A Course in
Real Analysis
SECOND EDITION

John N. McDonald
School of Mathematical and Statistical Sciences
Arizona State University

Neil A. Weiss
School of Mathematical and Statistical Sciences
Arizona State University

Biographies by Carol A. Weiss
To Pat and Carol
Photo Credits

Cover image courtesy of iStockphoto; page xii photo courtesy of Andrew Sherwood; page xiii photo courtesy of Carol Weiss; page 2 photo courtesy of Wikipedia; page 28 photo courtesy of Wikipedia; page 80 photo courtesy of Wikipédia; page 110 photo courtesy of Wikipedia; page 144 photo courtesy of Wikipedia; page 178 photo courtesy of Professor Catterina Dagnino, Dipartimento di Matematica, Università di Torino; page 224 photo courtesy of Sovfoto/Eastfoto, Inc.; page 268 photo courtesy of Dr. Brigitte Bukovics; page 356 photo courtesy of Universitätsbibliothek Bonn; page 396 photo courtesy of Wikipedia; page 424 photo courtesy of UP1/Corbis-Bettmann; page 452 photo courtesy of Wikipedia; page 496 photo reprinted with permission from E. Jakimowicz, A. Miranowicz (eds.), “Stefan Banach – Remarkable Life, Brilliant Mathematics” (Gdańsk University Press, Gdańsk, 2010); page 550 photo courtesy of Professor Ingrid Daubechies, Department of Mathematics, Princeton University; page 602 photo courtesy of Mary Elizabeth Shannon; page 632 photo courtesy of Nigel Lesmoir-Gordon.
## Contents

About the Authors xii  
Preface xv  

---  

**PART ONE**  
Set Theory, Real Numbers, and Calculus  

<table>
<thead>
<tr>
<th>1</th>
<th>SET THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Basic Definitions and Properties 3</td>
</tr>
<tr>
<td>1.2</td>
<td>Functions and Sets 10</td>
</tr>
<tr>
<td>1.3</td>
<td>Equivalence of Sets; Countability 17</td>
</tr>
<tr>
<td>1.4</td>
<td>Algebras, σ-Algebras, and Monotone Classes 22</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
<th>THE REAL NUMBER SYSTEM AND CALCULUS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The Real Number System 29</td>
</tr>
<tr>
<td>2.2</td>
<td>Sequences of Real Numbers 35</td>
</tr>
<tr>
<td>2.3</td>
<td>Open and Closed Sets 47</td>
</tr>
<tr>
<td>2.4</td>
<td>Real-Valued Functions 54</td>
</tr>
<tr>
<td>2.5</td>
<td>The Cantor Set and Cantor Function 61</td>
</tr>
<tr>
<td>2.6</td>
<td>The Riemann Integral 67</td>
</tr>
</tbody>
</table>
## PART TWO  □  Measure, Integration, and Differentiation

### 3 □ LEBESGUE MEASURE ON THE REAL LINE

*Biography:* Émile Félix-Édouard-Justin Borel 80

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Borel Measurable Functions and Borel Sets</td>
<td>81</td>
</tr>
<tr>
<td>3.2</td>
<td>Lebesgue Outer Measure</td>
<td>89</td>
</tr>
<tr>
<td>3.3</td>
<td>Further Properties of Lebesgue Outer Measure</td>
<td>94</td>
</tr>
<tr>
<td>3.4</td>
<td>Lebesgue Measure</td>
<td>101</td>
</tr>
</tbody>
</table>

### 4 □ THE LEBESGUE INTEGRAL ON THE REAL LINE

*Biography:* Henri Léon Lebesgue 110

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>The Lebesgue Integral for Nonnegative Functions</td>
<td>111</td>
</tr>
<tr>
<td>4.2</td>
<td>Convergence Properties of the Lebesgue Integral for Nonnegative Functions</td>
<td>121</td>
</tr>
<tr>
<td>4.3</td>
<td>The General Lebesgue Integral</td>
<td>129</td>
</tr>
<tr>
<td>4.4</td>
<td>Lebesgue Almost Everywhere</td>
<td>139</td>
</tr>
</tbody>
</table>

