CHAPTER 2

Stress and Strain Transformation

2.1 INTRODUCTION

In Chapter 1 we defined stress and strain states at any point within the solid body as having six distinctive components, i.e. three normal and three shear components, with respect to an arbitrary coordinate system. The values of these six components at the given point will change with the rotation of the original coordinate system. It is therefore important to understand how to perform stress or strain transformations between two coordinate systems, and to be able to determine the magnitudes and orientations of stress or strain components that result. One key reason for stress or strain transformation is that the strains are normally measured in the laboratory along particular directions, and they must be transformed into a new coordinate system before the relevant stresses can be re-calculated. In this chapter we discuss the stress/strain transformation principles and the key role they play in the stress calculation of a drilled well at any point of interest; whether vertical, horizontal or inclined.

2.2 TRANSFORMATION PRINCIPLES

Let's consider the cube of Figure 1.2, and cut it in an arbitrary way such that the remaining part will form a tetrahedron. The reason for choosing a tetrahedron for this analysis is that a shape with four sides has the least number of planes to enclose a point. Figure 2.1 shows the stresses acting on the side and cut planes of the tetrahedron. The stress acting on the cut plane is denoted by S, which can be resolved into three components along the respective coordinate axes, assuming n defines the directional normal to the cut plane.

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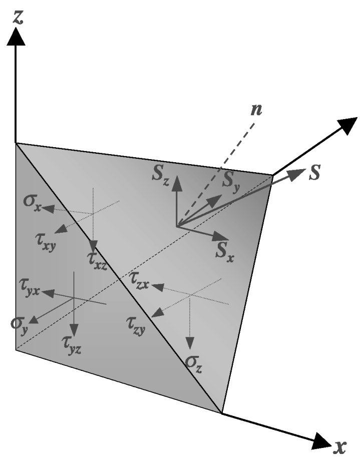

Figure 2.1 Normal and shear stresses acting on a tetrahedron.

Assuming the cut plane to have an area of unity, i.e. $A = 1$, the areas of the remaining cube sides can be expressed as (Figure 2.2):

$$\begin{aligned} A &= 1 \\ A_1 &= \cos(n, y) \\ A_2 &= \cos(n, x) \\ A_3 &= \cos(n, z) \end{aligned} \tag{2.1}$$

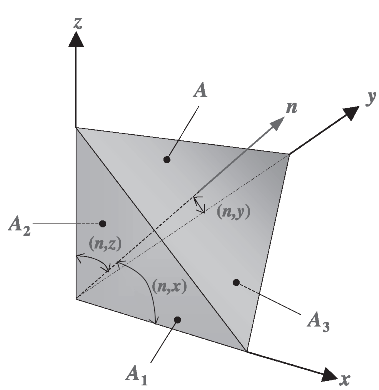

Figure 2.2 The areas of the side and cut planes.

Since the tetrahedron remains in equilibrium, we use the concept of force balance to determine the magnitude of the stresses acting on the cut plane. A force balance in the x direction is given by:

$$\sum F_x = 0$$

or

$$S_x A - \boldsymbol{\sigma}_x A_2 - \boldsymbol{\tau}_{xy} A_1 - \boldsymbol{\tau}_{xz} A_3 = 0$$

Repeating the force balance for the other two directions, and inserting the expressions for the areas given in Equation 2.1, the stresses acting on the cut plane, after some manipulation, can be given by:

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}_x & \boldsymbol{\tau}_{xy} & \boldsymbol{\tau}_{xz} \\ \boldsymbol{\tau}_{xy} & \boldsymbol{\sigma}_y & \boldsymbol{\tau}_{yz} \\ \boldsymbol{\tau}_{xz} & \boldsymbol{\tau}_{yz} & \boldsymbol{\sigma}_z \end{bmatrix} \begin{bmatrix} \cos(n, x) \\ \cos(n, y) \\ \cos(n, z) \end{bmatrix} \tag{2.2}$$

Equation 2.2 is known as *Cauchy's transformation law (principle)*, which can also be shortened to:

$$[S] = [\boldsymbol{\sigma}][n]$$

where $[S]$ represents the resulting stress vector acting on area A, assuming that the initial coordinate system of the cube will remain unchanged, and $[n]$ is the vector of *direction cosines*.

