1.1. INTRODUCTION

A physical phenomenon of a reasonably straight, slender member (or body) bending laterally (usually abruptly) from its longitudinal position due to compression is referred to as buckling. The term buckling is used by engineers as well as laypeople without thinking too deeply. A careful examination reveals that there are two kinds of buckling: (1) bifurcation-type buckling; and (2) deflection-amplification-type buckling. In fact, most, if not all, buckling phenomena in the real-life situation are the deflection-amplification type. A bifurcation-type buckling is a purely conceptual one that occurs in a perfectly straight (geometry) homogeneous (material) member subjected to a compressive loading of which the resultant must pass...
though the centroidal axis of the member (concentric loading). It is highly unlikely that any ordinary column will meet these three conditions perfectly. Hence, it is highly unlikely that anyone has ever witnessed a bifurcation-type buckling phenomenon. Although, in a laboratory setting, one could demonstrate setting a deflection-amplification-type buckling action that is extremely close to the bifurcation-type buckling. Simulating those three conditions perfectly even in a laboratory environment is not probable.

Structural members resisting tension, shear, torsion, or even short stocky columns fail when the stress in the member reaches a certain limiting strength of the material. Therefore, once the limiting strength of material is known, it is a relatively simple matter to determine the load-carrying capacity of the member. Buckling, both the bifurcation and the deflection-amplification type, does not take place as a result of the resisting stress reaching a limiting strength of the material. The stress at which buckling occurs depends on a variety of factors ranging from the dimensions of the member to the boundary conditions to the properties of the material of the member. Determining the buckling stress is a fairly complex undertaking.

If buckling does not take place because certain strength of the material is exceeded, then, why, one may ask, does a compression member buckle? Chajes (1974) gives credit to Salvadori and Heller (1963) for clearly elucidating the phenomenon of buckling, a question not so easily and directly explainable, by quoting the following from *Structure in Architecture*:

> A slender column shortens when compressed by a weight applied to its top, and, in so doing, lowers the weight’s position. The tendency of all weights to lower their position is a basic law of nature. It is another basic law of nature that, whenever there is a choice between different paths, a physical phenomenon will follow the easiest path. Confronted with the choice of bending out or shortening, the column finds it easier to shorten for relatively small loads and to bend out for relatively large loads. In other words, when the load reaches its buckling value the column finds it easier to lower the load by bending than by shortening.

Although these remarks will seem excellent to most laypeople, they do contain nontechnical terms such as choice, easier, and easiest, flavoring the subjective nature. It will be proved later that buckling is a phenomenon that can be explained with fundamental natural principles.

If bifurcation-type buckling does not take place because the aforementioned three conditions are not likely to be simulated, then why, one may ask, has so much research effort been devoted to study of this phenomenon? The bifurcation-type buckling load, the critical load, gives
the upper-bound solution for practical columns that hardly satisfies any one of the three conditions. This will be shown later by examining the behavior of an eccentrically loaded cantilever column.

1.2. NEUTRAL EQUILIBRIUM

The concept of the stability of various forms of equilibrium of a compressed bar is frequently explained by considering the equilibrium of a ball (rigid-body) in various positions, as shown in Fig. 1-1 (Timoshenko and Gere 1961; Hoff 1956).

Although the ball is in equilibrium in each position shown, a close examination reveals that there are important differences among the three cases. If the ball in part (a) is displaced slightly from its original position of equilibrium, it will return to that position upon the removal of the disturbing force. A body that behaves in this manner is said to be in a state of stable equilibrium. In part (a), any slight displacement of the ball from its position of equilibrium will raise the center of gravity. A certain amount of work is required to produce such a displacement. The ball in part (b), if it is disturbed slightly from its position of equilibrium, does not return but continues to move down from the original equilibrium position. The equilibrium of the ball in part (b) is called unstable equilibrium. In part (b), any slight displacement from the position of equilibrium will lower the center of gravity of the ball and consequently will decrease the potential energy of the ball. Thus in the case of stable equilibrium, the energy of the system is a minimum (local), and in the case of unstable equilibrium it is a maximum (local). The ball in part (c), after being displaced slightly, neither returns to its original equilibrium position nor continues to move away upon removal of the disturbing force. This type of equilibrium is called neutral equilibrium. If the equilibrium is neutral, there is no change in energy during a displacement in the conservative force system. The response of the column is very similar to that of the ball in Fig. 1-1. The straight configuration of the column is stable at small loads, but it is unstable at large loads. It is assumed that a state of neutral equilibrium exists at the

![Figure 1-1 Stability of equilibrium](image)
transition from stable to unstable equilibrium in the column. Then the load at which the straight configuration of the column ceases to be stable is the load at which neutral equilibrium is possible. This load is usually referred to as the critical load.

To determine the critical load, eigenvalue, of a column, one must find the load under which the member can be in equilibrium, both in the straight and in a slightly bent configuration. How slightly? The magnitude of the slightly bent configuration is indeterminate. It is conceptual. This is why the free body of a column must be drawn in a slightly bent configuration. The method that bases this slightly bent configuration for evaluating the critical loads is called the method of neutral equilibrium (neighboring equilibrium, or adjacent equilibrium).

At critical loads, the primary equilibrium path (stable equilibrium, vertical) reaches a bifurcation point and branches into neutral equilibrium paths (horizontal). This type of behavior is called the buckling of bifurcation type.

1.3. EULER LOAD

It is informative to begin the formulation of the column equation with a much idealized model, the Euler\(^1\) column. The axially loaded member shown in Fig. 1-2 is assumed to be prismatic (constant cross-sectional area) and to be made of homogeneous material. In addition, the following further assumptions are made:

1. The member’s ends are pinned. The lower end is attached to an immovable hinge, and the upper end is supported in such a way that it can rotate freely and move vertically, but not horizontally.
2. The member is perfectly straight, and the load \(P\), considered positive when it causes compression, is concentric.
3. The material obeys Hooke’s law.
4. The deformations of the member are small so that the term \((y')^2\) is negligible compared to unity in the expression for the curvature, \(\frac{y''}{[1 + (y')^2]^{3/2}}\). Therefore, the curvature can be approximated by \(y''\). \(^2\)

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\(^{1}\) The Euler (1707–1783) column is due to the man who, in 1744, presented the first accurate column analysis. A brief biography of this remarkable man is given by Timoshenko (1953). Although it is customary today to refer to a simply supported column as an Euler column, Euler in fact analyzed a flag-pole-type cantilever column in his famous treatise according to Chajes (1974).

\(^{2}\) \(y'\) and \(y''\) denote the first and second derivatives of \(y\) with respect to \(x\). Note: \(|y''| < |y'|\) but \(|y'| \approx \) thousandths of a radian in elastic columns.
From the free body, part (b) in Fig. 1-2, the following becomes immediately obvious:

\[ EIy'' = -M(x) = -Py \quad \text{or} \quad EIy'' + Py = 0 \]  

Equation (1.3.2) is a second-order linear differential equation with constant coefficients. Its boundary conditions are

\[ y = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell \]  

Equations (1.3.2) and (1.3.3) define a linear eigenvalue problem. The solution of Eq. (1.3.2) will now be obtained. Let \( k^2 = P/EI \), then \( y'' + k^2y = 0 \). Assume the solution to be of a form \( y = \alpha e^{mx} \) for which \( y' = \alpha me^{mx} \) and \( y'' = \alpha m^2 e^{mx} \). Substituting these into Eq. (1.3.2) yields

\( (m^2 + k^2)\alpha e^{mx} = 0 \).

Since \( \alpha e^{mx} \) cannot be equal to zero for a nontrivial solution, \( m^2 + k^2 = 0 \), \( m = \pm ki \). Substituting gives

\[ y = C_1\alpha e^{kix} + C_2\alpha e^{-kix} = A \cos kx + B \sin kx \]

\( A \) and \( B \) are integral constants, and they can be determined by boundary conditions.
\[ y = 0 \quad \text{at } x = 0 \Rightarrow A = 0 \]
\[ y = 0 \quad \text{at } x = \ell \Rightarrow B \sin k\ell = 0 \]

As \( B \neq 0 \) (if \( B = 0 \), then it is called a trivial solution; \( 0 = 0 \)), \( \sin k\ell = 0 \Rightarrow k\ell = n\pi \)
where \( n = 1, 2, 3, \ldots \) but \( n \neq 0 \). Hence, \( k^2 = P/EI = n^2\pi^2/\ell^2 \), from which it follows immediately

\[ P_{cr} = \frac{n^2\pi^2EI}{\ell^2}\quad (n = 1, 2, 3, \ldots) \quad (1.3.4) \]

The eigenvalues \( P_{cr} \), called critical loads, denote the values of load \( P \) for which a nonzero deflection of the perfect column is possible. The deflection shapes at critical loads, representing the eigenmodes or eigenvectors, are given by

\[ y = B \sin \frac{n\pi x}{\ell} \quad (1.3.5) \]

Note that \( B \) is undetermined, including its sign; that is, the column may buckle in any direction. Hence, the magnitude of the buckling mode shape cannot be determined, which is said to be immaterial.

The smallest buckling load for a pinned prismatic column corresponding to \( n = 1 \) is

\[ P_E = \frac{\pi^2EI}{\ell^2} \quad (1.3.6) \]

If a pinned prismatic column of length \( \ell \) is going to buckle, it will buckle at \( n = 1 \) unless external bracings are provided in between the two ends.

A curve of the applied load versus the deflection at a point in a structure such as that shown in part (a) of Fig. 1-3 is called the equilibrium path. Points along the primary (initial) path (vertical) represent configurations of the column in the compressed but straight shape; those along the secondary path (horizontal) represent bent configurations. Equation (1.3.4) determines a periodic bifurcation point, and Eq. (1.3.5) represents a secondary (adjacent or neighboring) equilibrium path for each value of \( n \). On the basis of Eq. (1.3.5), the secondary path extends indefinitely in the horizontal direction. In reality, however, the deflection cannot be so large and yet satisfies the assumption of rotations to be negligibly small. As \( P \) in Eq. (1.3.4) is not a function of \( y \), the secondary path is horizontal. A finite displacement formulation to be discussed later shows that the secondary equilibrium path for the column curves upward and has a horizontal tangent at the critical load.
Note that at $P_{cr}$ the solution is not unique. This appears to be at odds with the well-known notion that the solutions to problems of classical linear elasticity are unique. It will be recalled that the equilibrium condition is determined based on the deformed geometry of the structure in part (b) of Fig. 1-2. The theory that takes into account the effect of deflection on the equilibrium conditions is called the second-order theory. The governing equation, Eq. (1.3.2), is an ordinary linear differential equation. It describes neither linear nor nonlinear responses of a structure. It describes an eigenvalue problem. Any nonzero loading term on the right-hand side of Eq. (1.3.2) will induce a second-order (nonlinear) response of the structure.

Dividing Eq. (1.3.4) by the cross-sectional area $A$ gives the critical stress

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{\ell^2 A} = \frac{\pi^2 EA r^2}{\ell^2 A} = \frac{\pi^2 E}{(\ell/r)^2}$$

(1.3.7)

where $\ell/r$ is called the slenderness ratio and $r = \sqrt{I/A}$ is the radius of gyration of the cross section. Note that the critical load and hence, the critical buckling stress is independent of the yield stress of the material. They are only the function of modulus of elasticity and the column geometry. In Fig. 1-3(b), $C_c$ is the threshold value of the slenderness ratio from which elastic buckling commences.

$$\begin{cases}
\text{eigen pair} \\
\text{eigenvalue} = P_{cr} = \frac{n^2 \pi^2 EI}{\ell^2} \\
\text{eigenvector} = y = B \sin \frac{n \pi x}{\ell}
\end{cases}$$

Figure 1-3 Euler load and critical stresses
1.4. DIFFERENTIAL EQUATIONS OF BEAM-COLUMNS

Bifurcation-type buckling is essentially flexural behavior. Therefore, the free-body diagram must be based on the deformed configuration as the examination of equilibrium is made in the neighboring equilibrium position. Summing the forces in the horizontal direction in Fig. 1-4(a) gives

\[ \sum F_y = 0 = (V + dV) - V + qdx, \]

from which it follows immediately

\[ \frac{dV}{dx} = V' = -q(x) \]  

(1.4.1)

Summing the moment at the top of the free body gives

\[ \sum M_{\text{top}} = 0 = (M + dM) - M + Vdx + pdy - q(dx) \frac{dx}{2} \]

Figure 1-4 Free-body diagrams of a beam-column
Neglecting the second-order term, leads to

\[ \frac{dM}{dx} + p \frac{dy}{dx} = -V \]  \hspace{1cm} (1.4.2)

Taking derivatives on both sides of Eq. (1.4.2) gives

\[ M'' + (py')' = -V' \]  \hspace{1cm} (1.4.3)

Since the convex side of the curve (buckled shape) is opposite from the positive \( y \) axis, \( M = EIy'' \). From Eq. (1.4.1), \( V' = -q(x) \). Hence, \((EIy'')'' + (py')' = q(x)\). For a prismatic (\( EI = \text{const} \)) beam-column subjected to a constant compressive force \( P \), the equation is simplified to

\[ EIy'' + py'' = q(x) \]  \hspace{1cm} (1.4.4)

Equation (1.4.4) is the fundamental beam-column governing differential equation.

Consider the free-body diagram shown in Fig. 1-4(d). Summing forces in the \( y \) direction gives

\[ \sum F_y = 0 = -(V + dV) + V + qdx \Rightarrow \frac{dV}{dx} = V' = q(x) \]  \hspace{1cm} (1.4.5)

Summing moments about the top of the free body yields

\[ \sum M_{top} = 0 \]

\[ = -(M + dM) + M - Vdx - pdy - qdx dx/2 \Rightarrow \]

\[ - \frac{dM}{dx} - p \frac{dy}{dx} = V \]  \hspace{1cm} (1.4.6)

For the coordinate system shown in Fig. 1-4(d), the curve represents a decreasing function (negative slope) with the convex side to the positive \( y \) direction. Hence, \(-EIy'' = M(x)\). Thus,

\[ -(-EIy'')' - (-py') = V \]  \hspace{1cm} (1.4.7)

which leads to

\[ EIy''' + py' = V \quad \text{or} \quad EIy''' + py'' = q(x) \]  \hspace{1cm} (1.4.8)

It can be shown that the free-body diagrams shown in Figs. 1-4(b) and 1-4(c) will lead to Eq. (1.4.4). Hence, the governing differential equation is independent of the shape of the free-body diagram assumed.
The homogeneous solution of Eq. (1.4.4) governs the bifurcation buckling of a column (characteristic behavior). The concept of geometric imperfection (initial crookedness), material heterogeneity, and an eccentricity is equivalent to having nonvanishing $q(x)$ terms.

Rearranging Eq. (1.4.4) gives

$$EIy'''' + py'' = 0 \Rightarrow y'''' + k^2y'' = 0, \quad \text{where } k^2 = \frac{p}{EI}$$

Assuming the solution to be of a form $y = \alpha e^{mx}$, then $y' = \alpha me^{mx}$, $y'' = \alpha m^2e^{mx}, y''' = \alpha m^3e^{mx}$, and $y'''' = \alpha m^4e^x$. Substituting these derivatives back to the simplified homogeneous differential equation yields

$$\alpha m^4 e^{mx} + \alpha k^2 m^2 e^{mx} = 0 \Rightarrow \alpha e^{mx} (m^4 + k^2 m^2) = 0$$

Since $\alpha \neq 0$ and $e^{mx} \neq 0 \Rightarrow m^2(m^2 + k^2) = 0 \Rightarrow m = \pm 0, \pm ki$. Hence,

$$y_h = c_1 e^{kix} + c_2 e^{-kix} + c_3xe^0 + c_4 e^0$$

Know the mathematical identities

$$e^{ikx} = \cos kx + i \sin kx$$
$$e^{-ikx} = \cos kx - i \sin kx$$

Hence, $y_h = A \sin kx + B \cos kx + Cx + D$ where integral constants $A$, $B$, $C$, and $D$ can be determined uniquely by applying proper boundary conditions of the structure.

