2\pi^2 EI/l^2$. Use the graphical method.

3.45 Solve the can design problem formulated in Section 2.2 using the graphical method.

3.46 Consider the two-bar truss shown in Figure 2.5. Using the given data, design a minimum-mass structure where $W = 100$ kN; $\theta = 30^\circ$; $h = 1$ m; $s = 1.5$ m; modulus of elasticity $E = 210$ GPa; allowable stress $\sigma_a = 250$ MPa; mass density $\rho = 7850$ kg/m$^3$. Use Newtons and millimeters as units. The members should not fail in stress and their buckling should be avoided. Deflection at the top in either direction should not be more than 5 cm.

For Exercise 3.46, use hollow circular tubes as members with mean radius $R$ and wall thickness $t$ as design variables. Make sure that $R/t \geq 50$. Design the structure so that member 1 is symmetric with member 2. The radius and thickness must also satisfy the constraints $2 \leq t \leq 40$ mm and $2 \leq R \leq 40$ cm.

3.48 Design a symmetric structure defined in Exercise 3.46, treating cross-sectional area $A$ and height $h$ as design variables. The design variables must also satisfy the constraints $1 \leq A \leq 50$ cm$^2$ and $0.5 \leq h \leq 3$ m.

3.49 Design a symmetric structure defined in Exercise 3.46, treating cross-sectional area $A$ and span $s$ as design variables. The design variables must also satisfy the constraints $1 \leq A \leq 50$ cm$^2$ and $0.5 \leq s \leq 4$ m.

3.50 Design a minimum-mass symmetric three-bar truss (the area of member 1 and that of member 3 are the same) to support a load $P$, as was shown in Figure 2.9. The following notation may be used: $P_x = P \cos \theta$, $P_y = P \sin \theta$, $A_1 =$ cross-sectional area of members 1 and 3, $A_2 =$ cross-sectional area of member 2.

The members must not fail under the stress, and the deflection at node 4 must not exceed 2 cm in either direction. Use Newtons and millimeters as units. The data is given as $P = 50$ kN; $\theta = 30^\circ$; mass density, $\rho = 7850$ kg/m$^3$; $l = 1$ m; modulus of elasticity, $E = 210$ GPa; allowable stress, $\sigma_a = 150$ MPa. The design variables must also satisfy the constraints $50 \leq A_i \leq 5000$ mm$^2$.

*3.51 Design of a water tower support column. As an employee of ABC Consulting Engineers, you have been asked to design a cantilever cylindrical support column of minimum mass for a new water tank. The tank itself has already been designed in the teardrop shape, shown in Figure E3.51. The height of the base of the tank ($H$), the diameter of the tank ($D$), and the wind pressure on the tank ($w$) are given as $H = 30$ m, $D = 10$ m, and $w = 700$ N/m$^2$. Formulate the design optimization problem and then solve it graphically (created by G. Baenziger).

In addition to designing for combined axial and bending stresses and buckling, several limitations have been placed on the design. The support column must have an inside diameter of at least 0.70 m ($d_i$) to allow for piping and ladder access to the interior...
of the tank. To prevent local buckling of the column walls, the diameter/thickness ratio $(d_o/t)$ cannot be greater than 92. The large mass of water and steel makes deflections critical, as they add to the bending moment. The deflection effects, as well as an assumed construction eccentricity ($e$) of 10 cm, must be accounted for in the design process. Deflection at the center of gravity (C.G.) of the tank should not be greater than $\Delta$.

Limits on the inner radius and wall thickness are $0.35 \leq R \leq 2.0$ m and $1.0 \leq t \leq 20$ cm.