### 5 □ ELEMENTS OF MEASURE THEORY

*Biography:* Constantin Carathéodory 144

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Measure Spaces</td>
<td>145</td>
</tr>
<tr>
<td>5.2</td>
<td>Measurable Functions</td>
<td>151</td>
</tr>
<tr>
<td>5.3</td>
<td>The Abstract Lebesgue Integral for Nonnegative Functions</td>
<td>158</td>
</tr>
<tr>
<td>5.4</td>
<td>The General Abstract Lebesgue Integral</td>
<td>165</td>
</tr>
<tr>
<td>5.5</td>
<td>Convergence in Measure</td>
<td>174</td>
</tr>
</tbody>
</table>

### 6 □ EXTENSIONS TO MEASURES AND PRODUCT MEASURE

*Biography:* Guido Fubini 178

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Extensions to Measures</td>
<td>179</td>
</tr>
<tr>
<td>6.2</td>
<td>The Lebesgue-Stieltjes Integral</td>
<td>191</td>
</tr>
<tr>
<td>6.3</td>
<td>Product Measure Spaces</td>
<td>201</td>
</tr>
<tr>
<td>6.4</td>
<td>Iteration of Integrals in Product Measure Spaces</td>
<td>212</td>
</tr>
</tbody>
</table>

### 7 □ ELEMENTS OF PROBABILITY

*Biography:* Andrei Nikolaevich Kolmogorov 224

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>The Mathematical Model for Probability</td>
<td>225</td>
</tr>
<tr>
<td>7.2</td>
<td>Random Variables</td>
<td>236</td>
</tr>
<tr>
<td>7.3</td>
<td>Expectation of Random Variables</td>
<td>247</td>
</tr>
<tr>
<td>7.4</td>
<td>The Law of Large Numbers</td>
<td>257</td>
</tr>
</tbody>
</table>
8 ▪ DIFFERENTIATION AND ABSOLUTE CONTINUITY

Biography: Giuseppe Vitali 268
8.1 Derivatives and Dini-Derivates 269
8.2 Functions of Bounded Variation 281
8.3 The Indefinite Lebesgue Integral 285
8.4 Absolutely Continuous Functions 292

9 ▪ SIGNED AND COMPLEX MEASURES

Biography: Johann Radon 302
9.1 Signed Measures 303
9.2 The Radon-Nikodym Theorem 311
9.3 Signed and Complex Measures 321
9.4 Decomposition of Measures 332
9.5 Measurable Transformations and the General Change-of-Variable Formula 342

PART THREE ▪ Topological, Metric, and Normed Spaces

10 ▪ TOPOLOGIES, METRICS, AND NORMS

Biography: Felix Hausdorff 356
10.1 Introduction to Topological Spaces 357
10.2 Metrics and Norms 363
10.3 Weak Topologies 370
10.4 Closed Sets, Convergence, and Completeness 373
10.5 Nets and Continuity 379
10.6 Separation Properties 386
10.7 Connected Sets 391

11 ▪ SEPARABILITY AND COMPACTNESS

Biography: Maurice Fréchet 396
11.1 Separability, Second Countability, and Metrizability 397
11.2 Compact Metric Spaces 401
11.3 Compact Topological Spaces 406
11.4 Locally Compact Spaces 410
11.5 Function Spaces 415
## 12 □ COMPLETE AND COMPACT SPACES

*Biography: Marshall Harvey Stone*

12.1 The Baire Category Theorem 425
12.2 Contractions of Complete Metric Spaces 429
12.3 Compactness in the Space $C(\Omega, \Lambda)$ 433
12.4 Compactness of Product Spaces 438
12.5 Approximation by Functions from a Lattice 442
12.6 Approximation by Functions from an Algebra 446

## 13 □ HILBERT SPACES AND BANACH SPACES

*Biography: David Hilbert*

13.1 Preliminaries on Normed Spaces 453
13.2 Hilbert Spaces 458
13.3 Bases and Duality in Hilbert Spaces 468
13.4 $L^p$-Spaces 475
13.5 Nonnegative Linear Functionals on $C(\Omega)$ 483
13.6 The Dual Spaces of $C(\Omega)$ and $C_0(\Omega)$ 491