**Example 1** Consider a both-ends-fixed column shown in Fig. 1-5.
\[ y' = Ak \cos kx - Bk \sin kx + C \]

\[ y'' = -Ak^2 \sin kx - Bk^2kx \]

\[ y = 0 \quad \text{at} \quad x = 0 \Rightarrow B + D = 0 \]

\[ y' = 0 \quad \text{at} \quad x = 0 \Rightarrow Ak + C = 0 \]

\[ y = 0 \quad \text{at} \quad x = \ell \Rightarrow A \sin k\ell + B \cos k\ell + C\ell + D = 0 \]

\[ y' = 0 \quad \text{at} \quad x = \ell \Rightarrow Ak \cos k\ell - Bk \sin k\ell + C = 0 \]

For a nontrivial solution for \( A, B, C, \) and \( D \) (or the stability condition equation), the determinant of coefficients must vanish. Hence,

\[
\begin{vmatrix}
0 & 1 & 0 & 1 \\
k & 0 & 1 & 0 \\
\sin k\ell & \cos k\ell & \ell & 1 \\
k \cos k\ell & -k \sin k\ell & 1 & 0
\end{vmatrix} = 0
\]

Expanding the determinant (Maple®) gives

\[ 2(\cos k\ell - 1) + k\ell \sin k\ell = 0 \]

Know the following mathematical identities:

\[
\left\{
\begin{aligned}
\sin k\ell &= \sin \left(\frac{k\ell}{2} + \frac{k\ell}{2}\right) = \sin \frac{k\ell}{2} \cos \frac{k\ell}{2} + \cos \frac{k\ell}{2} \sin \frac{k\ell}{2} = 2 \sin \frac{k\ell}{2} \cos \frac{k\ell}{2} \\
\cos k\ell &= \cos \left(\frac{k\ell}{2} + \frac{k\ell}{2}\right) = \cos \frac{k\ell}{2} \cos \frac{k\ell}{2} - \sin \frac{k\ell}{2} \sin \frac{k\ell}{2} = 1 - 2 \sin^2 \frac{k\ell}{2} \\
\Rightarrow \cos k\ell - 1 &= -2 \sin^2 \frac{k\ell}{2}
\end{aligned}
\right.
\]

Rearranging the determinant given above yields:

\[
2 \left( -2 \sin^2 \frac{k\ell}{2} \right) + k\ell \left( 2 \sin \frac{k\ell}{2} \cos \frac{k\ell}{2} \right) = 0
\]

\[
\Rightarrow \sin \frac{k\ell}{2} \left( \frac{k\ell}{2} \cos \frac{k\ell}{2} - \sin \frac{k\ell}{2} \right) = 0
\]
Let \( u = k\ell/2 \), then the solution becomes \( \sin u = 0 \) and \( \tan u = u \). For \( \sin u = 0 \Rightarrow u = n\pi \) or \( k\ell = 2n\pi \Rightarrow p_{cr} = 4n^2\pi^2EI/\ell^2 \). Substituting the eigenvalue \( k = 2n\pi/\ell \) into the buckling mode shape yields

\[
y = \epsilon_1 \sin \frac{2n\pi x}{\ell} + \epsilon_2 \cos \frac{2n\pi x}{\ell} + \epsilon_3 x + \epsilon_4
\]

\[y = 0 \text{ at } x = 0 \Rightarrow 0 = \epsilon_2 + \epsilon_4 \Rightarrow \epsilon_4 = -\epsilon_2 \] Hence, \( y = \epsilon_1 \sin \left(\frac{2n\pi x}{\ell}\right) + \epsilon_2 \left(\cos \left(\frac{2n\pi x}{\ell}\right) - 1\right) + \epsilon_3 x \)

\[y = 0 \text{ at } x = \ell \Rightarrow 0 = \epsilon_1 \sin 2n\pi + \epsilon_2 \left(\cos 2n\pi - 1\right) + \epsilon_3 \ell \Rightarrow \epsilon_3 = 0 \]

\[
y' = -\frac{2n\pi}{\ell} \epsilon_2 \sin \frac{2n\pi x}{\ell} + \frac{2n\pi}{\ell} \epsilon_1 \cos \frac{2n\pi x}{\ell}
\]

\[y' = 0 \text{ at } x = 0 \Rightarrow y' = 0 + \frac{2n\pi}{\ell} \epsilon_1 \Rightarrow \epsilon_1 = 0 \]

Hence, \( y = \epsilon_2 \left(\cos \left(\frac{2n\pi x}{\ell}\right) - 1\right) \) \( \Leftarrow \) eigenvector or mode shape as shown in Fig. 1-6.

If \( n = 1 \),

\[
p_{cr} = \frac{\pi^2 EI}{\left(\frac{\ell}{2}\right)^2} = \frac{\pi^2 EI}{(\ell_e)^2}
\]

where \( \ell_e = \ell/2 \) is called the effective buckling length of the column. For \( \tan u = u \), the smallest nonzero root can be readily computed using Maple®. In the old days, it was a formidable task to solve such a simple transcendental equation. Hence, a graphical solution method was frequently employed, as shown in Fig. 1-7.
From Maple® output, the smallest nonzero root is

\[ u = 4.4934094 \Rightarrow \frac{k\ell}{2} = 4.493 \Rightarrow k\ell = 8.9868 \Rightarrow k^2\ell^2 = 80.763 \]

\[ P_{\sigma} = \frac{80.763EI}{\ell^2} = \frac{8.183\pi^2EI}{\ell^2} = \frac{\pi^2EI}{(0.349578\ell)^2} = \frac{\pi^2EI}{[0.699156(0.5\ell)]^2} \]

The corresponding mode shape is shown in Fig. 1-8.

**Example 2** Propped Column as shown in Fig. 1-9.

\[ y = A \sin kx + B \cos kx + Cx + D \]

\[ y' = Ak \cos kx - Bk \sin kx + C \]

\[ y'' = -Ak^2 \sin kx - Bk^2 \cos kx \]
\[ y = 0 \quad \text{at} \quad x = 0 \Rightarrow B + D = 0 \]
\[ y'' = 0 \quad \text{at} \quad x = 0 \Rightarrow B = 0 \Rightarrow D = 0 \]
\[ y = 0 \quad \text{at} \quad x = \ell \Rightarrow A \sin k\ell + C\ell = 0 \Rightarrow C = \frac{1}{\ell} \sin k\ell A \]
\[ y' = 0 \quad \text{at} \quad x = \ell \Rightarrow ak \cos k\ell + C = 0 \Rightarrow C = -Ak \cos k\ell \]

Equating for C gives \(-Ak \cos k\ell = -A\frac{1}{\ell} \sin k\ell \Rightarrow \tan k\ell = k\ell\)

Let \(u = k\ell \Rightarrow \tan u = u\), then from the previous example, \(u = 4.9340945\)

\[ kl = 4.934 = \sqrt{\frac{P}{EI}\ell} \]

\[ P_{cr} = \frac{20.19EI}{\ell^2} = \frac{2.04575\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(0.699155\ell)^2} \]

Substituting the eigenvalue of \(k = 4.934/\ell\) into the eigenvector gives

\[ y = A \sin kx - \left( \frac{A}{\ell} \sin k\ell \right) x = A \left[ \sin \left( \frac{4.934x}{\ell} \right) - \left( \frac{1}{\ell} \sin 4.934 \right) x \right] \]

\[ y_i|_{x=0.699\ell} = A \left[ -0.30246 - (-9.7755 \times 10^{-1} \times 0.699155) \right] \]
\[ = A(0.3796) > 0 \]
Summing the moment at the inflection point yields

$$\sum M|_{x=0.699\ell} = 0 = \frac{20.19EI}{\ell^2}A(0.3796) - R(0.699155\ell) \Rightarrow$$

$$R = 11 \frac{EI}{\ell^3} A \neq 0$$

For $W10 \times 49$, $I_y = 93.4 \text{ in}^4$, $r_y = 2.54 \text{ in}$, say $\ell = 25 \text{ ft} = 300 \text{ in}$, $Area = 14.4 \text{ in}^2$

If it is assumed that this column has initial imperfection of $\ell/250$ at the inflection point, then

$$y_x=0.699155\ell = \ell/250 = 300/250 = 1.2 \text{ in} \Rightarrow A = 1.758$$

Then, $R = 11 \times \left( \frac{(29 \times 10^3 \times 93.4)/300^3}{300} \right) \times 1.758 = 1.94 \text{ kips}$

$$\frac{k\ell}{r} = \frac{1 \times 0.699155 \times 300}{2.54} = 82.6 \Rightarrow F_{cr} = 15.6 \text{ ksi} \Rightarrow P_{cr} = 224.6 \text{ kips}$$

$$R = 1.94/224.6 \times 100 = 0.86% < 2\% \Leftarrow \text{rule of thumb}$$

1.5. EFFECTS OF BOUNDARY CONDITIONS ON THE COLUMN STRENGTH

The critical column buckling load on the same column can be increased in two ways.

1. Change the boundary conditions such that the new boundary condition will make the effective length shorter.
   - (a) pinned-pinned $\Rightarrow \ell_c = \ell$
   - (b) pinned-fixed $\Rightarrow \ell_c = 0.7 \ell$
   - (c) fixed-fixed $\Rightarrow \ell_c = 0.5 \ell$
   - (d) flag pole (cantilever) $\Rightarrow \ell_c = 2.0 \ell$, etc.

2. Provide intermediate bracing to make the column buckle in higher modes $\Rightarrow$ achieve shorter effective length.

Consider an elastically constrained column $AB$ shown in Fig. 1–10.

The two members, $AB$ and $BC$, are assumed to have identical member length and flexural rigidity for simplicity. The moments, $m$ and $M$, are due to the rotation at point $B$ and possibly due to the axial shortening of member $AB$.

Since $Q = (M + m)/\ell << p_{cr}$, $Q$ is set equal to zero and the effect of any axial shortening is neglected.
Summing moment at the top of the free body gives
(from the left free body) (from the right free body)
\[ M(x) + Py - \frac{mx}{\ell} = 0 \]
\[ M(x) - P(-y) - \frac{mx}{\ell} = 0 \]

\[ EIy'' = -M(x) = -\left(Py - \frac{mx}{\ell}\right) \quad EIy'' = M(x) = -Py + \frac{mx}{\ell} \]

As expected, the assumed deformed shape does not affect the Governing Differential Equation (GDE) of the behavior of member \(AB\).

\[ EIy'' + Py = \frac{mx}{\ell} \]

Let \( k^2 = \frac{P}{EI} \Rightarrow y'' + k^2 y = (\frac{mx}{\ell P}) k^2 \)

The general solution to this DE is given
\[ y = A \sin kx + B \cos kx + \frac{m}{\ell P}x \]
\[ y = 0 \quad \text{at} \quad x = 0 \Rightarrow B = 0 \]
\[ y = 0 \quad \text{at} \quad x = \ell \Rightarrow A = -\frac{m}{P \sin k \ell} \]
\[ y = \frac{m}{P} \left( \frac{x}{\ell} - \sin k x \right) \Rightarrow \text{buckling mode shape} \]

Since joint B is assumed to be rigid, continuity must be preserved. That is
\[ \frac{dy}{dx}|_{col} = \frac{dy}{dx}|_{bm} \]
for col
\[ \frac{dy}{dx}|_{x=\ell} = \frac{m}{P} \left( \frac{1}{\ell} - k \frac{\cos kx}{\sin k \ell} \right) = \frac{m}{P} \left( \frac{1}{\ell} - \frac{k}{\tan k \ell} \right) \]
\[ = \frac{m}{kEI} \left( \frac{1}{k \ell} - \frac{1}{\tan k \ell} \right) \]
for beam
\[ \frac{dy}{dx}|_{x=0} = \theta_N = \frac{m \ell}{4EI} \]

Recall the slope deflection equation:
\[ m = (2EI/\ell)(2\theta_N + \theta_F - \beta \phi) \Rightarrow \theta_N = m\ell/4EI \]

Equating the two slopes at joint B gives
\[ \frac{m\ell}{4EI} = -\frac{m}{kEI} \left( \frac{1}{k \ell} - \frac{1}{\tan k \ell} \right) \Leftarrow \text{Note the direction of rotation at joint B!} \]

If the frame is made of the same material, then
\[ \frac{\ell}{4I_b} = -\frac{1}{kI_c} \left( \frac{1}{k \ell} - \frac{1}{\tan k \ell} \right) \quad \text{or} \quad \frac{k\ell}{4} = -\frac{I_b}{I_c} \left( \frac{1}{k \ell} - \frac{1}{\tan k \ell} \right) \Rightarrow \text{stability condition equation} \]

Rearranging the stability condition equation gives
\[ \frac{k\ell I_c}{4I_b} = -\frac{1}{k \ell} + \frac{1}{\tan k \ell} \Rightarrow \frac{1}{k \ell} + \frac{k\ell I_c}{4I_b} = \frac{4I_b + (k\ell)^2 I_c}{k\ell 4I_b} \Rightarrow \frac{\tan k \ell}{4I_b + (k\ell)^2 I_c} \]
If \( I_b = 0 \) \( \Rightarrow \) \( P_{cr} = \frac{\pi^2 EI_c}{\ell^2} \)

If \( I_b = \infty \) \( \Rightarrow \) \( P_{cr} = \frac{2\pi^2 EI_c}{\ell^2} \)

For \( I_b = I_c \), then \( \tan k\ell = \frac{4k\ell}{4 + (k\ell)^2} \), the smallest root of this equation is \( k\ell = 3.8289 \).

\[
P_{cr} = \frac{14.66EI_c}{\ell^2} = \frac{1.485\pi^2 EI_c}{\ell^2} \Rightarrow \text{as expected } 1 < 1.485 < 2.
\]

1.6. INTRODUCTION TO CALCULUS OF VARIATIONS

The calculus of variations is a generalization of the minimum and maximum problem of ordinary calculus. It seeks to determine a function, \( y = f(x) \), that minimizes/maximizes a definite integral

\[
I = \int_{x_1}^{x_2} F(x, y, y', y'', \ldots) \, dx
\]

which is called a functional (function of functions) and whose integrand contains \( y \) and its derivatives and the independent variable \( x \).

Although the calculus of variations is similar to the maximum and minimum problems of ordinary calculus, it does differ in one important aspect. In ordinary calculus, one obtains the actual value of a variable for which a given function has an extreme point. In the calculus of variations, one does not obtain a function that provides extreme value for a given

![Figure 1-12 Deformed shape of column (in neighboring equilibrium)]
integral (functional). Instead, one only obtains the governing differential equation that the function must satisfy to make the given function have a stationary value. Hence, the calculus of variations is not a computational tool, but it is only a device for obtaining the governing differential equation of the physical stationary value problem.