### Pertinent constants and formulas

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height of water tank</td>
<td>$h = 10$ m</td>
</tr>
<tr>
<td>Allowable deflection</td>
<td>$\Delta = 20$ cm</td>
</tr>
<tr>
<td>Unit weight of water</td>
<td>$\gamma_w = 10$ kN/m$^3$</td>
</tr>
<tr>
<td>Unit weight of steel</td>
<td>$\gamma_s = 80$ kN/m$^3$</td>
</tr>
<tr>
<td>Modulus of elasticity</td>
<td>$E = 210$ GPa</td>
</tr>
<tr>
<td>Moment of inertia of the column</td>
<td>$I = \frac{t}{12} [d_o^4 - (d_o - 2t)^4]$</td>
</tr>
<tr>
<td>Cross-sectional area of column material</td>
<td>$A = \pi t (d_o - t)$</td>
</tr>
<tr>
<td>Allowable bending stress</td>
<td>$\sigma_b = 165$ MPa</td>
</tr>
<tr>
<td>Allowable axial stress</td>
<td>$\sigma_a = \frac{12t^2E}{92(H/r)^2}$ (calculated using the critical buckling load with a factor of safety of 23/12)</td>
</tr>
<tr>
<td>Radius of gyration</td>
<td>$r = \sqrt{I/A}$</td>
</tr>
<tr>
<td>Average thickness of tank wall</td>
<td>$t_r = 1.5$ cm</td>
</tr>
<tr>
<td>Volume of tank</td>
<td>$V = 1.2\pi D^2 h$</td>
</tr>
<tr>
<td>Surface area of tank</td>
<td>$A_s = 1.25\pi D^2$</td>
</tr>
<tr>
<td>Projected area of tank, for wind loading</td>
<td>$A_p = \frac{2Dh}{3}$</td>
</tr>
<tr>
<td>Load on the column due to weight of water and steel tank</td>
<td>$P = V\gamma_w + A_s\gamma_s$</td>
</tr>
<tr>
<td>Lateral load at the tank C.G. due to wind pressure</td>
<td>$W = wA_p$</td>
</tr>
</tbody>
</table>

### Figure E3.51

Graphic of a water tower support column.
Deflection at C.G. of tank

\[ \delta = \delta_1 + \delta_2, \text{ where} \]

\[ \delta_1 = \frac{WH^2}{12EI}(4H + 3h) \]

\[ \delta_2 = \frac{H}{2E}(0.5Wh + Pe)(H + h) \]

Moment at base

\[ M = WH + 0.5h) + (\delta + e)P \]

Bending stress

\[ f_b = \frac{M}{2I}d_o \]

Axial stress

\[ f_a = \frac{P}{A} = \frac{V\gamma_w + A\gamma_s h}{\pi(t(d_o - t))} \]

Combined stress constraint

\[ \frac{f_a}{\sigma_a} + \frac{f_b}{\sigma_b} \leq 1 \]

Gravitational acceleration

\[ g = 9.81 \text{ m/s}^2 \]

*3.52 Design of a flag pole. Your consulting firm has been asked to design a minimum-mass flag pole of height \( H \). The pole will be made of uniform hollow circular tubing with \( d_o \) and \( d_i \) as outer and inner diameters, respectively. The pole must not fail under the action of high winds.

For design purposes, the pole will be treated as a cantilever that is subjected to a uniform lateral wind load of \( w \) (kN/m). In addition to the uniform load, the wind induces a concentrated load of \( P \) (kN) at the top of the pole, as shown in Figure E3.52. The flag pole must not fail in bending or shear. The deflection at the top should not exceed 10 cm. The ratio of mean diameter to thickness must not exceed 60. The pertinent data are given in the table that follows. Assume any other data if needed. The minimum and maximum values of design variables are \( 5 \leq d_o \leq 50 \text{ cm} \) and \( 4 \leq d_i \leq 45 \text{ cm} \).

Formulate the design problem and solve it using the graphical optimization technique.

**Pertinent constants and equations**

<table>
<thead>
<tr>
<th>Cross-sectional area</th>
<th>( A = \frac{\pi}{4}(d_o^2 - d_i^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment of inertia</td>
<td>( I = \frac{\pi}{4}(d_o^4 - d_i^4) )</td>
</tr>
<tr>
<td>Modulus of elasticity</td>
<td>( E = 210 \text{ GPa} )</td>
</tr>
<tr>
<td>Allowable bending stress</td>
<td>( \sigma_b = 165 \text{ MPa} )</td>
</tr>
<tr>
<td>Allowable shear stress</td>
<td>( \tau_s = 50 \text{ MPa} )</td>
</tr>
<tr>
<td>Mass density of pole material</td>
<td>( \rho = 7800 \text{ kg/m}^3 )</td>
</tr>
<tr>
<td>Wind load</td>
<td>( w = 2.0 \text{ kN/m} )</td>
</tr>
<tr>
<td>Height of flag pole</td>
<td>( H = 10 \text{ m} )</td>
</tr>
<tr>
<td>Concentrated load at top</td>
<td>( P = 4.0 \text{ kN} )</td>
</tr>
<tr>
<td>Moment at base</td>
<td>( M = (PH + 0.5wH^2) ), kN \cdot m</td>
</tr>
<tr>
<td>Bending stress</td>
<td>( \sigma = \frac{M}{2I}d_o ), kPa</td>
</tr>
<tr>
<td>Shear at base</td>
<td>( S = (P + wH), \text{ kN} )</td>
</tr>
<tr>
<td>Shear stress</td>
<td>( \tau = \frac{S}{12I}(d_o^2 + d_o d_i + d_i^2), \text{ kPa} )</td>
</tr>
<tr>
<td>Deflection at top</td>
<td>( \delta = \frac{PH^3}{3EI} + \frac{wH^4}{8EI} )</td>
</tr>
<tr>
<td>Minimum and maximum thickness</td>
<td>( 0.5 ) and ( 2 \text{ cm} )</td>
</tr>
</tbody>
</table>

