CERTAIN CLASSES OF SETS, MEASURABILITY, AND POINTWISE APPROXIMATION

In this introductory chapter, the concepts of a field and of a $\sigma$-field are introduced, they are illustrated by means of examples, and some relevant basic results are derived. Also, the concept of a monotone class is defined and its relationship to certain fields and $\sigma$-fields is investigated. Given a collection of measurable spaces, their product space is defined, and some basic properties are established. The concept of a measurable mapping is introduced, and its relation to certain $\sigma$-fields is studied. Finally, it is shown that any random variable is the pointwise limit of a sequence of simple random variables.

1 MEASURABLE SPACES

Let $\Omega$ be an abstract set (or space) and let $C$ be a class of subsets of $\Omega$, i.e., $C \subseteq P(\Omega)$, the class of all subsets of $\Omega$.

**Definition 1**
$C$ is said to be a field, usually denoted by $\mathcal{F}$, if

(i) $C$ is non-empty.
(ii) If $A \in C$, then $A^c \in C$.
(iii) If $A_1, A_2 \in C$, then $A_1 \cup A_2 \in C$. ■

**Remark 1**
In view of (ii) and (iii), the union $A_1 \cup A_2$ may be replaced by the intersection $A_1 \cap A_2$. 1
EXAMPLES

(1) \( \mathcal{C} = \{ \emptyset, \Omega \} \) is a field called the \textit{trivial} field. (It is the smallest possible field.)

(2) \( \mathcal{C} = \{ \text{all subsets of } \Omega \} = \mathcal{P}(\Omega) \) is a field called the \textit{discrete} field. (It is the largest possible field.)

(3) \( \mathcal{C} = \{ \emptyset, A, A^c, \Omega \} \) for some \( A \) with \( \emptyset \subset A \subset \Omega \).

(4) Let \( \Omega \) be infinite (countably or not) and let \( \mathcal{C} = \{ A \subseteq \Omega; A \text{ is finite or } A^c \text{ is finite} \} \). Then \( \mathcal{C} \) is a field.

(5) Let \( \mathcal{C} \) be the class of all (finite) sums (unions of pairwise disjoint sets) of the partitioning sets of a finite partition of an arbitrary set \( \Omega \) (see Definition 2 below). Then \( \mathcal{C} \) is a field \( (\text{induced or generated by the underlying partition}) \).

Remark 2

In Example 4, it is to be observed that if \( \Omega \) is finite rather than infinite, then \( \mathcal{C} = \mathcal{P}(\Omega) \).

Consequences of Definition 1

(1) \( \Omega, \emptyset \in \mathcal{F} \) for every \( \mathcal{F} \).

(2) If \( A_j \in \mathcal{F}, j = 1, \ldots, n \), then \( \bigcup_{j=1}^{n} A_j \in \mathcal{F} \).

(3) If \( A_j \in \mathcal{F}, j = 1, \ldots, n \), then \( \bigcap_{j=1}^{n} A_j \in \mathcal{F} \).

Remark 3

It is shown by examples that \( A_j \in \mathcal{F}, j \geq 1 \), need not imply \( \bigcup_{j=1}^{\infty} A_j \in \mathcal{F} \), and similarly for \( \bigcap_{j=1}^{\infty} A_j \) (see Remark 5 below).

Definition 2

\( \mathcal{C} \) is said to be a \textit{\( \sigma \)-field}, usually denoted by \( \mathcal{A} \), if it is a field and (iii) in Definition 1 is strengthened to
(iii’) If $A_j \in \mathcal{C}$, $j = 1, 2, ...$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{C}$. ■

**Remark 4**

In view of (ii) and (iii), the union in (iii’) may be replaced by the intersection $\bigcap_{j=1}^{\infty} A_j$.

**EXAMPLES**

(6) $\mathcal{C} = \{\emptyset, \Omega\}$ is a $\sigma$-field called the **trivial $\sigma$-field**.

(7) $\mathcal{C} = \mathcal{P}(\Omega)$ is a $\sigma$-field called the **discrete $\sigma$-field**.

(8) Let $\Omega$ be uncountable and let $\mathcal{C} = \{A \subseteq \Omega; A \text{ is countable or } A^c \text{ is countable}\}$. Then $\mathcal{C}$ is a $\sigma$-field. (Of course, if $\Omega$ is countably infinite, then $\mathcal{C} = \mathcal{P}(\Omega)$).

(9) Let $\mathcal{C}$ be the class of all countable sums of the partitioning sets of a countable partition of an arbitrary set $\Omega$. Then $\mathcal{C}$ is a $\sigma$-field (*induced* or *generated* by the underlying partition). ■

**Remark 5**

A $\sigma$-field is always a field, but a field need not be a $\sigma$-field. In fact, in Example 4 take $\Omega = \mathbb{R}$ (real line), and let $A_j = \{k \text{ integer}; -j \leq k \leq j\}$, $j = 0, 1, ...$. Then $A_j \in \mathcal{C}$, $\bigcup_{j=0}^{\infty} A_j \in \mathcal{C}$ for any $n = 0, 1, ...$ but $\bigcup_{j=0}^{\infty} A_j (= \text{set of all integers}) \notin \mathcal{C}$.