The bifurcation buckling behavior of a both-end-pinned column shown in Fig. 1-12 may be examined in two different perspectives. Consider first that the static deformation prior to buckling has taken place and the examination is being conducted in the neighboring equilibrium position where the axial compressive load has reached the critical value and the column bifurcates (is disturbed) without any further increase of the load. The strain energy stored in the elastic body due to this flexural action is

\[
U = \frac{1}{2} \int_v \sigma^T \varepsilon \, dv = \frac{1}{2} \int_v \left( \frac{E h''}{I} y \right) (y''y) dv
\]

\[
= \frac{E}{2} \int_\ell (y'')^2 \int_\ell (y)^2 dAd\ell = \frac{EI}{2} \int_0^\ell (y'')^2 \, dx
\]

In calculating the strain energy, the contributions from the shear strains are generally neglected as they are very small compared to those from normal strains.\(^3\)

Neglecting the small axial shortening prior to buckling \((\Delta_s < \varepsilon \ell)\) where \(\varepsilon < 0.0005"/"\), hence \(\Delta_s < 0.05 \% \) of \(\ell\), the vertical distance, \(\Delta_b\), due to the flexural action can be computed as

\[
\Delta_b = \int_0^\ell ds - \ell = \int_0^\ell \sqrt{dx^2 + dy^2} - \ell = \int_0^\ell \sqrt{1 + (y')^2} \, dx - \ell
\]

\[
= \int_0^\ell \left[ 1 + \frac{1}{2} (y')^2 \right] \, dx - \ell = \frac{1}{2} \int_0^\ell (y')^2 \, dx
\]

Hence, the change (loss) in potential energy of the critical load is

\[
V = -\frac{1}{2} P \int_0^\ell (y')^2 \, dx
\]

\(^3\) Of course, the shear strains can be included in the formulation. The resulting equation is called the differential equation, considering the effect of shear deformations.
and the total potential energy functional becomes

\[ I = \Pi = U + V = \frac{EI}{2} \int_0^\ell (y'')^2 \, dx - \frac{1}{2} P \int_0^\ell (y')^2 \, dx \]  

(1.6.4)

Now the task is to find a function, \( y = f(x) \) which will make the total functional, \( \pi \), have a stationary value.

\[ \delta \Pi = \delta (U + V) \]

\[ = 0 \iff \text{necessary condition for equilibrium or stationary value} \]

\[ \delta^2 \Pi \]

\[ \begin{cases} > 0 \iff \text{minimum value or stable equilibrium} \\ < 0 \iff \text{maximum value or unstable equilibrium} \\ = 0 \iff \text{neutral or neutral equilibrium} \end{cases} \]

\[ \iff \text{sufficient condition} \]

If one chooses an arbitrary function, \( \bar{y}(x) \), which only satisfies the boundary conditions (geometric) and lets \( y(x) \) be the real exact function, then

\[ \bar{y}(x) = y(x) + \varepsilon \eta(x) \]  

(1.6.5)

where \( \varepsilon = \text{small number} \) and \( \eta(x) = \text{twice differentiable function satisfying the geometric boundary conditions} \). A graphical representation of the above statement is as follows:

\[ \text{Figure 1-13 Varied path} \]

If one expresses the total potential energy functional in terms of the generalized (arbitrarily chosen) displacement, \( \bar{y}(x) \), then

\[ \Pi = U + V = \int_0^\ell \left[ \frac{EI}{2} (y'')^2 + \varepsilon \eta'' \right]^2 - \frac{P}{2} (y' + \varepsilon \eta')^2 \, dx \]  

(1.6.6)

Note that \( \pi \) is a function of \( \varepsilon \) for a given \( \eta(x) \). If \( \varepsilon = 0 \), then \( \bar{y}(x) = y(x) \), which is the curve that provides a stationary value to \( \pi \). For this to happen
Differentiating Eq. (1.6.6) under the integral sign leads to

\[ \left. \frac{d(U + V)}{de} \right|_{e=0} = 0 \]  

(1.6.7)

Making use of Eq. (1.6.7) yields

\[ \int_0^\ell (EIy'' - Py') dx = 0 \]  

(1.6.8)

To simplify Eq. (1.6.8) further, use integration by parts. Consider the second term in Eq. (1.6.8).

Let \( u = y' \), \( du = y'' dx \), \( dv = \eta' dx \), \( v = \eta \) (\( \int u dv = uv - \int v du \))

\[ \int_0^\ell y'\eta' dx = y'\eta\bigg|_0^\ell - \int_0^\ell \eta y'' dx \]

\[ = - \int_0^\ell \eta y'' dx \quad (\eta \text{ satisfies the geometric bc's}) \quad (a) \]

Similarly,

\[ \int_0^\ell y''\eta'' dx = y''\eta\bigg|_0^\ell - \int_0^\ell \eta'y''' dx = y''\eta\bigg|_0^\ell - y'''\eta\bigg|_0^\ell + \int_0^\ell y''\eta'' dx \]

(b)

Equations (a) and (b) lead to

\[ \int_0^\ell (EIy'' + Py'') \eta dx + (EIy''') \eta' \bigg|_0^\ell = 0 \]  

(1.6.9)

Except \( \eta(0) = \eta(\ell) = 0 \), \( \eta(x) \) is completely arbitrary and therefore nonzero; hence, the only way to hold Eq. (1.6.9) to be true is that each part of Eq. (1.6.9) must vanish simultaneously. That is

\[ \int_0^\ell (EIy'' + Py'') \eta dx = 0 \quad \text{and} \quad (EIy''') \eta' \bigg|_0^\ell = 0 \]

Since \( \eta'(0) \), \( \eta'(\ell) \), and \( \eta(x) \) are not zero and \( \eta'(0) \neq \eta'(\ell) \), it follows that \( y(x) \) must satisfy
\[ EIy^{iv} + Py'' = 0 \quad \Leftarrow \text{Euler-Lagrange differential equation (1.6.10)} \]

\[ EIy''|_{x=0} = 0 \quad \Leftarrow \text{natural boundary condition (1.6.11)} \]

\[ EIy''|_{x=\ell} = 0 \quad \Leftarrow \text{natural boundary condition (1.6.12)} \]

It is recalled that one imposed the geometric boundary conditions, \( y(0) = y(\ell) = 0 \) at the beginning; however, it can be shown that these conditions are not necessarily required. Shames and Dym (1985) elegantly explain the case for the problem that has the properties of being self-adjoint and positive definite.

The governing differential equation can be obtained either by (1) considering the equilibrium of deformed elements of the system or (2) using the principle of stationary potential energy and the calculus of variations. For a simple system such as a simply supported column buckling, method (1) is much easier to apply, but for a complex system such as cylindrical or spherical shell or plate buckling, method (2) is preferred as the concept is almost automatic although the mathematical manipulations involved are fairly complex. In dealing with the total potential energy, the kinematic (or geometric) boundary conditions involve displacement conditions (deflection or slope) of the boundary, while natural boundary conditions involve internal force conditions (moment or shear) at the boundary.

**Example 1** Derive the Euler-Lagrange differential equation and the necessary kinematic (geometric) and natural boundary conditions for the prismatic cantilever column with a linear spring (spring constant \( a \)) attached to its free end shown in Fig. 1-14.

The strain energy stored in the deformed body is

\[ U = \frac{EI}{2} \int_0^\ell (y'')^2 \, dx + \frac{a}{2} (y_0)^2 \quad (1.6.13) \]
The loss of potential energy of the external load due to the deformation to the neighboring equilibrium position is

\[
V = -\frac{P}{2} \int_0^\ell (y')^2 \, dx
\]  

(1.6.14)

Hence, the total potential energy functional becomes

\[
\Pi = U + V = \frac{EI}{2} \int_0^\ell (y''')^2 \, dx + \frac{\alpha}{2} (y')^2 - \frac{P}{2} \int_0^\ell (y')^2 \, dx
\]

or

\[
\Pi = \int_0^\ell \left[ \frac{EI}{2} (y''')^2 - \frac{P}{2} (y')^2 \right] \, dx + \frac{\alpha}{2} (y')^2
\]  

(1.6.15)

The total potential energy functional must be stationary if the first variation \( \delta \Pi = 0 \). Since the differential operator and the variational operator are interchangeable, one obtains

\[
\delta \Pi = \int_0^\ell (EI''(\delta'y'') - Py'\delta y') \, dx + \alpha y_0 \delta y_\ell = 0
\]  

(1.6.16)

Integrating by parts each term in the parenthesis of Eq. (1.6.16) yields

\[
\int_0^\ell EI'y'y'' \, dx = \left[ EI'y'y' \right]_0^\ell - \left[ EI'y'''y' \right]_0^\ell + \int_0^\ell EI'y''y'' \, dx
\]  

(1.6.17)

\[
- \int_0^\ell Py'y' \, dx = -\left[ Py'y' \right]_0^\ell + \int_0^\ell Py''y' \, dx
\]  

(1.6.18)

It becomes obvious by inspection of the sketch that (1) the deflection and slope must be equal to zero due to the unyielding support at \( A (x = 0) \) and the variation will also be equal to zero, that is, \( y_0 = 0, y'_0 = 0 \) and \( \delta y_0 = 0, \delta y'_0 = 0 \), and (2) the moment and its variation must also be equal to zero due to the roller support at \( B (x = \ell) \), that is, \( y''_\ell = 0 \) and \( \delta y''_\ell = 0 \) where the subscripts 0 and \( \ell \) represent the values at \( A (x = 0) \) and \( B (x = \ell) \), respectively. The first and second term of Eq. (1.6.17) can be written, respectively, as

\[
\left[ EI'y'y' \right]_0^\ell = EI'y''_\ell \delta y'_\ell - EI'y''_0 \delta y'_0 = 0
\]
and

$$- [EIy'''\delta y]_0^\ell = -EIy'''\delta y_\ell + EIy'''_0\delta y_0 = -EIy'''\delta y_\ell.$$  

The first term of Eq. (1.6.18) can be written as

$$- [Py'\delta y]_0^\ell = -Py'_\ell\delta y_\ell + Py'_0\delta y_0 = -Py'_\ell\delta y_\ell$$

Equation (1.6.16) may now be rearranged

$$\delta \pi = (\alpha y_\ell - EIy'''_\ell - Py'_\ell)\delta y_\ell + \int_0^\ell (EIy''' + Py'')\delta y dx = 0 \quad (1.6.19)$$

It is noted here in Eq. (1.6.19) that $\delta y_\ell$ is not zero. In order for Eq. (1.6.19) to be equal to zero for all values of $\delta y$ between $x = 0$ and $x = \ell$, it is required that the function $y$ must satisfy the Euler-Lagrange differential equation (the integrand inside the parenthesis)

$$EIy''' + Py'' = 0 \quad (1.6.20)$$

and additional condition

$$\alpha y_\ell - EIy'''_\ell - Py'_\ell = 0 \quad (1.6.21)$$

must be met.

Equation (1.6.21), along with the condition $y_\ell'' = 0$, are the natural boundary conditions of the problem, and $y_0$ (and/or $\delta y_0$) = 0 $y'_0$ (and/or $\delta y'_0$) = 0 are the geometric boundary conditions of the problem. Hence, four boundary conditions are available as required for a fourth-order differential equation. The sum of all of the expanded integral terms at the end points consisting of a multiple of the geometric boundary conditions and/or the natural boundary conditions is collectively called a conjunct or a concomitant and is equal to zero for all positive definite and self-adjoint problems.

### 1.7. DERIVATION OF BEAM-COLUMN GDE USING FINITE STRAIN

Recall the following Green-Lagrange finite strain:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,j}u_{k,i}) \quad (1.7.1)$$

$$e_{xx} = \frac{du_x}{dx} + \frac{1}{2}\left[\left(\frac{du_x}{dx}\right)^2 + \left(\frac{du_y}{dx}\right)^2 + \left(\frac{du_z}{dx}\right)^2\right] \Leftrightarrow \text{axial strain} \quad (1.7.2)$$
where \( (\frac{d^2u_x}{dx^2})^2 \simeq 0 \) (considered to be a higher order term) and \( (\frac{d^2u_z}{dx^2})^2 = 0 \) (only uniaxial bending is considered here). For the given coordinate system in the sketch, the axial strain due to bending is

\[
e_b = -\frac{d^2u_y}{dx^2} y
\]  

(1.7.3)

where \( \frac{d^2u_y}{dx^2} = \frac{1}{\rho} \) is the curvature of the elastic curve. The sum of axial strains due to axial force and flexure constitutes the total normal strain. Hence,

\[
\varepsilon_{xx} = \varepsilon_a + \varepsilon_b = \frac{du_x}{dx} + \frac{1}{2} \left( \frac{du_y}{dx} \right)^2 - \frac{d^2u_y}{dx^2} y
\]  

(1.7.4)

The strain energy stored in the elastic body becomes

\[
U = \frac{1}{2} \int \sigma^T \varepsilon dv = \frac{E}{2} \int \varepsilon_{xx}^2 dv = \frac{E}{2} \int \left[ \frac{du_x}{dx} - y \frac{d^2u_y}{dx^2} + \frac{1}{2} \left( \frac{du_y}{dx} \right)^2 \right]^2 dv
\]

\[
= \frac{E}{2} \int_0^\ell \int_A \left[ \left( \frac{du_x}{dx} \right)^2 + \left( \frac{d^2u_y}{dx^2} \right)^2 y^2 + \frac{1}{4} \left( \frac{du_y}{dx} \right)^4 - 2 \frac{du_x}{dx} \frac{d^2u_y}{dx^2} y \right. \\
\left. - \frac{d^2u_y}{dx^2} \left( \frac{du_y}{dx} \right)^2 + du_x \left( \frac{du_y}{dx} \right)^2 \right] dAdx
\]  

(1.7.5)

Neglecting the higher order term and integrating over the cross-sectional area \( A \) while noting all integrals of the form \( \int ydA \) to be zero as \( y \) is measured from the centroidal axis, one gets

\[
U = \int_{\ell} \left[ \frac{EA}{2} \left( \frac{du_x}{dx} \right)^2 + \frac{EI}{2} \left( \frac{d^2u_y}{dx^2} \right)^2 + \frac{EA}{2} \frac{du_x}{dx} \left( \frac{du_y}{dx} \right)^2 \right] dx
\]  

(1.7.6)

The loss of potential energy of the applied transverse load is

\[
V = -\int_{\ell} w u_y dx
\]  

(1.7.7)
Hence, the total potential energy functional of the system becomes

\[ \Pi = U + V = \int_{\ell} \left[ \frac{EA}{2} \left( \frac{du_x}{dx} \right)^2 + \frac{EI}{2} \left( \frac{d^2 u_y}{dx^2} \right)^2 + \frac{EA}{2} \frac{du_x}{dx} \left( \frac{du_y}{dx} \right)^2 - w u_y \right] dx \]

or

\[ \Pi = U + V = \int_{\ell} \left[ \frac{EA}{2} \left( \frac{du_x}{dx} \right)^2 + \frac{EI}{2} \left( \frac{d^2 u_y}{dx^2} \right)^2 - \frac{P}{2} \left( \frac{du_y}{dx} \right)^2 - w u_y \right] dx \]

Note that \( P = \sigma A = E A e_a = EA (du_x / dx) \), which is called the stress resultant. The negative sign corresponds to the fact that \( P \) is in compression.

The quantity inside the square bracket, the integrand, is denoted by \( F \).

Applying the principle of the minimum potential energy (or applying the Euler-Lagrange differential equation), one obtains

\[ F = \frac{EA}{2} (u')^2 + \frac{EI}{2} (y'')^2 - \frac{P}{2} (y')^2 - wy \]

where \( u = u_x, y = u_y \).

Recall the Euler-Lagrange DE (see Bleich 1952, pp. 91–103):

\[ F_u - \frac{d}{dx} F_{u'} + \frac{d^2}{dx^2} F_{u''} - \ldots = 0 \]

\[ F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \ldots = 0 \]

\[ F_u = 0, \ F_{u'} = EA u' \Rightarrow - \frac{d}{dx} F_{u'} = -EA u'', \ F_{u''} = 0, \]

\[ E A u'' = 0 \]

\[ F_y = -w, \ F_{y'} = -P y' \Rightarrow - \frac{d}{dx} = P y'', \]

\[ F_{y''} = E I y'' \Rightarrow \frac{d^2}{dx^2} F_{y''} = E I y''', \]

\[ E I y''' + P y'' = w \]
It should be noted that the concept of finite axial strain implicitly implies the buckled shape (lateral displacement) and any prebuckling state is ignored.

1.8. GALERKIN METHOD

The requirement that the total potential energy of a hinged column has a stationary value is shown in the following equation:

$$
\int_{0}^{\ell} (EIy^{iv} + Py'') \delta y' dx + (EIy'') \delta y' \bigg|_{0}^{\ell} = 0
$$

(1.8.1)

where $\delta y$ is a virtual displacement.