I. THE BASIC CONCEPTS
3.53 Design of a sign support column. A company’s design department has been asked to design a support column of minimum weight for the sign shown in Figure E3.53. The height to the bottom of the sign \( H \), the width \( b \), and the wind pressure \( p \) on the sign are as follows: \( H = 20 \text{ m} \), \( b = 8 \text{ m} \), \( p = 800 \text{ N/m}^2 \).

The sign itself weighs 2.5 kN/m\(^2\)(w). The column must be safe with respect to combined axial and bending stresses. The allowable axial stress includes a factor of safety with respect to buckling. To prevent local buckling of the plate, the diameter/thickness ratio \( d_o/t \) must not exceed 92. Note that the bending stress in the column will increase as a result of the deflection of the sign under the wind load. The maximum deflection at the sign’s center of gravity should not exceed 0.1 m. The minimum and maximum values of design variables are \( 25 \leq d_o \leq 150 \text{ cm} \) and \( 0.5 \leq t \leq 10 \text{ cm} \) (created by H. Kane).
Pertinent constants and equations

Height of sign \( h = 4.0 \text{ m} \)

Cross-sectional area \( A = \frac{\pi}{4} \left( d_o^2 - (d_o - 2t)^2 \right) \)

Moment of inertia \( I = \frac{\pi}{64} \left( d_o^6 - (d_o - 2t)^4 \right) \)

Radius of gyration \( r = \sqrt{I/A} \)

Young’s modulus (aluminum alloy) \( E = 75 \text{ GPa} \)

Unit weight of aluminum \( \gamma = 27 \text{ kN/m}^3 \)

Allowable bending stress \( \sigma_b = 140 \text{ MPa} \)

Allowable axial stress \( \sigma_a = \frac{12\pi^2E}{92(H/r)^2} \)

Wind force \( F = pbh \)

Weight of sign \( W = wbh \)

Deflection at center of gravity of sign \( \delta = \frac{F}{EI} \left( \frac{H^3}{3} + \frac{H^2h}{2} + \frac{Hh^2}{4} \right) \)

Bending stress in column \( f_b = \frac{M}{2I} d_o \)

Axial stress \( f_a = \frac{W}{A} \)

Moment at base \( M = F \left( H + \frac{h}{2} \right) + W\delta \)

Combined stress requirement \( \frac{f_a}{\sigma_a} + \frac{f_b}{\sigma_b} \leq 1 \)

*3.54 Design of a tripod. Design a minimum mass tripod of height \( H \) to support a vertical load \( W = 60 \text{ kN} \). The tripod base is an equilateral triangle with sides \( B = 1200 \text{ mm} \). The struts have a solid circular cross-section of diameter \( D \) (Figure E3.54).

![Figure E3.54 Tripod.](image-url)
The axial stress in the struts must not exceed the allowable stress in compression, and the axial load in the strut $P$ must not exceed the critical buckling load $P_{cr}$ divided by a safety factor $FS = 2$. Use consistent units of Newtons and centimeters. The minimum and maximum values for the design variables are $0.5 \leq H \leq 5 \text{ m}$ and $0.5 \leq D \leq 50 \text{ cm}$. Material properties and other relationships are given next:

<table>
<thead>
<tr>
<th>Material</th>
<th>aluminum alloy 2014-T6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allowable compressive stress</td>
<td>$\sigma_a = 150 \text{ MPa}$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E = 75 \text{ GPa}$</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho = 2800 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Strut length</td>
<td>$l = \left(H^2 + \frac{1}{3}B^2\right)^{0.5}$</td>
</tr>
<tr>
<td>Critical buckling load</td>
<td>$P_{cr} = \frac{\pi^2 EI}{l^2}$</td>
</tr>
<tr>
<td>Moment of inertia</td>
<td>$I = \frac{\pi}{64}D^4$</td>
</tr>
<tr>
<td>Strut load</td>
<td>$P = \frac{Wl}{3Hl}$</td>
</tr>
</tbody>
</table>

I. THE BASIC CONCEPTS