## 14 □ NORMED SPACES AND LOCALLY CONVEX SPACES

*Biography: Stefan Banach*

14.1 The Hahn-Banach Theorem 497
14.2 Linear Operators on Banach Spaces 505
14.3 Compact Self-Adjoint Operators 511
14.4 Topological Linear Spaces 520
14.5 Weak and Weak* Topologies 530
14.6 Compact Convex Sets 537

## PART FOUR □ Harmonic Analysis, Dynamical Systems, and Hausdorff Measure

## 15 □ ELEMENTS OF HARMONIC ANALYSIS

*Biography: Ingrid Daubechies*

15.1 Introduction to Fourier Series 551
15.2 Convergence of Fourier Series 558
15.3 The Fourier Transform 565
15.4 Fourier Transforms of Measures 575
15.5 $L^2$-Theory of the Fourier Transform 583
15.6 Introduction to Wavelets 588
15.7 Orthonormal Wavelet Bases; The Wavelet Transform 593
## 16 MEASURABLE DYNAMICAL SYSTEMS

*Biography: Claude Elwood Shannon*

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.1</td>
<td>Introduction and Examples</td>
<td>603</td>
</tr>
<tr>
<td>16.2</td>
<td>Ergodic Theory</td>
<td>611</td>
</tr>
<tr>
<td>16.3</td>
<td>Isomorphism of Measurable Dynamical Systems; Entropy</td>
<td>618</td>
</tr>
<tr>
<td>16.4</td>
<td>The Kolmogorov-Sinai Theorem; Calculation of Entropy</td>
<td>625</td>
</tr>
</tbody>
</table>

## 17 HAUSDORFF MEASURE AND FRACTALS

*Biography: Benoit B. Mandelbrot*

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.1</td>
<td>Outer Measure and Measurability</td>
<td>633</td>
</tr>
<tr>
<td>17.2</td>
<td>Hausdorff Measure</td>
<td>638</td>
</tr>
<tr>
<td>17.3</td>
<td>Hausdorff Dimension and Topological Dimension</td>
<td>643</td>
</tr>
<tr>
<td>17.4</td>
<td>Fractals</td>
<td>648</td>
</tr>
</tbody>
</table>

Index 655
About the Authors:

John N. McDonald

After receiving his Ph.D. in mathematics from Rutgers University, John N. McDonald joined the faculty in the Department of Mathematics (now the School of Mathematical and Statistical Sciences) at Arizona State University, where he attained the rank of full professor. McDonald has taught a wide range of mathematics courses, including calculus, linear algebra, differential equations, real analysis, complex analysis, and functional analysis.

Known by colleagues and students alike as an excellent instructor, McDonald was honored by his department with the Charles Wexler Teaching Award. He also serves as a mentor in the prestigious Joaquin Bustoz Math-Science Honors Program, an intense academic program that provides motivated students an opportunity to commence university mathematics and science studies prior to graduating high school.

McDonald has numerous research publications, which span the areas of complex analysis, functional analysis, harmonic analysis, and probability theory. He is also a former Managing Editor of the Rocky Mountain Journal of Mathematics.

McDonald and his wife, Pat, have four children and six grandchildren. In addition to spending time with his family, he enjoys music, film, and staying physically fit through jogging and other exercising.
About the Authors:

Neil A. Weiss

Neil A. Weiss received his Ph.D. from UCLA and subsequently accepted an assistant-professor position at Arizona State University (ASU), where he was ultimately promoted to the rank of full professor. Weiss has taught mathematics, probability, statistics, and operations research from the freshman level to the advanced graduate level.

In recognition of his excellence in teaching, he received the Dean’s Quality Teaching Award from the ASU College of Liberal Arts and Sciences. He has also been runner-up twice for the Charles Wexler Teaching Award in the ASU School of Mathematical and Statistical Sciences. Weiss’s comprehensive knowledge and experience ensures that his texts are mathematically accurate, as well as pedagogically sound.

Weiss has published research papers in both theoretical and applied mathematics, including probability, engineering, operations research, numerical analysis, and psychology. He has also published several teaching-related papers.