Assume that it is possible to approximate the deflection of the column by a series of independent functions, $g_i(x)$, multiplied by undetermined coefficients, $a_i$.

$$
y_{\text{approx}} = a_1 g_1(x) + a_2 g_2(x) + \ldots + a_n g_n(x)
$$

(1.8.2)

If each $g_i(x)$ satisfies the geometric and natural boundary conditions, then the second term in Eq. (1.8.1) vanishes when it substitutes $y_{\text{approx}}$ to $y$. Also, the coefficients, $a_i$, must be chosen such that $y_{\text{approx}}$ will satisfy the first term. Let the operator be

$$
Q = EI \frac{d^4}{dx^4} + P \frac{d^2}{dx^2}
$$

(1.8.3)

and

$$
\phi = \sum_{i=1}^{n} a_i g_i(x)
$$

(1.8.4)

From Eqs. (1.8.3) and (1.8.4), the first term of Eq. (1.8.1.) becomes:

$$
\int_{0}^{\ell} Q(\phi) \delta \phi \ dx = 0
$$

(1.8.5)

Since $\phi$ is a function of $n$ parameters, $a_i$,

$$
\delta \phi = \frac{\partial \phi}{\partial a_1} \delta a_1 + \frac{\partial \phi}{\partial a_2} \delta a_2 + \ldots + \frac{\partial \phi}{\partial a_n} \delta a_n
$$

$$
= g_1 \delta a_1 + g_2 \delta a_2 + \ldots + g_n \delta a_n = \sum_{i=1}^{n} g_i \delta a_i
$$

(1.8.6)

$$
\int_{0}^{\ell} Q(\phi) \sum_{i=1}^{n} g_i(x) \delta a_i \ dx = 0
$$

(1.8.7)
Since it has been assumed that \( g_i(x) \) are independent of each other, the only way to hold Eq. (1.8.7) is that each integral of Eq. (1.8.7) must vanish, that is

\[
\int_0^\ell Q(\phi)g_i(x)\delta a_i\,dx = 0 \quad i = 1, 2, \ldots, n
\]

\( a_i \) are arbitrary; hence \( \delta a_i \neq 0 \).

\[
\int_0^\ell Q(\phi)g_i(x)\,dx = 0 \quad i = 1, 2, \ldots, n \quad (1.8.8)
\]

Equation (1.8.8) is somewhat similar to the weighted integral process in the finite element method.

**Example 1** Consider the axial buckling of a propped column. The Galerkin method is to be applied. For \( y_{\text{approx}} \), use the lateral displacement function of a propped beam subjected to a uniformly distributed load. Hence,

\[
y_{\text{approx}} = \phi = A(x\ell^3 - 3x^3\ell + 2x^4)
\]

\[
Q(\phi) = EI \frac{d^4\phi}{dx^4} + P \frac{d^2\phi}{dx^2} = A[48EI + P(24x^2 - 18\ell x)]
\]

\[
g(x) = (\ell^3x - 3\ell x^3 + 2x^4)
\]

\[
\int_0^\ell A[48EI + P(24x^2 - 18\ell x)](\ell^3x - 3\ell x^3 + 2x^4)\,dx = 0
\]

![Diagram of a propped column](image)
Carrying out the integration gives

\[
A \left( (36EI\ell^5/5) - (12P\ell^7/35) \right) = 0 \Rightarrow A \neq 0 \text{ for a nontrivial solution}
\]

\[
P_\sigma = 21EI/\ell^2 \Leftarrow 3.96\% \text{ greater than the exact value, } P_{\sigma \text{ exact}} = 20.2EI/\ell^2
\]

1.9. CONTINUOUS BEAM-COLUMNS RESTING ON ELASTIC SUPPORTS

A general method to evaluate the minimum required spring constants of a beam-column resting on an elastic support is to apply the slope-deflection equations with axial compression. In order to simplify the illustration, all beam-columns are assumed to be rigid and equal spans.

1.9.1. One Span

Assume that a small displacement occurs at \( b \), so that the bar becomes inclined to the horizontal by a small angle, \( \alpha \). As the stability of a system is examined in the neighboring equilibrium position, free body for equilibrium must be extracted from a deformed state. Owing to this displacement, the load \( P \) moves to the left by the amount

\[
L(1 - \cos \alpha) = \frac{L\alpha^2}{2}
\]

and the decrease in the potential energy of the load \( P \), equal to the work done by \( P \), is

\[
\frac{PL\alpha^2}{2}
\]

At the same time the spring deforms by the amount \( \alpha L \), and the increase in strain energy of the spring is

\[
\frac{k(\alpha L)^2}{2}
\]
where \( k \) denotes the spring constant. The system will be stable if
\[
\frac{k(\alpha L)^2}{2} > \frac{P L \alpha^2}{2}
\] (1.9.4)
and will be unstable if
\[
\frac{k(\alpha L)^2}{2} < \frac{P L \alpha^2}{2}
\] (1.9.5)
Therefore the critical value of the load \( P \) is found from the condition that
\[
\frac{k(\alpha L)^2}{2} = \frac{P L \alpha^2}{2}
\] (1.9.6)
from which
\[
k = \frac{\beta P \alpha}{L} \Rightarrow \beta = 1
\] (1.9.7)
The same conclusion can be reached by considering the equilibrium of the forces acting on the bar. However, if the system has three or more springs, simple statics may not be sufficient to determine the small displacement associated with each spring. Hence, the energy method appears to be better suited.

1.9.2. Two Span
For small deflection \( \delta \), the angle of inclination of the bar \( ab \) is \( \delta / L \), and the distance \( \lambda \) moved by the force \( P \) is found to be
\[
\lambda = 2 \left[ \frac{1}{2} L \left( \frac{\delta}{L} \right)^2 \right] = \frac{1}{L} \delta^2
\] (1.9.8)
and the work done by \( P \) is
\[
\Delta W = P \lambda = \frac{P \delta^2}{L}
\] (1.9.9)

Figure 1-18 Two-span model
The strain energy stored in the spring is

\[ \Delta U = \frac{k \delta^2}{2} \]  

(1.9.10)

The critical value of the load \( P \) is found from the equation

\[ \Delta U = \Delta W \]  

(1.9.11)

which represents the condition when the equilibrium configuration changes from stable to unstable. Hence,

\[ k = \frac{\beta P_{cr}}{L} = \frac{2P_{cr}}{L} \Rightarrow \beta = 2 \]  

(1.9.12)

1.9.3. Three Span

For small displacements, the rotation of bars \( ab \) and \( cd \) may be expressed as

\[ \alpha_1 = \frac{\delta_1}{L} \quad \text{and} \quad \alpha_2 = \frac{\delta_2}{L} \]  

(1.9.13)

and the rotation of bar \( bc \) is

\[ \frac{\delta_2 - \delta_1}{L} \]  

(1.9.14)

Figure 1-19 Three-span model
The distance $\lambda$ moved by the force $P$ is found to be

$$
\lambda = \frac{1}{2} L \left[ \left( \frac{\delta_1}{L} \right)^2 + \left( \frac{\delta_2 - \delta_1}{L} \right)^2 + \left( \frac{\delta_2}{L} \right)^2 \right]
$$

$$
= \frac{1}{2L} (\delta_1^2 + \delta_2^2 + \delta_1^2 - 2\delta_1\delta_2 + \delta_2^2) = \frac{1}{L} (\delta_1^2 - \delta_1\delta_2 + \delta_2^2) \quad (1.9.15)
$$

and the work done by the force $P$ is

$$
\Delta W = P\lambda = \frac{P}{L} (\delta_1^2 - \delta_1\delta_2 + \delta_2^2) \quad (1.9.16)
$$

The strain energy stored in the elastic supports during buckling is

$$
\Delta U = \frac{k}{2} (\delta_1^2 + \delta_2^2) \quad (1.9.17)
$$

The critical condition is found by equating these two expressions

$$
P \left( \frac{\delta_1^2 - \delta_1\delta_2 + \delta_2^2}{L} \right) = \frac{k}{2} (\delta_1^2 + \delta_2^2) \Rightarrow P = \frac{kL}{2} \frac{\delta_1^2 + \delta_2^2}{\delta_1^2 - \delta_1\delta_2 + \delta_2^2} = \frac{kL}{2} \frac{N}{D} \quad (1.9.18)
$$

where $N$ and $D$ represent the numerator and denominator of the fraction.

To find the critical value of $P$, one must adjust the deflections $\delta_1$ and $\delta_2$, which are unknown, so as to make $P$ a minimum value. This is accomplished by setting $\partial P / \partial \delta_1 = 0$ and $\partial P / \partial \delta_2 = 0$.

$$
\frac{\partial P}{\partial \delta_1} = \frac{kL}{2} \frac{D(\partial N / \partial \delta_1) - N(\partial D / \partial \delta_1)}{D^2} = 0 \Rightarrow
$$

$$
\frac{\partial N}{\partial \delta_1} - \frac{N}{D} \frac{\partial D}{\partial \delta_1} = \frac{\partial N}{\partial \delta_1} - \frac{2P}{kL} \frac{\partial D}{\partial \delta_1} = 0 \quad (1.9.19)
$$

Similarly,

$$
\frac{\partial N}{\partial \delta_2} - \frac{2P}{kL} \frac{\partial D}{\partial \delta_2} = 0 \quad (1.9.20)
$$

and

$$
\frac{\partial N}{\partial \delta_1} = 2\delta_1, \quad \frac{\partial N}{\partial \delta_2} = 2\delta_2, \quad \frac{\partial D}{\partial \delta_1} = 2\delta_1 - \delta_2, \quad \frac{\partial D}{\partial \delta_2} = 2\delta_2 - \delta_1 \quad (1.9.21)
$$

Substituting these values, one obtains

$$
2\delta_1 - \frac{2P}{kL} (2\delta_1 - \delta_2) = \delta_1 \left( 1 - \frac{2P}{kL} \right) + \delta_2 \frac{P}{kL} = 0 \quad (1.9.22)
$$
\[ 2\delta_2 - \frac{2P}{kL}(2\delta_2 - \delta_1) = \delta_1 \frac{P}{kL} + \delta_2 \left(1 - \frac{2P}{kL}\right) = 0 \]  \hspace{1cm} (1.9.23)

For nontrivial solutions, the coefficient determinant must vanish. Hence,
\[
\begin{vmatrix}
1 - \frac{2P}{kL} & \frac{P}{kL} \\
\frac{P}{kL} & 1 - \frac{2P}{kL}
\end{vmatrix} = 0 \Rightarrow \left(1 - \frac{2P}{kL}\right)^2 - \left(\frac{P}{kL}\right)^2 = 0 \Rightarrow P_1 = \frac{kL}{3}, P_2 = kL
\] \hspace{1cm} (1.9.24)

The critical load \( P_1 \) corresponds to the buckling mode shape shown in Fig. 1-19(b), and the critical load \( P_2 \) corresponds to the buckling mode shape shown in Fig. 1-19(c). For a given system, the critical load is the small one. Hence, \( P_1 \) is the correct solution. Hence,

\[ k = \frac{\beta P_{cr}}{L} = \frac{3P_{cr}}{L} \Rightarrow \beta = 3 \]  \hspace{1cm} (1.9.25)

The same problem can be solved readily by using equations of equilibrium. Noting that the reactive force of the spring is given by \( k\delta \), the end reactions are

\[ R_a = \frac{2}{3} k\delta_1 + \frac{1}{3} k\delta_2 \]  \hspace{1cm} (1.9.26)
\[ R_d = \frac{1}{3} k\delta_1 + \frac{2}{3} k\delta_2 \]  \hspace{1cm} (1.9.27)

Another equation for \( R_a \) is found by taking the moment about point \( B \) for bar \( ab \), which gives
\[ P\delta_1 = R_aL \]  \hspace{1cm} (1.9.28)

and similarly, for \( ad \)
\[ P\delta_2 = R_dL \]  \hspace{1cm} (1.9.29)

Combining these four equations yields
\[ \frac{P}{L} \delta_1 = \frac{2}{3} k\delta_1 + \frac{1}{3} k\delta_2 \Rightarrow \delta_1 \left(2 - \frac{3P}{kL}\right) + \delta_2 = 0 \]  \hspace{1cm} (1.9.30)
\[ \frac{P}{L} \delta_2 = \frac{1}{3} k\delta_1 + \frac{2}{3} k\delta_2 \Rightarrow \delta_1 + \delta_2 \left(2 - \frac{3P}{kL}\right) = 0 \]  \hspace{1cm} (1.9.31)
Setting the determinant equal to zero yields
\[
\left| \begin{array}{cc}
2 - \frac{3P}{kL} & 1 \\
1 & 2 - \frac{3P}{kL}
\end{array} \right| = \left( 2 - \frac{3P}{kL} \right)^2 - 1 = 0 \Rightarrow P_1 = \frac{kL}{3} \text{ and } P_2 = kL
\]
(1.9.32)

By definition, \( P_1 \) is the correct solution.

**1.9.4. Four Span**

For small displacements, the rotation of bars \( ab \) and \( de \) may be expressed as
\[
\alpha_1 = \frac{\delta_1}{L} \text{ and } \alpha_2 = \frac{\delta_3}{L}
\]
(1.9.33)

and the angles of rotation of bar \( bc \) and \( cd \) are
\[
\frac{\delta_2 - \delta_1}{L} \text{ and } \frac{\delta_3 - \delta_2}{L}
\]
(1.9.34)

The distance \( \lambda \) moved by the force \( P \) is found to be
\[
\lambda = \frac{1}{2L} \left[ \left( \frac{\delta_1}{L} \right)^2 + \left( \frac{\delta_2 - \delta_1}{L} \right)^2 + \left( \frac{\delta_3 - \delta_2}{L} \right)^2 + \left( \frac{\delta_3}{L} \right)^2 \right]
\]
\[
= \frac{1}{2L} \left( \delta_1^2 + \delta_2^2 + \delta_1^2 - 2\delta_1\delta_2 + \delta_2^2 + \delta_2^2 - 2\delta_2\delta_3 + \delta_3^2 \right)
\]
(1.9.35)
\[
= \frac{1}{L} \left( \delta_1^2 - \delta_1\delta_2 + \delta_2^2 - \delta_2\delta_3 + \delta_3^2 \right)
\]

and the work done by the force \( P \) is
\[
\Delta W = P\lambda = \frac{P}{L} \left( \delta_1^2 - \delta_1\delta_2 + \delta_2^2 - \delta_2\delta_3 + \delta_3^2 \right)
\]
(1.9.36)

![Figure 1-20 Four-span model](image-url)
The strain energy stored in the elastic supports during buckling is

\[ \Delta U = \frac{k}{2}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) \]  

(1.9.37)

The critical condition is found by equating these two expressions

\[ \frac{P}{L}(\Delta_1^2 - \Delta_1 \Delta_2 + \Delta_2^2 - \Delta_2 \Delta_3 + \Delta_3^2) = \frac{k}{2}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) \Rightarrow \]

\[ P = \frac{kL}{2} \frac{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}{\Delta_1^2 - \Delta_1 \Delta_2 + \Delta_2^2 - \Delta_2 \Delta_3 + \Delta_3^2} = \frac{kLN}{2D} \]  

(1.9.38)

where \( N \) and \( D \) represent the numerator and denominator of the fraction. To find the critical value of \( P \), one must adjust the deflections \( \Delta_1, \Delta_2 \) and \( \Delta_3 \), which are unknown, so as to make \( P \) a minimum value. This is accomplished by setting \( \partial P/\partial \Delta_1 = 0, \partial P/\partial \Delta_2 \) and \( \partial P/\partial \Delta_3 = 0 \).

\[ \frac{\partial P}{\partial \Delta_1} = \frac{kL}{2} D \frac{\partial N}{\partial \Delta_1} - N \frac{\partial D}{\partial \Delta_1} = 0 \Rightarrow \]

\[ \frac{\partial N}{\partial \Delta_1} - \frac{N \partial D}{\partial \Delta_1} = \frac{\partial N}{\partial \Delta_1} - \frac{2P \partial D}{kL \partial \Delta_1} = 0 \]  

(1.9.39)

Similarly,

\[ \frac{\partial N}{\partial \Delta_2} - \frac{2P \partial D}{kL \partial \Delta_2} = 0 \]  

(1.9.40)

\[ \frac{\partial N}{\partial \Delta_3} - \frac{2P \partial D}{kL \partial \Delta_3} = 0 \]  

(1.9.41)

and

\[ \frac{\partial N}{\partial \Delta_1} = 2\Delta_1, \quad \frac{\partial N}{\partial \Delta_2} = 2\Delta_2, \quad \frac{\partial N}{\partial \Delta_3} = 2\Delta_3, \quad \frac{\partial D}{\partial \Delta_1} = 2\Delta_1 - \Delta_2, \]

\[ \frac{\partial D}{\partial \Delta_2} = 2\Delta_2 - \Delta_1 - \Delta_3, \quad \frac{\partial D}{\partial \Delta_3} = 2\Delta_3 - \Delta_2 \]  

(1.9.42)

Substituting these values, one obtains

\[ 2\Delta_1 - \frac{2P}{kL}(2\Delta_1 - \Delta_2) = \Delta_1 \left( 1 - \frac{2P}{kL} \right) + \Delta_2 \frac{P}{kL} + 0\Delta_3 = 0 \]  

(1.9.43)
\[ 2\delta_2 - \frac{2P}{kL}(2\delta_2 - \delta_1 - \delta_3) = \delta_1 \frac{P}{kL} + \delta_2 \left( 1 - \frac{2P}{kL} \right) + \delta_3 \frac{P}{kL} = 0 \quad (1.9.44) \]

\[ 2\delta_3 - \frac{2P}{kL}(2\delta_3 - \delta_2) = 0 \quad \delta_1 + \delta_2 \frac{P}{kL} + \delta_3 \left( 1 - \frac{2P}{kL} \right) = 0 \quad (1.9.45) \]

For nontrivial solutions, the coefficient determinant must vanish. Hence,

\[
\begin{vmatrix}
1 - \frac{2P}{kL} & \frac{P}{kL} & 0 \\
\frac{P}{kL} & 1 - \frac{2P}{kL} & \frac{P}{kL} \\
0 & \frac{P}{kL} & 1 - \frac{2P}{kL}
\end{vmatrix} = \left( 1 - \frac{2P}{kL} \right)^3 - 2 \left( 1 - \frac{2P}{kL} \right) \left( \frac{P}{kL} \right)^2 = 0
\]

(1.9.46)

The smallest critical load \( P_1 = 0.29289kL \) corresponds to the buckling mode shape shown in sketch.

\[ k = \frac{\beta P_{cr}}{L} = \frac{P_{cr}}{0.29289L} = \frac{3.414P_{cr}}{L} \Rightarrow \beta = 3.414 \quad (1.9.47) \]

The equilibrium method cannot be applied to problems with three or more elastic supports as there are only two equations of equilibrium available, that is, \( \sum \text{moment} = 0 \) and \( \sum \text{vertical force} = 0 \). It is further noted that \( \beta \) varies from 1 for one span to 4 for infinite equal spans. Since \( \beta \) equals 3.414 for four equal spans, the use of \( \beta = 4 \) for multistory frames would seem justified.