Let $I$ be any index set. Then

**Theorem 1**

(i) If $\mathcal{F}_j$, $j \in I$ are fields, so is $\bigcap_{j \in I} \mathcal{F}_j = \{A \subseteq \Omega; A \in \mathcal{F}_j, j \in I\}$.

(ii) If $\mathcal{A}_j$, $j \in I$ are $\sigma$-fields, so is $\bigcap_{j \in I} \mathcal{A}_j = \{A \subseteq \Omega; A \in \mathcal{A}_j, j \in I\}$. ■

**Proof**

Immediate. ■
Let $C$ be any class of subsets of $\Omega$. Then

**Theorem 2**

(i) There is a unique minimal field containing $C$. This is denoted by $F(C)$ and is called the **field generated by $C$**.

(ii) There is a unique minimal $\sigma$-field containing $C$. This is denoted by $\sigma(C)$ and is called the **$\sigma$-field generated by $C$**. ■

**Proof**

(i) $F(C) = \bigcap_{j \in I} F_j$, where $\{F_j, j \in I\}$ is the non-empty class of all fields containing $C$.

(ii) $\sigma(C) = \bigcap_{j \in I} A_j$, where $\{A_j, j \in I\}$ is the non-empty class of all $\sigma$-fields containing $C$. ■

**Remark 6**

Clearly, $\sigma(F(C)) = \sigma(C)$. Indeed, $C \subseteq F(C)$, which implies $\sigma(C) \subseteq \sigma(F(C))$. Also, every $\sigma$-field $A_i \supseteq C$ and therefore $A_i \supseteq F(C)$, since $A_i$ is a field (being a $\sigma$-field), and $F(C)$ is the minimal field (over $C$). Hence $\sigma(C) = \bigcap A_i \supseteq F(C)$. Since $\sigma(C)$ is a $\sigma$-field, it contains the minimal $\sigma$-field over $F(C)$, $\sigma(F(C))$; i.e., $\sigma(C) \supseteq \sigma(F(C))$. Hence $\sigma(C) = \sigma(F(C))$.

**Application 1**

Let $\Omega = \mathbb{R}$ and $C_0 = \{\text{all intervals in } \mathbb{R}\} = \{(x, y), (x, y], [x, y), (-\infty, a), (-\infty, a], [b, \infty), (b, \infty), x, y \in \mathbb{R}, x < y, a, b \in \mathbb{R}\}$. Then $\sigma(C_0)$ is denoted by $\mathcal{B}$ and is called the **Borel $\sigma$-field** over the real line. The sets in $\mathcal{B}$ are called **Borel sets**. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. $\bar{\mathbb{R}}$ is called the **extended real line** and the $\sigma$-field $\mathcal{B}$ generated by $\mathcal{B} \cup \{-\infty\} \cup \{\infty\}$ the **extended Borel $\sigma$-field**.

**Remark 7**

$\{x\} \in \mathcal{B}$ for every $x \in \mathbb{R}$. Indeed, $\{x\} = \bigcap_{n=1}^{\infty} [x, x + \frac{1}{n}]$ with $[x, x + \frac{1}{n}] \in \mathcal{B}$. Hence $\bigcap_{n=1}^{\infty} [x, x + \frac{1}{n}] \in \mathcal{B}$, or $\{x\} \in \mathcal{B}$. 


Definition 3

The pair $(\Omega, A)$ is called a measurable space and the sets in $A$ measurable sets. In particular, $(\mathbb{R}, B)$ is called the Borel real line, and $(\bar{\mathbb{R}}, \mathcal{B})$ the extended Borel real line.

Let $\mathcal{C}$ again be a class of subsets of $\Omega$. Then

Definition 4

$\mathcal{C}$ is called a monotone class if $A_j \in \mathcal{C}$, $j = 1, 2, ...$ and $A_j \uparrow$ (i.e., $A_1 \subseteq A_2 \subseteq ...$) or $A_j \downarrow$ (i.e., $A_1 \supseteq A_2 \supseteq ...$), then $\lim_{j \to \infty} A_j \overset{def}{=} \bigcup_{j=1}^{\infty} A_j \in \mathcal{C}$ and $\lim_{j \to \infty} A_j \overset{def}{=} \bigcap_{j=1}^{\infty} A_j \in \mathcal{C}$, respectively.

Theorem 3

A $\sigma$-field $\mathcal{A}$ is a monotone field (i.e., a field that is also a monotone class) and conversely.