By rotating our coordinate system, all stress components may change in order to maintain the force balance. For simplicity, we will first study a coordinate transformation and its effect on the stress components in a two-dimensional domain, before proceeding towards a general three-dimensional analysis.

2.3 TWO-DIMENSIONAL STRESS TRANSFORMATION

Figure 2.3 shows a steel bar under a tensile load F, where the tensile stress can simply be determined across the plane p-q, normal to the applied load. Although, this is a one-dimensional loading problem, the stress state is two-dimensional where a side load of zero actually exists. To develop the idea of coordinate transformation, we examine the stresses acting on plane m-n, which has an arbitrary orientation relative to the applied load.

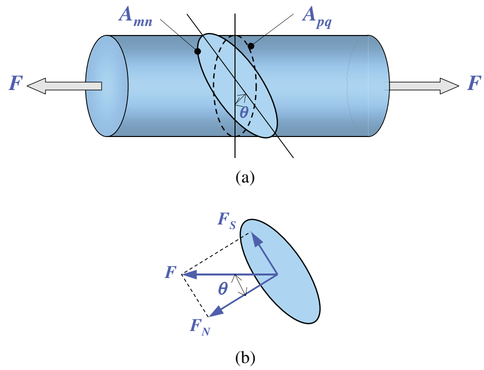

Figure 2.3 (a) Tensile force applied on a bar, (b) Projected forces parallel and normal to surface *m-n*.

The stress acting on plane *p-q* is simply expressed as:

$$\sigma_{pq} = \frac{F}{A_{pq}}$$

Applying force balance on plane *m-n* will project the applied force F into the normal force F_N and shear force F_S, i.e.:

$$F_N = F\cos\theta \qquad , \qquad F_S = F\sin\theta$$

The resulting normal and shear stresses acting on plane *m-n* will be:

$$\sigma_{mn} = \frac{F_N}{A_{mn}} = \frac{F\cos\theta}{\frac{A_{pq}}{\cos\theta}} = \sigma_{pq}\cos^2\theta$$

$$\tau_{mn} = \frac{F_S}{A_{mn}} = \frac{F\sin\theta}{\frac{A_{pq}}{\cos\theta}} = \sigma_{pq}\sin\theta\cos\theta$$

Introducing the following trigonometric identities:

$$2\sin\theta\cos\theta = \sin 2\theta$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

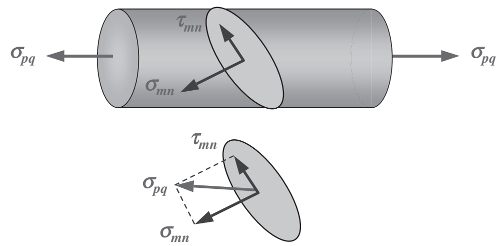

Figure 2.4 Normal and shear stresses acting in the bar.

The above two stress components can therefore be simplified to (Figure 2.4):

$$\sigma_{mn} = \tfrac{1}{2}\,\sigma_{pq}(1 + \cos 2\theta)$$

$$\tau_{mn} = \tfrac{1}{2}\sigma_{pq}\sin 2\theta \qquad (2.3)$$

The relation between the normal and shear stresses can most easily be illustrated using *Mohr's Circle* in which normal stress appears on the horizontal axis, shear stress corresponds to the vertical axis and the circle diameter extends to σ_{pq} as shown in Figure 2.5. By rotating the imaginary plane, any combination of shear and normal stresses can be found. Mohr's circle is used to determine the principal stresses, as well as for implementing failure analysis using Mohr-Coulomb criterion, which is going to be introduced and discussed in detail in Chapter 5.

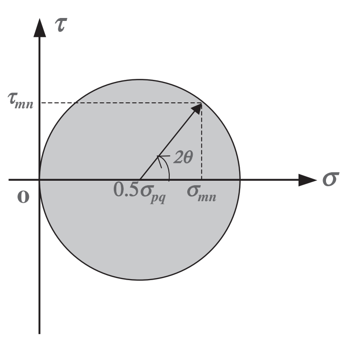

Figure 2.5 Mohr's circle for the stress state of plane *p-q*.