Compression members in real structures are not perfectly straight (sweep, camber), perfectly aligned, or concentrically loaded as is assumed in design calculations; there is always an initial imperfection. Examining the single-story column of Fig. 1-17 assuming there is an initial deflection \( \delta_0 \) reveals that the following equilibrium equation is required:

\[(k\delta)L = P(\delta + \delta_0) \quad (1.9.48)\]

for \( P = P_{cr} \)

\[ k_{reqd} = \frac{P_{cr}}{L} \left( 1 + \frac{\delta_0}{\delta} \right) \quad (1.9.49) \]
Since $k_{\text{ideal}} = P_{cr}/L$, Eq. (1.9.48) becomes

$$k_{\text{reqd}} = k_{\text{ideal}} \left(1 + \frac{\delta_0}{\delta}\right)$$  \hfill (1.9.50)

which is the stiffness requirement for compression members having initial imperfection $\delta_0$. The stiffness requirement is

$$Q = k_{\text{reqd}}\delta = k_{\text{ideal}} \left(1 + \frac{\delta_0}{\delta}\right)\delta = k_{\text{ideal}}(\delta + \delta_0)$$  \hfill (1.9.51)

Winter (1960) has suggested $\delta = \delta_0 = L/500$. Substitution of this into Eqs. (1.9.49) and (1.9.50) gives the following design equations:

For stiffness, $k_{\text{reqd}} = 2k_{\text{ideal}}$  \hfill (1.9.52)

For nominal strength

$$Q_n = k_{\text{ideal}}(2\delta_0) = k_{\text{ideal}}(0.004L) = \frac{\beta P_{cr}L}{0.004L}$$  \hfill (1.9.53)

**Example 1** Turn-buckled threaded rods ($F_y = 50$ ksi, $F_u = 70$ ksi) are to be provided for the bracing system for a single-story frame shown in Fig. 1-21. The typical loading on each girder consists of three concentrated loads. The factored loads are: $P_1 = 200$ kips and $P_2 = 100$ kips. Determine the diameter of the rod by the AISC (2005) Specification for Structural Steel Building, 13th edition.
\[
\sum P = 4 \times (200 + 2 \times 100) = 1,600 \text{ kips}, \beta = 1, A_e = UA_n
\]
\[
= 1 \times A_n
\]

\[Q_u = 1 \times 1,600 \times 0.004 = 6.4 \text{ kips}, \cos \theta = \frac{25}{\sqrt{(25^2 + 15^2)}} = 0.8575\]

Design for strength

\[Q_n \text{ for yielding}, \quad Q_n = \frac{Q_u}{0.9} = \frac{6.4}{0.9} = 7.11 \text{ kips}\]

\[Q_n \text{ for fracture}, \quad Q_n = \frac{Q_u}{0.75} = \frac{6.4}{0.75} = 8.53 \text{ kips}\]

The required diameter of the rod against yielding is
\[7.11 = \frac{\pi}{4} \times d^2 \times 50 \times 0.8575, \quad d = 0.46 \text{ in.}\]

The required diameter of the rod against fracture is
\[8.53 = \frac{\pi}{4} \times \left( d - \frac{0.9743}{11} \right)^2 \times 70 \times 0.8575, \quad d = 0.154 \text{ in.} \text{ (11 threads per inch is justified)}\]

Design for stiffness
\[k_{\text{reqd}} = 2k_{\text{ideal}} = \frac{2\beta P_r}{L} = \frac{2 \times 1 \times 1,600}{25 \times 12} = \frac{EA}{L} \cos^2 \theta\]
\[10.67 = \frac{29,000 \times \pi \times d^2 \times 0.8575^2}{4 \times 25 \times 12}, \quad d = 0.44 \text{ in.} < 0.514 \text{ in.}, \text{ use}\]
\[d = 5/8 \text{ in.} (= 0.625 \text{ in.})\]

1.10. ELASTIC BUCKLING OF COLUMNS SUBJECTED TO DISTRIBUTED AXIAL LOADS

When a column is subjected to distributed compressive forces along its length, the governing differential equation of the deflected curve is no longer a differential equation with constant coefficients.

The solution to this problem may be considered in three different ways: (1) application of infinite series such as Bessel functions, (2) one of the approximate methods, such as the energy method, and (3) the finite element method (the solution converges to the exact one following the grid
refinement). The energy methods and the finite element analysis will be illustrated in the next chapter.

Consider the problem of elastic buckling of a prismatic column subjected to its own weight.\(^4\)\(^5\) Figure 1-22 shows a flagpole-type cantilever column. The lower end of the column is built in, the upper end is free, and the weight is uniformly distributed along the column length. Assuming the buckled shape of the column as shown in Fig. 1–22, the differential equation of the deflected curve can be shown as:

\[
EI \frac{d^2y}{dx^2} = \int_x^\ell q(\eta - y) \, d\xi
\]

(1.10.1)

where the integral on the right-hand side of the equation represents the bending moment at any cross section \(mn\) produced by the uniformly distributed load of intensity \(q\). Likewise, the shearing force at any cross section \(mn\) can be expressed as

\[
EI \frac{d^3y}{dx^3} = -q(\ell - x)\frac{dy}{dx}
\]

(1.10.2)


Note that the moment given in Eq. (1.10.1) is a decreasing function against the $x$-axis, and hence, the rate of change of the moment must be negative as shown in Eq. (1.10.2). Equation (1.10.2) is an ordinary differential equation with a variable coefficient. Many differential equations with variable coefficients can be reduced to Bessel equations. In order to facilitate the solution, a new independent variable $z$ is introduced such that

$$z = \frac{2}{3} \sqrt{\frac{q}{EI}} (\ell - x)^3$$  \hspace{1cm} (1.10.3)$$

By taking successive derivatives, one obtains

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = -\frac{dy}{dz} \sqrt{\frac{3}{2 EI}}$$  \hspace{1cm} (1.10.4)$$

$$\frac{d^2 y}{dx^2} = \left( \frac{3}{2 EI} \right)^{\frac{3}{2}} \left( \frac{1}{3} z^{-1} \frac{dy}{dz} + z^2 \frac{d^2 y}{dz^2} \right)$$  \hspace{1cm} (1.10.5)$$

$$\frac{d^3 y}{dx^3} = \frac{3}{2 EI} \left( \frac{1}{9} z^{-3} \frac{dy}{dz} - \frac{d^2 y}{dz^2} - z \frac{d^3 y}{dz^3} \right)$$  \hspace{1cm} (1.10.6)$$

Substituting Eqs. (1.10.4) and (1.10.5) into Eq. (1.10.2) and letting

$$\frac{dy}{dz} = u$$  \hspace{1cm} (1.10.7)$$

One obtains

$$\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \left( 1 - \frac{1}{9 z^2} \right) u = \frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} + \left( 1 - \frac{p^2}{z^2} \right) u = 0$$  \hspace{1cm} (1.10.8)$$

Equation (1.10.8) is a Bessel equation, and its solution can be expressed in terms of Bessel functions.

Invoking the method of Frobenius, it is assumed that a solution of the form

$$u(z) = \sum_{n=0}^{\infty} c_n z^{r+n}$$  \hspace{1cm} (1.10.9)$$

exists for Bessel's equation, Eq. (1.10.8) of order $p$ ($\pm 1/3$ in this case).

Substituting Eq. (1.10.9) into Eq. (1.10.8), one obtains:

6 Frobenius (1848–1917) was a German mathematician.

\[ \sum_{n=0}^{\infty} c_n (r+n)(r+n-1) z^{r+n-2} + \sum_{n=0}^{\infty} e_n (r+n) z^{r+n-2} + \sum_{n=0}^{\infty} (-p^2) e_n z^{r+n-2} + \sum_{n=2}^{\infty} c_{n-2} z^{r+n-2} = 0 \]

or

\[ a_0 (r^2 - p^2) z^{r-2} + a_1 [(r+1)^2 - p^2] z^{r-1} + \sum_{n=2}^{\infty} \{c_n [(n+r)^2 - p^2] + c_{n-2}\} z^{r+n-2} = 0 \quad (1.10.10) \]

The indicial equation is \( r^2 - p^2 = 0 \) with roots \( r_1 = p = 1/3 \) and \( r_2 = -p = -1/3 \). Setting \( r = p \) in Eq. (1.10.10) yields

\[ (1 + 2p)c_1 z^{p-1} + \sum_{n=2}^{\infty} [n(n+2p)c_n + c_{n-2}] z^{n+p-2} = 0 \]

indicating that \( c_1 = 0 \) and \( c_n = \frac{-c_{n-2}}{n(n+2p)} \), for \( n \geq 2 \). \quad (1.10.11)

Hence, all the coefficients with odd-numbered subscripts equal to zero. Letting \( n = 2j + 2 \) one sees that the coefficients with even-numbered subscripts satisfy

\[ c_{2(j+1)} = \frac{-c_{2j}}{2^2(j+1)(p+j+1)}, \quad \text{for} \quad j \geq 0, \]

which yields

\[ c_2 = \frac{-a_0}{2^2(p+1)}, \quad c_4 = \frac{-c_2}{2^2(2)(p+2)} = \frac{c_0}{2^4(2!)(p+1)(p+2)}, \]

\[ c_6 = \frac{-c_4}{2^2(3)(p+3)} = \frac{-c_0}{2^6(3!)(p+1)(p+2)(p+3)}, \ldots \]

Hence, the series of Eq. (1.10.9) becomes

\[ u_1 = z^p \left[ a_0 - \frac{a_0}{2^2(p+1)} z^2 + \frac{a_0}{2^42!(p+1)(p+2)} z^4 - \ldots \right] \]

\[ = a_0 z^p \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^{2n} n!(p+1)(p+2) \ldots (p+n)} \quad (1.10.12) \]
It is customary in Eq. (1.10.12) to let the integral constant, 
\( c_0 = \left[ 2^p \Gamma(p + 1) \right]^{-1} \) in which \( \Gamma(p + 1) \) is the gamma function. Then, Eq. (1.10.12) becomes

\[
J_p(z) = (z/2)^p \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(p + n + 1)}
\]

which is known as the Bessel function of the first kind of order \( p \). Thus \( J_p(z) \) is the first solution of Eq. (1.10.8). One will again be able to apply the method of Frobenius with \( r = -p \) to find the second solution. From Eq. (1.10.10), one immediately obtains

\[
(1 - 2p)c_1 z^{-p-1} + \sum_{n=2}^{\infty} [n(n - 2p)\epsilon_n + \epsilon_{n-2}] z^{n-p-2} = 0 \tag{1.10.13}
\]

indicating \( \epsilon_1 = 0 \) as before and

\[
\epsilon_n = \frac{-\epsilon_{n-2}}{n(n - 2p)} \tag{1.10.14}
\]

With algebraic operations similar to those done earlier, one obtains the second solution of Eq. (1.10.8)

\[
J_{-p}(z) = (z/2)^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n - p + 1)} \tag{1.10.15}
\]

Hence, the complete solution of Eq. (1.10.8) is

\[
u(z) = u_1(z) + u_2(z) = AJ_p(z) + BJ_{-p}(z) \tag{1.10.16}
\]

In Eq. (1.10.16), \( A \) and \( B \) are constants of integration, and they must be determined from the boundary conditions of the column. Since the upper end of the column is free, the condition yields

\[
\left( \frac{d^2 y}{dx^2} \right)_{x=\ell} = 0
\]

Observing that \( z = 0 \) at \( x = \ell \) and using Eqs. (1.10.5) and (1.10.7), one can express this condition as

\[
\left( \frac{1}{3} z^{-\frac{1}{2}}u + \frac{1}{2} z^\frac{1}{2} \frac{du}{dz} \right)_{z=0} = 0
\]

Substituting Eq. (1.10.16) into this equation, one obtains \( A = 0 \) and hence

\[
u(z) = BJ_{-p}(z) \tag{1.10.17}
\]
At the lower end of the column the condition is
\[
\left( \frac{dy}{dx} \right)_{x=0} = 0
\]
With the use of Eqs. (1.10.3), (1.10.4), and (1.10.7), this condition is expressed in the form
\[
u = 0 \quad \text{when} \quad z = \frac{2}{3} \sqrt{\frac{q\ell^3}{EI}}.
\]
The value of \( z \) which makes \( u = 0 \) can be found from Eq. (1.10.17) by trial and error, from a table of the Bessel function of order \(-\frac{1}{3}\), or from a computerized symbolic algebraic code such as Maple®. The lowest value of \( z \) which makes \( u = 0 \), corresponding to the lowest buckling load, is found from Maple® to be \( z = 1.866350859 \), and hence
\[
z = \frac{2}{3} \sqrt{\frac{q\ell^3}{EI}} = 1.866
\]
or
\[
(ql)_c = \frac{7.837EI}{\ell^2}.
\] (1.10.18)
This is the critical value of the uniform load for the column shown in Fig. 1-22.

Equation (1.10.2) above is differentiated once more to derive the governing equation of the buckling of the column under its own weight as
\[
EI \frac{d^2}{dx^2} \left( \frac{d^2y}{dx^2} \right) + q \frac{d}{dx} \left[ (\ell - x) \frac{dy}{dx} \right] = 0 \quad (1.10.19)
\]
Equation (1.10.19) is accompanied by appropriate boundary conditions. For the column that is pinned, clamped, and free at its end, the boundary conditions are, respectively
\[
y = 0, \quad \frac{d^2y}{dx^2} = 0 \quad (1.10.20a)
\]
\[
y = 0, \quad \frac{dy}{dx} = 0 \quad (1.10.20b)
\]
\[
\frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0 \quad (1.10.20c)
\]
As the differential equation is an ordinary homogeneous equation with a variable constant, the power series method, or a combination of Bessel and Lommel functions, are used after a clever transformation. Elishakoff (2005) gives credit to Dinnik (1912) for the solution of the pin-ended column as

\[(q\ell)_{cr} = \frac{18.6EI}{\ell^2} \quad (1.10.21)\]

and to Engelhardt (1954) for the solution of the column that is clamped at one end (bottom) and pinned at the other (top) as

\[(q\ell)_{cr} = \frac{52.5EI}{\ell^2} \quad (1.10.22)\]

as well as for the column that is clamped at both ends as

\[(q\ell)_{cr} = \frac{74.6EI}{\ell^2} \quad (1.10.23)\]

Structural Stability (STSTB)\(^9\) computes critical load for the column that is clamped at one end (top) and pinned at the other (bottom) as

\[(q\ell)_{cr} = \frac{30.0EI}{\ell^2} \quad (1.10.24)\]

Solutions given by Eqs. (1.10.18), (1.10.21), (1.10.22), (1.10.23), and (1.10.24) can be duplicated closely (within the desired accuracy) by most present-day computer programs, for example, STSTB. Wang et al. (2005) present exact solutions for columns with other boundary conditions. A case of considerable practical importance, in which the moment of inertia of the column section varies along its length, has been investigated. However, these problems can be effectively treated by the present-day computer programs, and efforts associated with the complex mathematical manipulations can now be diverted into other endeavors.