In addition to his numerous research publications, Weiss has authored or coauthored books in real analysis, probability, statistics, and finite mathematics. His texts—well known for their precision, readability, and pedagogical excellence—are used worldwide.

In his spare time, Weiss enjoys walking and studying and practicing meditation. He is married and has two sons and three grandchildren.
Preface

This book is about real analysis, but it is not an ordinary real analysis book. Written with the student in mind, it incorporates pedagogical techniques not often found in books at this level.

In brief, *A Course in Real Analysis* is a modern graduate-level or advanced-undergraduate-level textbook about real analysis that engages its readers with motivation of key concepts, hundreds of examples, over 1300 exercises, and applications to probability and statistics, Fourier analysis, wavelets, measurable dynamical systems, Hausdorff measure, and fractals.

What Makes This Book Unique

*A Course in Real Analysis* contains many features that are unique for a real analysis text. Here are some of those features.

**Motivation of key concepts.** All key concepts are motivated. The importance and rationale behind ideas such as measurable functions, measurable sets, and Lebesgue integration are made transparent.

**Detailed theoretical discussion.** Detailed proofs of most results (i.e., lemmas, theorems, corollaries, and propositions) are provided. To fully engage the reader, proofs or parts of proofs are sometimes assigned as exercises.

**Illustrative examples.** Following most definitions and results, one or more examples, most of which consist of several parts, are presented that illustrate the concept or result in order to solidify it in the reader's mind and provide a concrete frame of reference.
Abundant and varied exercises. The book contains over 1300 exercises, not including parts, that vary widely with regard to application and level.

Applications. Diverse applications appear throughout the text, some as examples and others as entire sections or chapters. For instance, applications to probability theory are ubiquitous. Other applications include those to statistics, Fourier analysis, wavelets, measurable dynamical systems, Hausdorff measure, and fractals.

Careful referencing. As an aid to effective use of the book, references (including page numbers) to definitions, examples, exercises, and results are consistently provided. Additionally, post-referenced exercises are marked with a star (★); all such exercises are strongly recommended for solution by the reader.

Biographies. Each chapter begins with a biography of a famous mathematician. In addition to being of general interest, these biographies help the reader obtain a perspective on how real analysis and its applications have developed.

New to the Second Edition

In this second edition of *A Course in Real Analysis*, besides fine tuning the material from the first edition, several significant revisions have been made, many of which are based on feedback from instructors, students, and reviewers. Some of the most important revisions are as follows.

Chapter Splits. To make the chapters more uniform, long chapters in the first edition have been split into two chapters in the second edition.

Riemann Integrability. A detailed proof of the fact that a bounded function on a closed bounded interval is Riemann integrable if and only if its set of points of discontinuity has measure zero has been supplied.

Extensions to Measures. The treatment of extensions to measures has been simplified and improved by stating and proving a result that shows that the collection of all Cartesian products of sets formed from a finite number of semi-algebras is also a semialgebra.

Classical Change-of-Variable Formula. A subsection has been included that applies the general change-of-variable formula for measurable transformations to get a result that contains as a special case the classical change-of-variable formula in Euclidean $n$-space.

Self-Adjoint Operators. A section on self-adjoint operators on Hilbert space has been added that includes the statement and proof of the spectral theorem for compact self-adjoint operators.

The Schwartz Class. A subsection that discusses the Schwartz class—the collection of all complex-valued rapidly decreasing functions—and the behavior of the Fourier transform on that class has been provided.

Hausdorff Measure and Fractals. Another applications chapter has been added that includes a discussion of outer measure and metric outer measure, Hausdorff measure, Hausdorff dimension and topological dimension, and fractals.
Organization

A *Course in Real Analysis* offers considerable flexibility in the choice of material to cover. The following list is a brief explanation of the organization of the text.

- Chapters 1 and 2 present prerequisite material that may be review for many students but provide a common ground for all readers. At the option of the instructor, these two chapters can be covered either briefly or in detail; they can also be assigned to the students for independent reading.

- Chapters 3–6 present the elements of measure and integration by first discussing the Lebesgue theory on the line (Chapters 3 and 4) and then the abstract theory (Chapters 5 and 6). This material is prerequisite to all subsequent chapters.