Note 2.1: Stresses are transformed according to a *squared trigonometric law.* This is because: (a) The criterion for transformation is *force balance* and not stress balance according to Newton's second law; and (b) Both the force and the area have to be transformed in space. This results in a squared transformation law.

2.4 STRESS TRANSFORMATION IN SPACE

We have presented how the tractions are transformed using the same coordinate system. We will now develop a formulation for the stress transformation in a three-dimensional domain – from the coordinate system (x, y, z) to a new system (x', y', z'), as shown in Figure 2.6.

The transformation is performed in two stages. Firstly, the x' axis is rotated to align with the cut plane normal n, and then the stress components are calculated (see Figure 2.6).

The tractions for the element along the old coordinate axes, i.e. x, y and z can be written using the Cauchy's transformation law as follows:

$$\begin{bmatrix} S_{x'x} \\ S_{y'y} \\ S_{z'z} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos(x', x) \\ \cos(y', y) \\ \cos(z', z) \end{bmatrix} \tag{2.4}$$

We then transform the tractions into the new coordinate system, i.e. (x', y', z'), which is $S_{x'y} \rightarrow S_{x'y'}$.

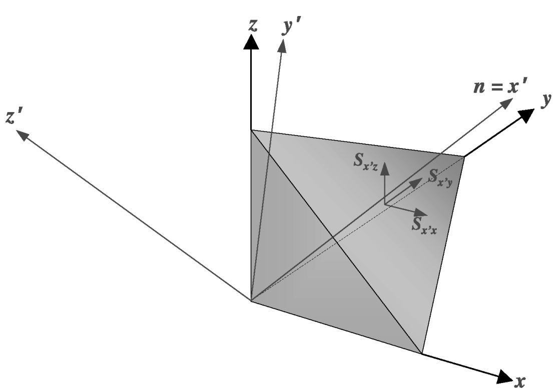

Figure 2.6 Stress transformation from one coordinate system to another.

It can be noted that the Cauchy's transformation law is similar to the transformation of the force components. Now the transformation of the area remains, which is carried out when the stresses related to the new coordinate system are found.

Considering the equilibrium condition of the tetrahedron, and using Newton's second law for the first stress component, we can write:

$$\sum F_{x'} = 0$$

or

$$S_{x'x'} = S_{x'x}\cos(x', x) + S_{x'y}\cos(x', y) + S_{x'z}\cos(x', z)$$

Assuming $\boldsymbol{\sigma}_{x'} \equiv S_{x'x'}$, the above equation can be expressed in a matrix form as:

$$\boldsymbol{\sigma}_{x'} = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \end{bmatrix} \begin{bmatrix} S_{x'x} \\ S_{x'y} \\ S_{x'z} \end{bmatrix} \quad (2.5)$$

By combining Equations 2.4 and 2.5, $\boldsymbol{\sigma}_{x'}$ can be given by:

$$\boldsymbol{\sigma}_{x'} = \begin{bmatrix} \cos(x', x) \\ \cos(y', y) \\ \cos(z', z) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\sigma}_x & \boldsymbol{\tau}_{xy} & \boldsymbol{\tau}_{xz} \\ \boldsymbol{\tau}_{xy} & \boldsymbol{\sigma}_y & \boldsymbol{\tau}_{yz} \\ \boldsymbol{\tau}_{xz} & \boldsymbol{\tau}_{yz} & \boldsymbol{\sigma}_z \end{bmatrix} \begin{bmatrix} \cos(x', x) \\ \cos(y', y) \\ \cos(z', z) \end{bmatrix} \quad (2.6)$$

Equation 2.6 presents a general stress transformation relationship for one of the stress components. To find the rest, the above method will be repeated five times. The final three-dimensional stress transformation equation becomes:

$$\left[\boldsymbol{\sigma}'\right] = [q][\boldsymbol{\sigma}][q]^T \quad (2.7)$$

where:

$$\left[\boldsymbol{\sigma}'\right] = \begin{bmatrix} \boldsymbol{\sigma}_{x'} & \boldsymbol{\tau}_{x'y'} & \boldsymbol{\tau}_{x'z'} \\ \boldsymbol{\tau}_{x'y'} & \boldsymbol{\sigma}_{y'} & \boldsymbol{\tau}_{y'z'} \\ \boldsymbol{\tau}_{x'z'} & \boldsymbol{\tau}_{y'z'} & \boldsymbol{\sigma}_{z'} \end{bmatrix} \quad \text{and}$$

$$[q] = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \\ \cos(y', x) & \cos(y', y) & \cos(y', z) \\ \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix} \quad (2.8)$$

Note 2.2: The complex derivation of the general stress transformation equation is the result of two processes: (1) determining traction along a new plane, and (2) rotation of the coordinate system. This is equivalent to performing a *force balance*, and also *transforming the area*.