1.11. LARGE DEFLECTION THEORY (THE ELASTICA)

Although it is not likely to be encountered in the construction of buildings and bridges, a very slender compression member may exhibit a nonlinear elastic large deformation so that a simplifying assumption of the small

---


displacement theory may not be valid, as illustrated by Timoshenko and Gere (1961) and Chajes (1974). Consider the simply supported wiry column shown in Fig. 1-23. Aside from the assumption of small deflections, all the other idealizations made for the Euler column are assumed valid. The member is assumed perfectly straight initially and loaded along its centroidal axis, and the material is assumed to obey Hooke’s law.

From an isolated free body of the deformed configuration of the column, it can be readily observed that the external moment, \( P_y \), at any section is equal to the internal moment, \(-EI/\rho\).

Thus

\[ Py = -\frac{EI}{\rho} \quad (1.11.1) \]

where \( 1/\rho \) is the curvature. Since the curvature is defined by the rate of change of the unit tangent vector of the curve with respect to the arc length of the curve, the curvature and slope relationship is established.

\[ \frac{1}{\rho} = \frac{d\theta}{ds} \quad (1.11.2) \]

Substituting Eq. (1.11.1) into Eq. (1.11.2) yields

\[ EI \frac{d\theta}{ds} + Py = 0 \quad (1.11.3) \]

Introducing \( k^2 = P/EI \), Eq. (1.11.3) transforms into

\[ \frac{d\theta}{ds} + k^2 y = 0 \quad (1.11.4) \]

Differentiating Eq. (1.11.4) with respect to \( s \) and replacing \( dy/ds \) by \( \sin \theta \) yields

\[ \frac{d^2\theta}{ds^2} + k^2 \sin \theta = 0 \quad (1.11.5) \]
Multiplying each term of Eq. (1.11.5) by $2 \, d\theta$ and integrating gives

$$
\int \frac{d^2 \theta}{ds^2} \cdot 2 \, d\theta \, ds + \int 2k^2 \sin \theta \, d\theta = 0 \quad (1.11.6)
$$

Recalling the following mathematical identities

$$
\frac{d}{ds} \left( \frac{d\theta}{ds} \right)^2 = 2 \left( \frac{d\theta}{ds} \right) \left( \frac{d^2 \theta}{ds^2} \right) \quad \text{and} \quad \sin \theta \, d\theta = -d(\cos \theta),
$$

it follows immediately that

$$
\int d \left( \frac{d\theta}{ds} \right)^2 - 2k^2 \int d(\cos \theta) = 0 \quad (1.11.7)
$$

Carrying out the integration gives

$$
\left( \frac{d\theta}{ds} \right)^2 - 2k^2 \cos \theta = C \quad (1.11.8)
$$

The integral constant $C$ can be determined from the proper boundary condition. That is

$$
\frac{d\theta}{ds} = 0 \quad \text{at} \quad x = 0,
$$

$$
\left( \text{moment} = 0 \Rightarrow \frac{1}{\rho} = 0 \quad \text{or} \quad \rho = \infty, \text{straight line} \right) \quad \text{and} \quad \theta = \theta_0
$$

Hence,

$$
C = -2k^2 \cos \theta_0
$$

and Eq. (1.11.8) becomes

$$
\left( \frac{d\theta}{ds} \right)^2 - 2k^2(\cos \theta - \cos \theta_0) = 0 \quad (1.11.9)
$$

Taking the square root of Eq. (1.11.9) and rearranging gives

$$
ds = -\frac{d\theta}{\sqrt{2k\sqrt{\cos \theta - \cos \theta_0}}} \quad (1.11.10)
$$

Notice the negative sign in Eq. (1.11.10), which implies that $\theta$ decreases as $s$ increases. Carrying out the integral of Eq. (1.11.10) gives
\[
\int_0^{\ell/2} ds = -\frac{1}{\sqrt{2k}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \quad \text{or} \quad \frac{\ell}{2} = \frac{1}{\sqrt{2k}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}
\]

or

\[
\ell = \frac{2}{k} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2 \cos \theta - 2 \cos \theta_0}} \quad (1.11.11)
\]

Notice the negative sign is eliminated by reversing the limits of integration.

Making use of mathematical identities

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \quad \text{and} \quad \cos \theta_0 = 1 - 2 \sin^2 \frac{\theta_0}{2}
\]

in Eq. (1.11.11) yields:

\[
\ell = \frac{1}{k} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \theta}} \quad (1.11.12)
\]

In order to simplify Eq. (1.11.12) further, let

\[
\sin \frac{\theta_0}{2} = \alpha \quad (1.11.13)
\]

and introduce a new variable \(\phi\) such that

\[
\sin \frac{\theta}{2} = \alpha \sin \phi \quad (1.11.14)
\]

Then \(\theta = 0 \Rightarrow \phi = 0\) and \(\theta = \theta_0 \Rightarrow \sin \phi = 1 \Rightarrow \phi = \pi/2\).

Differentiating Eq. (1.11.14) yields

\[
\frac{1}{2} \cos \frac{\theta}{2} d\theta = \alpha \cos \phi \, d\phi \quad (1.11.15)
\]

which can be rearranged to show

\[
d\theta = \frac{2\alpha \cos \phi \, d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \frac{2\alpha \cos \phi \, d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \quad (1.11.16)
\]

Substituting Eqs. (1.11.13), (1.11.14), (1.11.15), and (1.11.16) into Eq. (1.11.12) yields
\[ \ell = \frac{1}{k} \int_{0}^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \theta_0 - \sin^2 \theta}} = \frac{1}{k} \int_{0}^{\pi/2} \frac{1}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi}} \sqrt{1 - \alpha^2 \sin^2 \phi} \ \frac{2\alpha \cos \phi \ d\phi}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi}} \]

\[ = \frac{2}{k} \int_{0}^{\pi/2} \frac{1}{\alpha \cos \phi} \ \frac{\alpha \cos \phi \ d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \]

\[ \ell = \frac{2}{k} \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} = \frac{2K}{k} \]

(1.11.17)

where:

\[ K = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \]

(1.11.18)

Equation (1.11.18) is known as the complete elliptic integral of the first kind. Its value can be readily evaluated from a computerized symbolic algebraic code such as Maple®. Equation (1.11.17) can be rewritten in the form

\[ \ell = \frac{2K}{k} = \frac{2K}{\sqrt{P/EI}} \]  

as \( k^2 = \frac{P}{EI} \)

or

\[ \frac{P}{P_{cr}} = \frac{4K^2}{\pi^2} \]

(1.11.19)

as

\[ P = \frac{4K^2}{\ell^2/EI} = \frac{4EIK}{\ell^2} \] and \( P_{cr} = \frac{\pi^2 EI}{\ell^2} \)

If the lateral deflection of the member is very small (just after the initial bulge), then \( \theta_0 \) is small and consequently \( \alpha^2 \sin^2 \phi \) in the denominator of \( K \) becomes negligible. The value of \( K \) approaches \( \pi/2 \) and from Eq. (1.11.19) \( P = P_{cr} = \pi^2 EI/\ell^2 \).

The midheight deflection, \( y_m \) (or \( \delta \)), can be determined from \( dy = ds \sin \theta \).

![Figure 1-24 Postbuckling behavior](image)
Substituting Eq. (1.11.10) into the above equation yields

$$dy = -\frac{\sin \theta \, d\theta}{\sqrt{2k\cos \theta - \cos \theta_0}}$$

Integrating the above equation gives

$$\int_0^{y_m} dy = -\frac{1}{2k} \int_0^{\theta_0} \frac{\sin \theta \, d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \quad \text{or} \quad y_m = \frac{1}{2k} \int_0^{\theta_0} \frac{\sin \theta \, d\theta}{\sqrt{\frac{\sin^2 \theta_0}{2} - \frac{\sin^2 \theta}{2}}}$$

Recall \( \sin (\theta/2) = \alpha \sin \phi \) and \( d\theta = 2\alpha \cos \phi \, d\phi / \sqrt{1 - \alpha^2 \sin^2 \phi} \)

Hence,

$$y_m = \frac{1}{2k} \int_0^{\theta_0} \sin \theta \, d\theta \sqrt{\frac{\sin^2 \theta_0}{2} - \frac{\sin^2 \theta}{2}} = \frac{1}{2k} \int_0^{\pi/2} 2\alpha \sin \phi \sqrt{1 - \alpha^2 \sin^2 \phi} \, 2\alpha \cos \phi \, d\phi$$

$$= \frac{1}{2k} \int_0^{\pi/2} \frac{2\alpha \sin \phi \sqrt{1 - \alpha^2 \sin^2 \phi} \, 2\alpha \cos \phi \, d\phi}{\alpha^2 - \alpha^2 \sin^2 \phi \sqrt{1 - \alpha^2 \sin^2 \phi}}$$

$$y_m = \delta = \frac{2\alpha}{k} \int_0^{\pi/2} \sin \phi \, d\phi = \frac{2\alpha}{k} \quad \text{or} \quad \frac{y_m}{\ell} = \frac{2\alpha}{\pi} \sqrt{\frac{k}{P/E}}$$

The distance between the two load points (x-coordinates) can be determined from

$$dx = ds \cos \theta$$

Substituting Eq. (1.11.10) into the above equation yields

$$dx = -\frac{\cos \theta \, d\theta}{\sqrt{2k\cos \theta - \cos \theta_0}}$$

Integrating (\( x_m \) is the x-coordinate at the midheight) the above equation gives

$$\int_0^{x_m} dx = -\frac{1}{\sqrt{2k}} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = -\frac{1}{\sqrt{k}} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{2 \cos \theta - 2 \cos \theta_0}} \quad \text{or}$$

$$x_m = \frac{1}{2k} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\frac{\sin^2 \theta_0}{2} - \frac{\sin^2 \theta}{2}}}$$
Recall \( \sin(\theta/2) = \alpha \sin \phi \) and \( d\theta = 2\alpha \cos \phi \, d\phi / \sqrt{1 - \alpha^2 \sin^2 \phi} \)

and \( \cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2) = 1 - 2\alpha^2 \sin^2 \phi \)

\[
x_m = \frac{1}{2k} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}
= \frac{1}{2k} \int_0^{\pi/2} \frac{(1 - 2\alpha^2 \sin^2 \phi)2\alpha \cos \phi \, d\phi}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi} \sqrt{1 - \alpha^2 \sin^2 \phi}}
= \frac{1}{k} \int_0^{\pi/2} \frac{(1 - 2\alpha^2 \sin^2 \phi) \, d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}}
\]

\[
x_0 = 2x_m = \frac{2}{k} \int_0^{\pi/2} \frac{[2(1 - \alpha^2 \sin^2 \phi) - 1] \, d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}}
= \frac{4}{k} \int_0^{\pi/2} \sqrt{1 - \alpha^2 \sin^2 \phi} \, d\phi
- \frac{2}{k} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} = \frac{4}{k} E(\alpha) - \ell
\]

where \( E(\alpha) \) is the complete elliptic integral of the second kind

\[
\frac{x_0}{\ell} = \frac{4E(\alpha)}{\ell \sqrt{\frac{P}{EI}}} - 1 = \frac{4E(\alpha)}{\pi \sqrt{\frac{P}{P_E}}} - 1
\]

The complete elliptic integral of the first kind can be evaluated by an infinite series given by

\[
K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}}
= \frac{\pi}{2} \left[ 1 + \frac{1}{2} \alpha^2 + \frac{1 \cdot 3}{2 \cdot 4} \alpha^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \alpha^6 + \cdots \right] \text{ with } \alpha^2 < 1
\]

Summing the first four terms of the above infinite series for \( \alpha = 0.5 \) yields \( K = 1.685174 \).
Likewise, the complete elliptic integral of the second kind can be evaluated by an infinite series given by

\[
E = \int_0^{\pi/2} \sqrt{1 - \alpha^2 \sin^2 \phi} \, d\phi = \frac{\pi}{2} \left[ 1 - \left(\frac{1}{2}\right)^2 \alpha^2 - \left(\frac{1\cdot3}{2\cdot4}\right)^2 \alpha^4 - \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 \alpha^6 - \cdots \right]
\]

with \( \alpha^2 < 1 \)

Summing the first four terms of the above infinite series for \( \alpha = 0.5 \) yields \( E = 1.46746 \). These two infinite series can be programmed as shown or can be evaluated by commercially available symbolic algebraic codes such as Maple\textsuperscript{®}, Matlab\textsuperscript{®}, and/or MathCAD\textsuperscript{®}.
Consider the postbuckling shape of the wiry column. This type of postbuckling behavior may only be imagined for a very thin high-strength wire. Notice that the two end support positions are reversed. The $\theta_0$ to make the two end points contact ($x_0$) is found to be 130.6 degrees by trial and error. Many ordinary materials may not be able to withstand the high-stress level required to develop a shape similar to that shown in Fig. 1-25 in an elastic manner, and the stresses in the critical column sections are likely to be extended well into the plastic region. Therefore, the practical value of the large deflection theory at large deflections is questionable.

### 1.12. ECCENTRICALLY LOADED COLUMNS—SECANT FORMULA

In the derivation of the Euler model, a both-end pinned column, it is assumed that the member is perfectly straight and homogeneous, and that
the loading is assumed to be concentric at every cross section so that the structure and loading are symmetric. These idealizations are made to simplify the problem. In real life, however, a perfect column that satisfies all three conditions does not exist. It is, therefore, interesting to study the behavior of an imperfect column and compare it with the behavior predicted by the Euler theory. The imperfection of a monolithic slender column is predominantly affected by the geometry and eccentricity of loading. As an imperfect column begins to bend as soon as the initial amount of the incremental load is applied, the behavior of an imperfect column can be investigated successfully by considering either an initial imperfection or an eccentricity of loading.

Consider the eccentrically loaded slender column shown in Fig. 1-26. From equilibrium of the isolated free body of the deformed configuration, Eq. (1.12.1) becomes obvious

$$EIy'' + P(e + y) = 0 \quad (1.12.1)$$

or

$$y'' + k^2 y = -k^2 e \quad \text{with} \quad k^2 = \frac{P}{EI} \quad (1.12.2)$$

It should be noted in Eq. (1.12.2) that the system (both-end pinned prismatic column of length $l$ with constant $EI$) eigenvalue remains unchanged from the Euler critical load as it is evaluated from the homogeneous differential equation.