- Chapter 7 presents an introduction to probability theory that includes the mathematical model for probability, random variables, expectation, and laws of large numbers. Although optional, this chapter is recommended as it provides a myriad of examples and applications for other topics.

- In Chapters 8 and 9, differentiation of functions and differentiation of measures are discussed, respectively. Topics examined include differentiability, bounded variation, absolute continuity of functions, signed and complex measures, the Radon-Nikodym theorem, decomposition of measures, and measurable transformations.

- Chapters 10 and 11 provide the fundamentals of topological and metric spaces. These chapters can be covered relatively quickly when the students have a background in topology from other courses. In addition to topics traditionally found in an introduction to topology, a discussion of weak topologies and function spaces is included.

- Completeness, compactness, and approximation comprise the topics for Chapter 12. Examined therein are the Baire category theorem, contractions of complete metric spaces, compactness in function and product spaces, and the Stone-Weierstrass theorem.

- Hilbert spaces and the classical Banach spaces are presented in Chapter 13. Among other things, bases and duality in Hilbert space, completeness and duality of $L^p$-spaces, and duality in spaces of continuous functions are discussed.

- The basic theory of normed and locally convex spaces is introduced in Chapter 14. Topics include the Hahn-Banach theorem, linear operators on Banach spaces, fundamental properties of locally convex spaces, and the Krein-Milman theorem.

- Chapter 15 provides applications of previous chapters to harmonic analysis. The elements of Fourier series and transforms and the $L^2$-theory of the Fourier transform are examined. In addition, an introduction to wavelets and the wavelet transform is presented.

- Chapter 16 introduces measurable dynamical systems and includes a discussion of ergodic theorems, isomorphisms of measurable dynamical systems, and entropy.
• Chapter 17, which is new to this edition, presents outer measure and measurability, Hausdorff measure, Hausdorff dimension and topological dimension, and an introduction to fractals.

Figure P.1 on the next page summarizes the preceding discussion and depicts the interdependence among chapters. In the flowchart, the prerequisites for a given chapter consist of all chapters that have a path leading to that chapter.

Acknowledgments

It is our pleasure to thank the following reviewers of the first and second editions of the book. Their comments and suggestions resulted in significant improvements to the text.

Bruce A. Barnes  
University of Oregon

Todd Kemp  
University of California, San Diego

Dennis D. Berkey  
Boston University

Yon-Seo Kim  
University of Chicago

Courtney Coleman  
Harvey Mudd College

Michael Klass  
University of California, Berkeley

Peter Duren  
University of Michigan

Enno Lenzmann  
University of Copenhagen

Wilfrid Gangbo  
Georgia Institute of Technology

Mara D. Neusel  
Texas Tech University

Maria Girardi  
University of South Carolina

Duong H. Phong  
Columbia University

Sigurdur Helgason  
Massachusetts Institute of Technology

Bert Schreiber  
Wayne State University

Our special thanks go to Bruce Barnes, who undertook a detailed reading and critiquing of the entire manuscript. We also thank the graduate students who furnished valuable feedback, in particular, Mohammed Alhodaly, Hamed Alsulami, Jimmy Mopecha, Lynn Tobin, and, especially, Jim Andrews, Trent Buskirk, Menassie Ephrem, Ken Peterson, John Williams, and Xiangrong Yin.

We thank Arizona State University for its support and those chairs of the ASU Mathematics Department who provided encouragement for the project: Rosemary Renaut, Christian Ringhofer, Nevin Savage, and William T. Trotter.

We also thank all of those at Elsevier/Academic Press for helping make this book a reality, in particular, Jeff Freeland, Katy Morrissey, Patricia Osborn, Marilyn Rash, and Lauren Schultz Yuhasz. These people ensured that the process of publishing the second edition of the book went smoothly and efficiently.

Our appreciation goes as well to our proofreaders, Cindy Scott and Carol Weiss. Finally, we would like to express our heartfelt thanks to Carol Weiss. Apart from writing the text, she was involved in every aspect of development and production. Moreover, Carol researched and wrote the biographies.

J.N.M

N.A.W.
FIGURE P.1 Interdependence among chapters