It can easily be shown that the direction cosines will satisfy the following equations:

$$\begin{aligned}
\cos^2(x', x) + \cos^2(x', y) + \cos^2(x', z) &= 1 \\
\cos^2(y', x) + \cos^2(y', y) + \cos^2(y', z) &= 1 \\
\cos^2(z', x) + \cos^2(z', y) + \cos^2(z', z) &= 1
\end{aligned} \tag{2.9}$$

As an example, we now assume that stresses are known in the coordinate system (x, y, z), and we would like to find the transformed stresses in the new coordinate system (x', y', z') where the first coordinate system is rotated by an angle of θ around the z-axis to create the second one.

This is a two-dimensional case, because the z-axis remains unchanged as shown in Figure 2.7.

Using Equation 2.8 and Figure 2.7, it can be seen that:

$$\begin{array}{lll}
\left(x', x\right) = \theta & \left(x', y\right) = 90^\circ - \theta & \left(x', z\right) = 90^\circ \\
\left(y', x\right) = 90^\circ + \theta & \left(y', y\right) = \theta & \left(y', z\right) = 90^\circ \\
\left(z', x\right) = 90^\circ & \left(z', y\right) = 90^\circ & \left(z', z\right) = 0^\circ
\end{array}$$

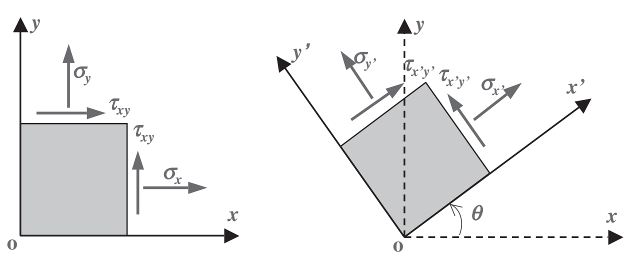

Figure 2.7 Stress components before and after transformation.

Assuming $\cos(90^\circ - \theta) = \sin\theta$ and $\cos(90^\circ + \theta) = -\sin\theta$ and inserting the above angles into Equation 2.8 gives the following transformation matrix:

$$[q] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

The transformed stresses can then be determined by means of Equation 2.7:

$$\left[\sigma'\right] = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \tag{2.11}$$

where the transformed stress components will become:

$$\begin{aligned} \sigma_{x'} &= \sigma_x\cos^2\theta + \tau_{xy}\sin2\theta + \sigma_y\sin^2\theta \\ \sigma_{y'} &= \sigma_x\sin^2\theta - \tau_{xy}\sin2\theta + \sigma_y\cos^2\theta \\ \sigma_{z'} &= \sigma_z \\ \tau_{x'y'} &= -\tfrac{1}{2}\,\sigma_x\sin2\theta + \tau_{xy}\cos2\theta + \tfrac{1}{2}\sigma_y\sin2\theta \\ \tau_{x'z'} &= \tau_{xz}\cos\theta + \tau_{yz}\sin\theta \\ \tau_{y'z'} &= -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta \end{aligned} \tag{2.12}$$

2.5 TENSOR OF STRESS COMPONENTS

A tensor is defined as an operator with physical properties, which satisfies certain laws for transformation. A tensor has 3^n components in space, where n represents the order of the tensor. Examples are: (i) temperature and mass which are scalars represented by $3^0 = 1$ component, (ii) velocity and force which are vectors represented by $3^1 = 3$ components, and, (iii) stress and strain which are three-dimensional tensor represented by $3^2 = 9$ components. The stress components can be written in a tensor as:

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.13}$$

where τ_{ij} is a normal stress if $i = j$ and τ_{ij} is a shear stress if $i \neq j$. The tensor in equation 2.8 is symmetric, i.e. $\tau_{ij} = \tau_{ji}$.