The general solution of Eq. (1.12.2) is

$$y = y_h + y_p = A \sin kx + B \cos kx - e \quad (1.12.3)$$

![Figure 1-26 Eccentrically loaded column](image-url)
The integral constants are evaluated from the boundary conditions. (The notion of solving an $n$th order ordinary differential equation implies that a direct or an indirect integral process is applied $n$ times and hence there should be $n$ integral constants in the solution of an $n$th order equation.) Thus the condition

$$y = 0 \quad \text{at} \quad x = 0$$

leads to

$$B = e$$

and the condition

$$y = 0 \quad \text{at} \quad x = \ell$$

gives

$$A = e \frac{1 - \cos k\ell}{\sin k\ell}$$

Substituting $A$ and $B$ into Eq. (1.12.3) yields

$$y = e \left( \cos kx + \frac{1 - \cos k\ell}{\sin k\ell} \sin kx - 1 \right) \quad (1.12.4)$$

Letting $x = \ell/2$ in Eq. (1.12.4) for the midheight deflection, $\delta$, gives

$$y \bigg|_{x=\ell/2} = \delta = e \left( \cos \frac{k\ell}{2} + \frac{1 - \cos k\ell}{\sin k\ell} \sin \frac{k\ell}{2} - 1 \right)$$

$$= e \left( \cos \frac{k\ell}{2} + \frac{1 - 1 + 2 \sin^2 \frac{k\ell}{2}}{2 \sin \frac{k\ell}{2} \cos \frac{k\ell}{2}} - 1 \right) \quad (1.12.5)$$

$$\delta = e \left( \sec \frac{k\ell}{2} - 1 \right) = e \left[ \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_E}} \right) - 1 \right] \text{ with } P_E = \frac{\pi^2 EI}{\ell^2}$$

The same deflection curve can be obtained using a fourth-order differential equation,

$$y = A \cos kx + B \sin kx + Cx + D$$

with

$$y = 0, \quad EIy'' = -Pe \quad \text{at} \quad x = 0 \quad \text{and}$$

$$y = 0, \quad EIy'' = -Pe \quad \text{at} \quad x = \ell.$$
Figure 1-27 shows the variation of the midheight deflection for two values of eccentricity, $e$.

The behavior of an eccentrically loaded column is essentially the same as that of an initially bent column except there will be the nonzero initial deflection at the no-load condition in the case of a column initially bent. A slightly imperfect column begins to bend as soon as the load is applied. The bending remains small until the load approaches the critical load, after which the bending increases very rapidly. Hence, the Euler theory provides a reasonable design criterion for real imperfect columns if the imperfections are small.

The maximum stress in the extreme fiber is due to the combination of the axial stress and the bending stress. Hence,

$$
\sigma_{\text{max}} = \frac{P}{A} + \frac{M_{\text{max}} c}{I} = \frac{P}{A} + \frac{P(\delta + e)c}{I} = \frac{P}{A} + \frac{ce \sec \left( \frac{\ell}{2} \sqrt{\frac{P}{EI}} \right)}{I}
$$

$$
= \frac{P}{A} \left[ 1 + \frac{ceA}{I} \sec \left( \frac{\ell}{2} \sqrt{\frac{P}{EI}} \right) \right] \quad (1.12.6)
$$

$$
\sigma_{\text{max}} = \frac{P}{A} \left[ 1 + \frac{ce}{r^2} \sec \left( \frac{\ell}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1.12.7)
$$

Equation (1.12.7) is known as the secant formula. In an old edition of *Standard Specification of Highway Bridges*, American Association of State...
Highway and Transportation Official (AASHTO) stipulated a constant value of 0.25 to account for a minimum initial imperfection usually encountered in practice, as shown in Eq. (1.12.8)

\[
\sigma_{AASHTO} = \frac{P}{A} \left[ 1 + \left( 0.25 + \frac{\varepsilon c}{r^2} \right) \sec \left( \frac{\ell}{2r} \sqrt{\frac{P}{EA}} \right) \right]
\]  
(1.12.8)

1.13. INELASTIC BUCKLING OF STRAIGHT COLUMN

In the discussions presented heretofore, the assumption has been made that the material obeys Hooke’s law. For this assumption to be valid, the stresses in the column must be below the proportional limit of the material. The linear elastic analysis is correct for slender columns. On the other hand, the axial stress in a short column will exceed the proportional limit. Consequently, the elastic analysis is not valid for short columns, and the limiting load for short columns must be determined by taking inelastic behavior into account. Before proceeding to consider the development of the theory of inelastic column behavior, it would be informative to review its historic perspective. The Euler hyperbola was derived by Leonhard Euler in 1744. It was believed at the time that the formula applied to all columns, short and slender. It was soon discovered that the formula was grossly unconservative for short columns; the Euler formula was considered to be completely erroneous and was discarded for a lengthy period of time, approximately 150 years. An anecdotal story reveals that people ridiculed Euler when he could not adequately explain why a coin (a compression member with an extremely small slenderness ratio) on an anvil smashed by a hammer yielded (flattened) instead of carrying an infinitely large stress. It is of interest to note that the concept of flexural rigidity, \(EI\), was not clearly defined at the time, and the modulus of elasticity of steel was determined by Thomas Young in 1807.\textsuperscript{10}

However, Theodore von Kármán developed the double-modulus theory in 1910 in his doctoral dissertation at Göttingen University under Ludwig Prandtl direction. It gained widespread acceptance and the validity of Euler’s work reestablished if the constant modulus \(E\) is replaced by an effective modulus for short columns. Later in 1947, F.R. Shanley\textsuperscript{11} demonstrated that the tangent modulus and not the double modulus is the correct effective modulus, which leads to lower buckling load than the double-modulus

theory and agrees better than the double-modulus theory with test results. These inelastic buckling analyses using effective modulus are just academic history today. The present-day finite element codes capable of conducting incremental analyses of the geometric and material nonlinearities, as refined in their final form in the 1980s, can correctly evaluate the inelastic column strengths, including the effects of initial imperfections, inelastic material properties including strain hardening, and residual stresses.

1.13.1. Double-Modulus (Reduced Modulus) Theory

Assumptions

1) Small displacement theory holds.
2) Plane sections remain plane. Bernoulli, or Euler, or Navier hypothesis.
3) The relationship between the stress and strain in any longitudinal fiber is given by the stress-strain diagram of the material (compression and tension, the same relationship).
4) The column section is at least singly symmetric, and the plane of bending is the plane of symmetry.
5) The axial load remains constant as the member moves from the straight to the deformed position.

\[ P = P_{cr} \]

\[ \sigma = \sigma_{cr} \]

\[ \sigma_1 = E\varepsilon_1 \]

\[ \sigma_2 = E\varepsilon_2 \]

\[ E_{cr} \text{ the slope of stress-strain curve at } \sigma = \sigma_{cr} \]

Figure 1-28 Reduced modulus model
In small displacement theory, the curvature of the bent column is

\[ \frac{1}{R} = \frac{d^2y}{dx^2} = \frac{d\phi}{dx} \]  

(1.13.1)

From a similar triangle relationship, the flexural strains are computed

\[ \varepsilon_1 = z_1y'' \]  

(1.13.2)

\[ \varepsilon_2 = z_2y'' \]  

(1.13.3)

and the corresponding stresses are

\[ \sigma_1 = Eh_1y'' \]  

(1.13.4)

\[ \sigma_2 = Eh_2y'' \]  

(1.13.5)

where \( E_t \) = tangent modulus, \( z_1 \) (tension) = \( Ez_1y'' \) and \( z_2 \) (compression) = \( E_tz_2y'' \).

The pure bending portion (no net axial force) requires

\[ \int_0^{h_1} s_1 dA + \int_0^{h_2} s_2 dA = 0 \]  

(1.13.6)

Equating the internal moment to the external moment yields

\[ \int_0^{h_1} s_1 z_1 dA + \int_0^{h_2} s_2 z_2 dA = Py \]  

(1.13.7)

Equation (1.13.6) is expanded to

\[ Ey'' \int_0^{h_1} z_1 dA + E_ty'' \int_0^{h_2} z_2 dA = 0 \]  

(1.13.8)

Let \( Q_1 = \int_0^{h_1} z_1 dA \) and \( Q_2 = \int_0^{h_2} z_2 dA \) \( \Rightarrow EQ_1 + E_tQ_2 = 0 \)  

(1.13.9)

Equation (1.13.7) is expanded to

\[ y'' \left( E \int_0^{h_1} z_1^2 dA + E_t \int_0^{h_2} z_2^2 dA \right) = Py \]  

(1.13.10)
Let \( \overline{E} = \frac{EI_1 + E_t I_2}{I} \) \hspace{1cm} (1.13.11)

which is called the reduced modulus that depends on the stress-strain relationship of the material and the shape of the cross section. \( I_1 \) is the moment of inertia of the tension side cross section about the neutral axis and \( I_2 \) is the moment of inertia of the compression side cross section such that

\[
I_1 = \int_0^{h_1} z_1^2 dA \quad \text{and} \quad I_2 = \int_0^{h_2} z_2^2 dA
\] \hspace{1cm} (1.13.12)

Equation (1.13.10) takes the form

\[
\overline{E} I y'' + Py = 0
\] \hspace{1cm} (1.13.13)

Equation (1.13.13) is the differential equation of a column stressed into the inelastic range identical to Eq. (1.3.3) except that \( E \) has been replaced by \( \overline{E} \), the reduced modulus. If it can be assumed that \( \overline{E} \) is constant, then Eq. (1.13.13) is a linear differential equation with constant coefficients, and its solution is identical to that of Eq. (1.3.3), except that \( E \) is replaced by \( \overline{E} \).

Corresponding critical load and critical stress based on the reduced modulus are

\[
P_{r,cr} = \frac{\pi^2 \overline{E} I}{\ell^2}
\] \hspace{1cm} (1.13.14)

and

\[
\sigma_{r,cr} = \frac{\pi^2 \overline{E}}{\left(\frac{\ell}{r}\right)^2}
\] \hspace{1cm} (1.13.15)

Introducing

\[
\tau_r = \frac{\overline{E}}{E} = \frac{E_t}{E} \frac{I_2}{I} + \frac{I_1}{I} < 1.0 \quad \text{and} \quad \tau = \frac{E_t}{E} < 1.0
\] \hspace{1cm} (1.13.16)

the differential equation based on the reduced modulus becomes

\[
EI \tau_r y'' + Py = 0
\] \hspace{1cm} (1.13.17)

and

\[
\tau_r = \frac{I_2}{I} + \frac{I_1}{I} \quad \text{and} \quad \sigma_{r,cr} = \frac{P_{r,cr}}{A} = \frac{\pi^2 E \tau_r}{\left(\frac{\ell}{r}\right)^2}
\] \hspace{1cm} (1.13.18)
The procedure for determining $\sigma_{t,cr}$ may be summarized as follows:

1) For $\sigma - \varepsilon$ diagram, prepare $\sigma - \tau$ diagram.
2) From the result of step 1, prepare $\tau - \sigma$ curve.
3) From the result of step 2, prepare $\sigma_t - (\ell/r)$ curve.

### 1.13.2. Tangent-Modulus Theory

**Assumptions**

The assumptions are the same as those used in the double-modulus theory, except assumption 5. The axial load increases during the transition from the straight to slightly bent position, such that the increase in average stress in compression is greater than the decrease in stress due to bending at the extreme fiber on the convex side. The compressive stress increases at all points; the tangent modulus governs the entire cross section.

If the load increment is assumed to be negligibly small such that

$$\Delta P << P$$  \hspace{1cm} (1.13.19)

then

$$E_i I y'' + Py = 0$$  \hspace{1cm} (1.13.20)

and the corresponding critical stress is

$$\sigma_{t,cr} = \frac{P_{t,cr}}{A} = \frac{\pi^2 E \tau}{\left(\frac{\ell}{r}\right)^2} \quad \text{with} \quad \tau = \frac{E_i}{E}$$  \hspace{1cm} (1.13.21)

Hence, $\sigma_t$ vs $\ell/r$ curve is not affected by the shape of the cross section.

The procedure for determining the $\sigma_t - (\ell/r)$ curve may be summarized as follows:

1) From $\sigma - \varepsilon$ diagram, establish $\sigma - \tau$ curve.
2) From the result of step 1, prepare $\sigma_t - (\ell/r)$.

![Figure 1-29 Tangent-modulus model](image)
Example 1
An axially loaded, simply supported column is made of structural steel with the following mechanical properties: $E = 30 \times 10^3$ ksi, $\sigma_p = 28.0$ ksi, $\sigma_y = 36$ ksi, and tangent moduli given in Table 1-2.
Determine the following:

1) The value of $\ell/r$, which divides the elastic buckling range and the inelastic buckling range

2) The value of $\tau_r$ and $\ell/r$ for $P/A = 28, 30, 32, 34, 35, 35.5$ ksi using the double-modulus theory and assuming that the cross section of the column is a square of side $h$.

3) The critical average stress $P/A$ for $\ell/r = 20, 40, 60, 80, 100, 120, 140, 160, 180,$ and $200$ using the tangent-modulus theory in the inelastic range. 

From the results of 1), 2), and 3), plot

4) The “$(P/A) - \tau_r$” curve for the double-modulus theory.

5) The “$(P/A) - (\ell/r)$” curves, distinguishing the portion of the curve derived by the tangent-modulus theory from that derived by the double-modulus theory. Present short discussions.

6) The current AISC LRFD Specification specifies (Chapter E) that the critical value of $P/A$ for axially loaded column shall not exceed the following:

(i) For $\lambda_c \leq 1.5$ $F_{cr} = (0.658 \lambda_c^2)F_y$

(ii) For $\lambda_c > 1.5$ $F_{cr} = \left[0.877/\lambda_c^2\right]F_y$

Plot these curves and superimpose them on the graph in 5 using double arguments ($\ell/r$ and $\lambda_c$) on the horizontal axis.

Table 1-2 Tangent moduli measured

<table>
<thead>
<tr>
<th>$\sigma_t$ or $\sigma_r$ (ksi)</th>
<th>$\tau = E_t/E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>28.0</td>
<td>1.00</td>
</tr>
<tr>
<td>29.0</td>
<td>0.98</td>
</tr>
<tr>
<td>30.0</td>
<td>0.96</td>
</tr>
<tr>
<td>31.0</td>
<td>0.93</td>
</tr>
<tr>
<td>32.0</td>
<td>0.88</td>
</tr>
<tr>
<td>33.0</td>
<td>0.77</td>
</tr>
<tr>
<td>34.0</td>
<td>0.55</td>
</tr>
<tr>
<td>35.0</td>
<td>0.31</td>
</tr>
<tr>
<td>35.5</td>
<td>0.16</td>
</tr>
<tr>
<td>36.0</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Locations of NA at various stages

\[ EQ_1 + E_t Q_2 = 0 \]  \hspace{1cm} (1.13.9)

\[ h_1 + h_2 = h \]

\[ Q_1 = \int_0^{h_1} z_1 dA \]

\[ Q_2 = \int_0^{h_2} z_2 dA \]

\[ Q_1 = \int_0^{h_1} z_1 h dz_1 = \frac{h}{2} z_1^2 \bigg|_0^{h_1} = \frac{hh_1^2}{2} \]

Likewise \( Q_2 = -(hh_2^2/2) \)

\[ Q_1 + \frac{E_t}{E} Q_2 = \frac{hh_1^2}{2} - \tau \frac{hh_2^2}{2} = h_1^2 - \tau (h - h_1)^2 = 0 \]

\[ h_1^2 + 2\tau hh_1 - \tau h^2 - \tau h_1^2 = 0 \]

\[ (1 - \tau)h_1^2 + 2\tau hh_1 - \tau h^2 = 0 \]

