DIFFERENTIAL EQUATIONS, DYNAMICAL SYSTEMS, AND AN INTRODUCTION TO CHAOS

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The main new features in this edition consist of a number of additional explo-
lations together with numerous proof simplifications and revisions. The new
explorations include a sojourn into numerical methods that highlights how
these methods sometimes fail, which in turn provides an early glimpse of
chaotic behavior. Another new exploration involves the previously treated
SIR model of infectious diseases, only now considered with zombies as the
infected population. A third new exploration involves explaining the motion
of a glider.

This edition has benefited from numerous helpful comments from a variety
of readers. Special thanks are due to Jamil Gomes de Abreu, Eric Adams, Adam
Leighton, Tiennyu Ma, Lluis Fernand Mello, Bogdan Przeradzki, Charles
Pugh, Hal Smith, and Richard Venti for their valuable insights and corrections.
Preface

In the thirty years since the publication of the first edition of this book, much has changed in the field of mathematics known as dynamical systems. In the early 1970s, we had very little access to high-speed computers and computer graphics. The word chaos had never been used in a mathematical setting. Most of the interest in the theory of differential equations and dynamical systems was confined to a relatively small group of mathematicians.

Things have changed dramatically in the ensuing three decades. Computers are everywhere, and software packages that can be used to approximate solutions of differential equations and view the results graphically are widely available. As a consequence, the analysis of nonlinear systems of differential equations is much more accessible than it once was. The discovery of complicated dynamical systems, such as the horseshoe map, homoclinic tangles, the Lorenz system, and their mathematical analysis, convinced scientists that simple stable motions such as equilibria or periodic solutions were not always the most important behavior of solutions of differential equations. The beauty and relative accessibility of these chaotic phenomena motivated scientists and engineers in many disciplines to look more carefully at the important differential equations in their own fields. In many cases, they found chaotic behavior in these systems as well.

Now dynamical systems phenomena appear in virtually every area of science, from the oscillating Belousov–Zhabotinsky reaction in chemistry to the chaotic Chua circuit in electrical engineering, from complicated motions in celestial mechanics to the bifurcations arising in ecological systems.
As a consequence, the audience for a text on differential equations and
dynamical systems is considerably larger and more diverse than it was in the
1970s. We have accordingly made several major structural changes to this
book, including:

1. The treatment of linear algebra has been scaled back. We have dispensed
with the generalities involved with abstract vector spaces and normed lin-
ear spaces. We no longer include a complete proof of the reduction of all
$n \times n$ matrices to canonical form. Rather, we deal primarily with matrices
no larger than $4 \times 4$.
2. We have included a detailed discussion of the chaotic behavior in the
Lorenz attractor, the Shil’nikov system, and the double-scroll attractor.
3. Many new applications are included; previous applications have been
updated.
4. There are now several chapters dealing with discrete dynamical systems.
5. We deal primarily with systems that are $C^\infty$, thereby simplifying many of
the hypotheses of theorems.

This book consists of three main parts. The first deals with linear systems of
differential equations together with some first-order nonlinear equations. The
second is the main part of the text: here we concentrate on nonlinear systems,
primarily two-dimensional, as well as applications of these systems in a wide
variety of fields. Part three deals with higher dimensional systems. Here we
emphasize the types of chaotic behavior that do not occur in planar systems,
as well as the principal means of studying such behavior—the reduction to a
discrete dynamical system.

Writing a book for a diverse audience whose backgrounds vary greatly poses
a significant challenge. We view this one as a text for a second course in differ-
etial equations that is aimed not only at mathematicians, but also at scientists
and engineers who are seeking to develop sufficient mathematical skills to
analyze the types of differential equations that arise in their disciplines.

Many who come to this book will have strong backgrounds in linear algebra
and real analysis, but others will have less exposure to these fields. To make
this text accessible to both groups, we begin with a fairly gentle introduction
to low-dimensional systems of differential equations. Much of this will be a
review for readers with a more thorough background in differential equations,
so we intersperse some new topics throughout the early part of the book for
those readers.

For example, the first chapter deals with first-order equations. We begin
it with a discussion of linear differential equations and the logistic popula-
tion model, topics that should be familiar to anyone who has a rudimentary
acquaintance with differential equations. Beyond this review, we discuss the
logistic model with harvesting, both constant and periodic. This allows us to
introduce bifurcations at an early stage as well as to describe Poincaré maps
and periodic solutions. These are topics that are not usually found in elementary differential equations courses, yet they are accessible to anyone with a background in multivariable calculus. Of course, readers with a limited background may wish to skip these specialized topics at first and concentrate on the more elementary material.

Chapters 2 through 6 deal with linear systems of differential equations. Again we begin slowly, with Chapters 2 and 3 dealing only with planar systems of differential equations and two-dimensional linear algebra. Chapters 5 and 6 introduce higher dimensional linear systems; however, our emphasis remains on three- and four-dimensional systems rather than completely general \( n \)-dimensional systems, even though many of the techniques we describe extend easily to higher dimensions.

The core of the book lies in the second part. Here, we turn our attention to nonlinear systems. Unlike linear systems, nonlinear systems present some serious theoretical difficulties such as existence and uniqueness of solutions, dependence of solutions on initial conditions and parameters, and the like. Rather than plunge immediately into these difficult theoretical questions, which require a solid background in real analysis, we simply state the important results in Chapter 7 and present a collection of examples that illustrate what these theorems say (and do not say). Proofs of all of the results are included in the final chapter of the book.

In the first few chapters in the nonlinear part of the book, we introduce important techniques such as linearization near equilibria, nullcline analysis, stability properties, limit sets, and bifurcation theory. In the latter half of this part, we apply these ideas to a variety of systems that arise in biology, electrical engineering, mechanics, and other fields.

Many of the chapters conclude with a section called “Exploration.” These sections consist of a series of questions and numerical investigations dealing with a particular topic or application relevant to the preceding material. In each Exploration we give a brief introduction to the topic at hand and provide references for further reading about this subject. But, we leave it to the reader to tackle the behavior of the resulting system using the material presented earlier. We often provide a series of introductory problems as well as hints as to how to proceed, but in many cases, a full analysis of the system could become a major research project. You will not find “answers in the back of the book” for the questions; in many cases, nobody knows the complete answer. (Except, of course, you!)

The final part of the book is devoted to the complicated nonlinear behavior of higher dimensional systems known as chaotic behavior. We introduce these ideas via the famous Lorenz system of differential equations. As is often the case in dimensions three and higher, we reduce the problem of comprehending the complicated behavior of this differential equation to that of understanding the dynamics of a discrete dynamical system or iterated
function. So we then take a detour into the world of discrete systems, dis-
cussing along the way how symbolic dynamics can be used to describe certain
chaotic systems completely. We then return to nonlinear differential equations
to apply these techniques to other chaotic systems, including those that arise
when homoclinic orbits are present.

We maintain a website at math.bu.edu/hsd devoted to issues regarding
this text. Look here for errata, suggestions, and other topics of interest to
teachers and students of differential equations. We welcome any contributions
from readers at this site.