Equation 2.11, which defines the general three-dimensional stress transformation, seems to be too complicated for simple calculations. To avoid this, we use Equation 2.7 and introduce the following simple expression as a tensor:

$$\tau_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \tau_{kl} q_{ik} q_{lj} \tag{2.14}$$

As an example, Equation 2.14 can be used to express $\sigma_{x'}$ in terms of the non-transformed stress components, i.e. $\sigma_x, \sigma_y, \sigma_z$, τ_{xy}, τ_{xz} and τ_{yz}:

$$\begin{aligned} \sigma_{x'} &= \tau_{1'1'} = \sum_{k=1}^{3}\sum_{l=1}^{3} \tau_{kl} q_{1k} q_{l1} \\ &= \tau_{11} q_{11}^2 + \tau_{22} q_{21}^2 + \tau_{33} q_{31}^2 + 2\tau_{12} q_{11} q_{12} + 2\tau_{13} q_{11} q_{13} + 2\tau_{23} q_{21} q_{31} \end{aligned}$$

or

$$\begin{aligned} \sigma_{x'} &= \sigma_x \cos^2\theta + \sigma_y (-\sin\theta)^2 + \sigma_z \times 0^2 + 2\tau_{xy} \cos\theta \sin\theta \\ &\quad + 2\tau_{xz} \cos\theta \times 0 + 2\tau_{yz} (-\sin\theta) \times 0 \end{aligned}$$

or

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta$$

which is similar to that of $\sigma_{x'}$ given in Equation 2.12. The same procedure can be used to find other stress components of the transformed stress state.

2.6 STRAIN TRANSFORMATION IN SPACE

Strain can be transformed in the same way as stress. By comparing Equations 1.2 and 1.6 it can be seen that stress and strain matrices have identical structures. This means that by replacing σ with ε and τ with $\gamma/2$, the same transformation method can be used for strain. Using the method defined in Sections 2.3 and 2.4 for stress transformation, the general strain transformation may therefore be expressed by:

$$\varepsilon_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{kl} q_{ik} q_{lj} \tag{2.15}$$

where the directional cosines are given in Equation 2.7. As an example, Equation 2.15 can be used to express ε'_{xy} in terms of the non-transformed strain components, i.e.:

$$\varepsilon'_{xy} = \tfrac{1}{2}\gamma'_{xy} = \varepsilon'_{12} = \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{kl} q_{1k} q_{l2}$$

$$= \varepsilon_{11} q_{11} q_{12} + \varepsilon_{22} q_{12} q_{22} + \varepsilon_{33} q_{13} q_{33}$$

$$+ \varepsilon_{12}(q_{11}q_{22} + q_{21}q_{12}) + \varepsilon_{13}(q_{11}q_{32} + q_{31}q_{12}) + \varepsilon_{23}(q_{21}q_{32} + q_{31}q_{23})$$

or

$$= \varepsilon_{11}\cos\theta\sin\theta + \varepsilon_{11}\sin\theta\cos\theta + \varepsilon_{11} \times 0 \times 1$$

$$+\varepsilon_{12}(\cos^2\theta - \sin^2\theta) + \varepsilon_{13}(\cos\theta \times 0 + 0 \times \sin\theta) + \varepsilon_{23}(-\sin\theta \times 0 + 0 \times 0)$$

or

$$\varepsilon'_{xy} = \tfrac{1}{2}\gamma'_{xy} = \varepsilon_x \sin 2\theta + \tfrac{1}{2}\gamma_{xy}\cos 2\theta$$

Example

2.1: A plane stress condition exists at a point on the surface of a loaded rock, where the stresses have the magnitudes and directions as given below (where in this case, minus implies a tension and plus a compression):

$$\sigma_x = -6600\text{psi}$$
$$\sigma_y = 1700\text{psi}$$
$$\tau_{xy} = -2700\text{psi}$$

Determine the stresses acting on an element that is oriented at a clockwise angle of 45° with respect to the original element.

Solution: Referring to norm where the counter-clockwise angles are positive, the element of the loaded rock oriented at a clockwise angle of 45° would indicate $\theta = -45°$ (as shown in Figure 2.8).