### Table 1-3 Cross-sectional properties vs. shifting neutral axis

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( h_1/h )</th>
<th>( l_1/l )</th>
<th>( l_2/l )</th>
<th>( (1) \times (4) )</th>
<th>( \tau_r )</th>
<th>( \sigma_r, \sigma_t, \sigma_r )</th>
<th>( \sigma/\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>.5000</td>
<td>.5000</td>
<td>.5000</td>
<td>.5000</td>
<td>1.0000</td>
<td>28.0</td>
<td>28.00</td>
</tr>
<tr>
<td>0.98</td>
<td>.4975</td>
<td>.4925</td>
<td>.5076</td>
<td>.4975</td>
<td>0.9899</td>
<td>29.0</td>
<td>29.60</td>
</tr>
<tr>
<td>0.96</td>
<td>.4950</td>
<td>.4848</td>
<td>.5155</td>
<td>.4948</td>
<td>0.9797</td>
<td>30.0</td>
<td>31.25</td>
</tr>
<tr>
<td>0.93</td>
<td>.4910</td>
<td>.4733</td>
<td>.5277</td>
<td>.4908</td>
<td>0.9640</td>
<td>31.0</td>
<td>33.33</td>
</tr>
<tr>
<td>0.88</td>
<td>.4840</td>
<td>.4536</td>
<td>.5495</td>
<td>.4835</td>
<td>0.9371</td>
<td>32.0</td>
<td>36.36</td>
</tr>
<tr>
<td>0.77</td>
<td>.4674</td>
<td>.4084</td>
<td>.6044</td>
<td>.4654</td>
<td>0.8738</td>
<td>33.0</td>
<td>42.86</td>
</tr>
<tr>
<td>0.55</td>
<td>.4258</td>
<td>.3088</td>
<td>.7572</td>
<td>.4165</td>
<td>0.7253</td>
<td>34.0</td>
<td>61.82</td>
</tr>
<tr>
<td>0.31</td>
<td>.3576</td>
<td>.1830</td>
<td>1.0602</td>
<td>.3287</td>
<td>0.5116</td>
<td>35.0</td>
<td>112.90</td>
</tr>
<tr>
<td>0.16</td>
<td>.2857</td>
<td>.0933</td>
<td>1.4577</td>
<td>.2332</td>
<td>0.3265</td>
<td>35.5</td>
<td>221.88</td>
</tr>
<tr>
<td>0.00</td>
<td>.0000</td>
<td>.0000</td>
<td>4.0000</td>
<td>.0000</td>
<td>0.0000</td>
<td>36.0</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

\[
h_1 = \frac{-\tau h + \sqrt{\tau^2 h^2 + (1 - \tau)(\tau h^2)}}{(1 - \tau)} = \frac{h(-\tau + \sqrt{\tau})}{(1 - \tau)}
\]

\[
\sigma_r = \frac{\pi^2 E \tau_r}{\left(\frac{\ell}{r}\right)^2} = \frac{\pi^2 \times 30 \times 10^3 \tau_r}{\left(\frac{\ell}{r}\right)^2}
\]

\[
\left(\frac{\ell}{r}\right) = \sqrt{\frac{30 \times 10^3 \pi^2 \tau_r}{\sigma_r}} = 544.14 \sqrt{\frac{\tau_r}{\sigma_r}}
\]

\[
\sigma_p = \frac{\pi^2 E}{\left(\frac{\ell}{r}\right)^2} \Rightarrow \frac{\ell}{r} = \pi \sqrt{\frac{E}{\sigma_p}} = 102.83
\]

1) and 3)

### Table 1-4 Slenderness ratio vs. critical stress

<table>
<thead>
<tr>
<th>( \ell/r )</th>
<th>( \sigma_r )</th>
<th>( \ell/r )</th>
<th>( \lambda_c )</th>
<th>( F_{cr, AISC} )</th>
<th>( \sigma_d/\tau )</th>
<th>( \sigma_t )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>102.83</td>
<td>28.0</td>
<td>200</td>
<td>2.21</td>
<td>6.49</td>
<td>7.402</td>
<td>7.402</td>
<td>{ elastic }</td>
</tr>
<tr>
<td>100.53</td>
<td>29.0</td>
<td>180</td>
<td>1.98</td>
<td>8.01</td>
<td>9.138</td>
<td>9.138</td>
<td></td>
</tr>
<tr>
<td>98.33</td>
<td>30.0</td>
<td>160</td>
<td>1.76</td>
<td>10.14</td>
<td>11.566</td>
<td>11.566</td>
<td></td>
</tr>
<tr>
<td>95.96</td>
<td>31.0</td>
<td>140</td>
<td>1.54</td>
<td>13.25</td>
<td>15.107</td>
<td>15.107</td>
<td></td>
</tr>
</tbody>
</table>

(Continued)
\[ \sigma_t = \frac{\pi^2 E \tau}{(\frac{l}{r})^2} \Rightarrow \sigma_t = \frac{\pi^2 E}{(\frac{l}{r})^2} \] in the elastic range,

\[ \tau = 1.0 \left( \sigma_t = \sigma_r = \sigma_E \right) \]

4) and 5)

<table>
<thead>
<tr>
<th>( \ell/r )</th>
<th>( \sigma_r )</th>
<th>( \ell/r )</th>
<th>( \lambda_c )</th>
<th>( F_{cr \text{ AISC}} )</th>
<th>( \sigma_r/\tau )</th>
<th>( \sigma_t )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>93.12</td>
<td>32.0</td>
<td>120</td>
<td>1.32</td>
<td>18.03</td>
<td>20.562</td>
<td>20.562</td>
<td>( \sigma_t ), from graph</td>
</tr>
<tr>
<td>88.54</td>
<td>33.0</td>
<td>100</td>
<td>1.10</td>
<td>21.64</td>
<td>29.609</td>
<td>29.000</td>
<td></td>
</tr>
<tr>
<td>79.47</td>
<td>34.0</td>
<td>80</td>
<td>0.88</td>
<td>25.99</td>
<td>46.264</td>
<td>33.200</td>
<td></td>
</tr>
<tr>
<td>65.79</td>
<td>35.0</td>
<td>60</td>
<td>0.66</td>
<td>29.97</td>
<td>82.247</td>
<td>34.200</td>
<td></td>
</tr>
<tr>
<td>52.18</td>
<td>35.5</td>
<td>40</td>
<td>0.44</td>
<td>33.18</td>
<td>185.055</td>
<td>35.300</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>36.0</td>
<td>20</td>
<td>0.22</td>
<td>35.27</td>
<td>740.220</td>
<td>35.990</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>36.00</td>
<td>( \infty )</td>
<td>36.000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4-1 Slenderness ratio vs. critical stress—cont’d

Figure 1-31 Stress vs. tangential-modulus ratio

6) (i) For \( \lambda_c \leq 1.5 \Rightarrow F_{cr} = (0.658 \lambda_c^2) F_y \) (ii) For \( \lambda_c > 1.5 \Rightarrow F_{cr} = \left[ \frac{0.877}{\lambda_c^2} \right] F_y \)

where \( \lambda_c = (k \ell / r \pi) \sqrt{(F_y / E)} \)

Compression members (or elements) may be classified into three different regions depending on their slenderness ratios (or width-to-thickness ratios): yield zone, inelastic transition zone, and elastic buckling.
As can be seen from Fig. 1-34, the tangent-modulus theory reduces the critical compressive stress only slightly compared to that by the reduced modulus theory in the inelastic transition zone. Furthermore, both theories give the inelastic critical stresses much higher (unconservative) for a solid square cross section considered herein than those computed from the AISC LRFD formulas that are considered to be representative (Salmon and Johnson 1996) of many test data scattered over the world reported by Hall (1981). Experience (Yoo et al. 2001; Choi and Yoo 2005) has shown that the effect of the initial imperfections is significant in columns of intermediate slenderness, whereas the presence of residual stresses reduces the elastic buckling strength. The lowest

**Figure 1-32** Stress vs. reduced modulus ratio

**Figure 1-33** Stress vs. modular ratio
slenderness columns, which fail by yielding in compression, are hardly affected by the presence of either the initial imperfections or the residual stresses. Any nonlinear residual stress distributions in girder shapes having the residual tensile stress reaching up to the yield stress can readily be examined by present-day finite element codes.

1.14. METRIC SYSTEM OF UNITS

Dimensions in this book are given in English units. Hard conversion factors to the metric system are given in Table 1-5. The unit of force in the International System of units (Système International) is the Newton (N). In European countries and Japan, however, the commonly used unit is kilogram-force (kgf). Both units are included in the table. Metrication is the process of converting from the various other systems of units used throughout the world to the metric or SI (Système International) system. Although the process was begun in France in the 1790s and is currently converted 95% throughout the world, it is confronting stubborn resistance in a handful of countries. The main large-scale popular opposition to metrication appears to be based on tradition, aesthetics, cost, and distaste for a foreign system. Even in some countries where the international system is officially adopted, some sectors of the industry or in a special product line, old tradition units are still being practiced.
Table 1-5  Conversion Factors

<table>
<thead>
<tr>
<th>SI to English</th>
<th>English to SI</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Length</strong></td>
<td></td>
</tr>
<tr>
<td>1 mm = 0.03937 in</td>
<td>1 in = 25.4 mm</td>
</tr>
<tr>
<td>1 m = 3.281 ft</td>
<td>1 ft = 0.3048 m</td>
</tr>
<tr>
<td>1 km = 0.6214 mi</td>
<td>1 mi = 1.609 km</td>
</tr>
<tr>
<td><strong>Area</strong></td>
<td></td>
</tr>
<tr>
<td>1 mm$^2$ = $1.55 \times 10^{-3}$ in$^2$</td>
<td>1 in$^2$ = $0.6452 \times 10^3$ mm$^3$</td>
</tr>
<tr>
<td>1 cm$^2$ = $1.55 \times 10^{-1}$ in$^2$</td>
<td>1 in$^2$ = 6.452 cm$^2$</td>
</tr>
<tr>
<td>1 m$^2$ = 10.76 ft$^2$</td>
<td>1 ft$^2$ = 0.0929 m$^2$</td>
</tr>
<tr>
<td>1 m$^2$ = 1.196 yd$^2$</td>
<td>1 yd$^2$ = 0.836 m$^2$</td>
</tr>
<tr>
<td><strong>Volume</strong></td>
<td></td>
</tr>
<tr>
<td>1 mm$^3$ = $6.102 \times 10^{-5}$ in$^3$</td>
<td>1 in$^3$ = $16.387 \times 10^3$ mm$^3$</td>
</tr>
<tr>
<td>1 cm$^3$ = $6.102 \times 10^{-2}$ in$^3$</td>
<td>1 in$^3$ = 6.102 cm$^3$</td>
</tr>
<tr>
<td>1 m$^3$ = 35.3 ft$^3$</td>
<td>1 ft$^3$ = 0.0283 m$^3$</td>
</tr>
<tr>
<td>1 m$^3$ = 1.308 yd$^3$</td>
<td>1 yd$^3$ = 0.765 m$^3$</td>
</tr>
<tr>
<td><strong>Moment of inertia</strong></td>
<td></td>
</tr>
<tr>
<td>1 in$^4$ = $41.62 \times 10^4$ mm$^4$</td>
<td>1 mm$^4$ = $0.024 \times 10^{-4}$ in$^4$</td>
</tr>
<tr>
<td>1 in$^4$ = $41.62$ cm$^4$</td>
<td>1 cm$^4$ = $0.024$ in$^4$</td>
</tr>
<tr>
<td>1 in$^4$ = $41.62 \times 10^{-8}$ m$^4$</td>
<td>1 m$^4$ = $0.024 \times 10^8$ in$^4$</td>
</tr>
<tr>
<td><strong>Mass</strong></td>
<td></td>
</tr>
<tr>
<td>1 kg = 2.205 lb</td>
<td>1 lb = 0.454 kg</td>
</tr>
<tr>
<td>1 kg = $1.102 \times 10^{-3}$ ton</td>
<td>1 ton (2000 lb) = 907 kg</td>
</tr>
<tr>
<td>1 Mg = 1.102 ton</td>
<td>1 tonne (metric) = 1000 kg</td>
</tr>
<tr>
<td><strong>Force</strong></td>
<td></td>
</tr>
<tr>
<td>1 N = 0.2248 lbf</td>
<td>1 lbf = 4.448 N</td>
</tr>
<tr>
<td>1 kgf = 2.205 lbf</td>
<td>1 kip = 4.448 kN</td>
</tr>
<tr>
<td><strong>Stress</strong></td>
<td></td>
</tr>
<tr>
<td>1 kgf/cm$^2$ = 14.22 psi</td>
<td>1 psi = 0.0703 kgf/cm$^2$</td>
</tr>
<tr>
<td>1 kN/m$^2$ = 0.145 psi</td>
<td>1 psi = 6.895 kPa (kN/m$^2$)</td>
</tr>
<tr>
<td>1 MN/m$^2$ = 0.145 ksi</td>
<td>1 ksi = 6.895 MN/m$^2$ (MPa)</td>
</tr>
</tbody>
</table>

**GENERAL REFERENCES**

Some of the more general references on the stability of structures are collected in this section for convenience. References cited in the text are listed at the ends of the respective chapters. References requiring further
details are given in the footnotes. Relatively recent textbooks and reference books include those by Bleich (1952), Timoshenko and Gere (1961), Ziegler (1968), Britvec (1973), Chajes (1974), Brush and Almroth (1975), Allen and Bulson (1980), Chen and Lui (1991), Bazant and Cedolin (1991), Godoy (2000), Simitses and Hodges (2006), and Galambos and Surovek (2008). Some of these books address only the elastic stability of framed structures, while others extend the coverage into the stability of plates and shells, including dynamic stability and stability of nonconservative force systems.

The design of structural elements and components is beyond the scope of this book. For stability design criteria for columns and plates, Guide to Stability Design Criteria (Galambos, 1998) is an excellent reference. The design of highway bridge structures is to be carried out based on AASHTO (2007) specifications, and steel building frames are to follow AISC (2005) specifications. In the case of ship structures, separate design rules are stipulated for different vessel types such as IACS (2005) and IACS (2006). A variety of organizations and authorities are claiming jurisdiction over the certificates of airworthiness of civil aviation aircrafts.

REFERENCES


**PROBLEMS**

1.1 For structures shown in Fig. P1-1, determine the following:

(a) Using fourth-order DE, determine the lowest three critical loads.

(b) Determine the lowest two critical loads.
1.2 Two rigid bars are connected with a linear rotational spring of stiffness $C = M/\theta$ as shown in Fig. P1-2. Determine the critical load of the structure in terms of the spring constant and the bar length.

1.3 For the structure shown in Fig. P1-2, plot the load versus transverse deflection in a qualitative sense when:
   (a) the transverse deflections are large,
   (b) the load is applied eccentrically, and
   (c) the model has an initial transverse deflection $d_0$.

1.4 Determine the critical load of the structure shown in Fig. P1-4.
1.5 Derive the Euler-Lagrange differential equation and the necessary geometric and natural boundary conditions for a prismatic column of length $\ell$ and elastically supported by a rotational spring of constant $\beta$ at A and a linear spring of constant $\alpha$ at B as shown in Fig. P1-5. Determine the critical load, $P_{cr}$.

![Figure P1-5](image)

1.6 Turn-buckled threaded rods ($F_y = 50$ ksi, $F_u = 70$ ksi) are to be provided for the bracing system for a single-story frame shown in Fig. P1-4. Determine the diameter of the rod by the AISC Specifications, 13th edition, for each loading.

![Figure P1-6](image)
when the typicalFactored loads on each girder are $P_1 = 250$ kips and $P_2 = 150$ kips, and

(b) when the frame is subjected to a horizontal wind load of intensity 20 psf on the vertical projected area.

1.7 Equation (1.10.22) gives the critical uniformly distributed axial compressive load as $q_{cr} = 52.5EI/(\ell^3)$ for a bottom fixed and top pinned column. Using any appropriate computer program available, including STSTB, verify that the critical uniformly distributed compressive load is $q_{cr} = 30.0EI/(\ell^3)$ for a top fixed and bottom pinned column.

1.8 An axially loaded, simply supported column is made of structural steel with the following mechanical properties: $E = 30 \times 10^3$ ksi, $\sigma_p = 28.0$ ksi, $\sigma_y = 36$ ksi, and tangent moduli given in Table 1-2.

Determine the following:

(a) The value of $\ell/r$, which divides the elastic buckling range and the inelastic buckling range.

(b) The value of $\tau_r$ and $\ell/r$ for $P/A = 28, 30, 32, 34, 35, 35.5$ ksi using the double-modulus theory and assuming that the cross section of the column is a rectangle of side $b$ and $h = 2b$.

(c) The critical average stress $P/A$ for $\ell/r = 20, 40, 60, 80, 100, 120, 140, 160, 180$, and $200$ using the tangent-modulus theory in the inelastic range.

From the results of a), (b), and (c), plot:
(d) The “\( P/A - \tau_r \)” curve for the double-modulus theory.

(e) The “\( P/A - (\ell/r) \)” curves, distinguishing the portion of the curve derived by the tangent-modulus theory from that derived by the double-modulus theory. Present short discussions.

(f) The current AISC LRFD Specification specifies (Chapter E) that the critical value of \( P/A \) for axially loaded column shall not exceed the following:

(i) For \( \lambda_c \leq 1.5 \) \( F_{cr} = (0.658^{\lambda_c^2})F_y \)

(ii) For \( \lambda_c > 1.5 \) \( F_{cr} = \left[0.877/\lambda_c^2\right]F_y \)

Plot these curves and superimpose them on the graph in (e) using double arguments (\( \ell/r \) and \( \lambda_c \)) on the horizontal axis.