Now, using Equation 2.12, we can readily calculate the stresses in the new coordinate system of (x',y') as below:

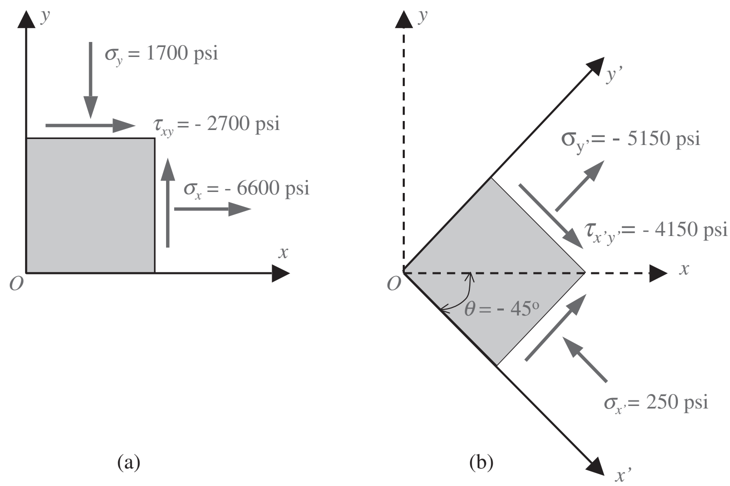

Figure 2.8 An element under plane stress, (a) in *x*-*y* coordinate system, (b) in *x*'-*y*' coordinate system.

$$\sigma_{x'} = -6600 \times \cos^2(-45) - 2700 \times \sin[2 \times (-45)] + 1700 \times \sin^2(-45)$$

$$\sigma_{x'} = 250 \text{ psi}$$

$$\sigma_{y'} = -6600 \times \sin^2(-45) + 2700 \times \sin[2 \times (-45)] + 1700\cos^2(-45)$$

$$\sigma_{y'} = -5150 \text{ psi}$$

$$\tau_{x'y'} = -\tfrac{1}{2} \times (-6600) \times \sin[2 \times (-45)] - 2700 \times \cos[2 \times (-45)] + \tfrac{1}{2} \times 1700 \times \sin[2 \times (-45)]$$

$$\tau_{x'y'} = -4150 \text{ psi}$$

Problems

2.1: Using the in-plane stresses of example 2.1, explain and show how the accuracy of the stress results in the new coordinate system can be verified.

2.2: Derive the general stress transformation equations for $\sigma_{y'}$, $\sigma_{z'}$, $\tau_{x'y'}$, $\tau_{x'z'}$ and $\tau_{y'z'}$ using Equation 2.14 and compare them with those of Equation 2.12.

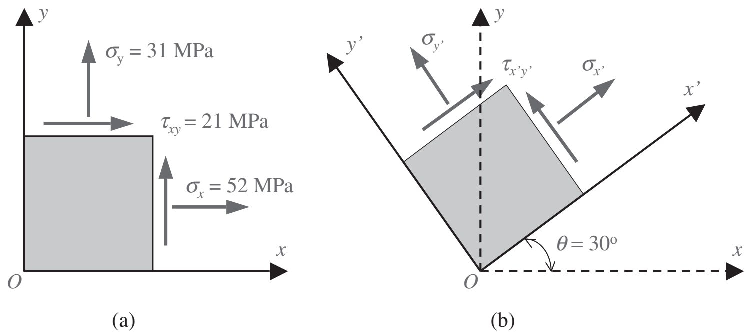

Figure 2.9 An element under plane stress, (a) in *x*-*y* coordinate system, (b) in *x*'-*y*' coordinate system.

2.3: Derive the general stress transformation equations for $\varepsilon_{x'}$, $\varepsilon_{y'}$, $\varepsilon_{z'}$, $\varepsilon_{x'z'}$ and $\varepsilon_{y'z'}$ using Equation 2.15.

2.4: An element in plane stress is subjected to stresses $\sigma_x = 52$ MPa, $\sigma_y = 31$ MPa, $\tau_{xy} = 21$ MPa, as shown in Figure 2.9. Determine the stresses acting on an element oriented at an angle θ $\{\tau\sigma\} = 30°$ from the *x* axis.

2.5: Using [q] in Equation 2.8, first find its transpose, i.e. $[q]^T$, and then derive direction cosines as given in Equation 2.9.