Proof

One direction is immediate. As for the other, let $\mathcal{F}$ be a monotone field and let any $A_j \in \mathcal{F}$, $j = 1, 2, ...$. To show that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$. We have: $\bigcup_{j=1}^{\infty} A_j = A_1 \cup (A_1 \cup A_2) \cup ... \cup (A_1 \cup ... \cup A_n) \cup ... = \bigcup_{n=1}^{\infty} B_n$, where $B_n = \bigcup_{j=1}^{n} A_j$, and hence $B_n \in \mathcal{F}$, $n = 1, 2, ...$ and $B_n \uparrow$. Thus $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

Theorem 4

If $\mathcal{M}_j$, $j \in I$, are monotone classes, so is $\bigcap_{j \in I} \mathcal{M}_j = \{A \subseteq \Omega; A \in \mathcal{M}_j, j \in I\}$.

Proof

Immediate.

Theorem 5

There is a unique minimal monotone class $\mathcal{M}$ containing $\mathcal{C}$. 


Proof

\[ \mathcal{M} = \bigcap_{j \in I} \mathcal{M}_j, \] where \{\mathcal{M}_j, j \in I\} is the class of all monotone classes containing \( \mathcal{C} \). \[ \blacksquare \]

Remark 8

\{\mathcal{M}_j, j \in I\} is non-empty since \( \sigma(\mathcal{C}) \) or \( \mathcal{P}(\Omega) \) belong in it.

Remark 9

It may be seen by means of examples (see Exercise 10) that a monotone class need not be a field.

However, see the next lemma, as well as Theorem 6.

Lemma 1

Let \( \mathcal{C} \) be a field and \( \mathcal{M} \) be the minimal monotone class containing \( \mathcal{C} \). Then \( \mathcal{M} \) is a field.

Proof

In order to prove that \( \mathcal{M} \) is a field, it suffices to prove that relations (*) hold, where

\[
(*) \left\{ \begin{array}{l}
\text{for every } A, B \in \mathcal{M}, \text{ we have:} \\
\text{(i) } A \cap B \in \mathcal{M} \\
\text{(ii) } A^c \cap B \in \mathcal{M} \\
\text{(iii) } A \cap B^c \in \mathcal{M}
\end{array} \right.
\]

(That is, for every \( A, B \in \mathcal{M} \), their intersection is in \( \mathcal{M} \), and so is the intersection of any one of them by the complement of the other.)

In fact, \( \mathcal{M} \supseteq \mathcal{C} \), implies \( \Omega \in \mathcal{M} \). Taking \( B = \Omega \), we get that for every \( A \in \mathcal{M}, A^c \cap \Omega = A^c \in \mathcal{M} \) (by (ii)). Since also \( A \cap B \in \mathcal{M} \) (by (i)) for all \( A, B \in \mathcal{M} \), the proof would be completed.

In order to establish (*), we follow the following three steps:

Step 1. For any \( A \in \mathcal{M} \), define \( \mathcal{M}_A = \{ B \in \mathcal{M} ; \text{ (*) holds} \} \), so that \( \mathcal{M}_A \subseteq \mathcal{M} \). Obviously \( A \in \mathcal{M}_A \), since \( \emptyset \in \mathcal{M} \). It is asserted that \( \mathcal{M}_A \) is a monotone class. Let \( B_j \in \mathcal{M}_A, j = 1, 2, ..., \) \( B_j \uparrow \). To show that \( \bigcup_{j=1}^{\infty} B_j \defeq B \in \mathcal{M}_A \); i.e., to show that (*) holds. We have: \( A \cap B = \)

\[
A \cap \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (A \cap B_j)
\]

Since \( A \subseteq \bigcup_{j=1}^{\infty} B_j \), it follows that \( A \cap B \in \mathcal{M}_A \). Therefore, (*) holds for \( A \cap B \). By the same reasoning, it follows that (*) holds for any \( A \subseteq \bigcup_{j=1}^{\infty} B_j \). Thus, (*) holds for \( \bigcup_{j=1}^{\infty} B_j \). Therefore, \( \mathcal{M}_A \) is a field.
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\[ A \cap \left( \bigcup_{j} B_j \right) = \bigcup_{j} (A \cap B_j) \in \mathcal{M}, \text{ since } \mathcal{M} \text{ is monotone and } A \cap B_j \uparrow. \]

Next, \( A^c \cap B = A^c \cap \left( \bigcup_{j} B_j \right) = \bigcup_{j} (A^c \cap B_j) \in \mathcal{M} \) since \( A^c \cap B_j \in \mathcal{M} \) by (\( \ast \))(ii), and \( A^c \cap B_j \uparrow \). Finally, \( A \cap B^c = A \cap \left( \bigcup_{j} B_j \right)^c = \bigcap_{j} (A \cap B_j^c) \in \mathcal{M} \) since \( \mathcal{M} \) is monotone. The case that \( B_j \downarrow \) is treated similarly, and the proof that \( \mathcal{M}_A \) is a monotone class is complete.

\textbf{Step 2.} If \( A \in \mathcal{C} \), then \( \mathcal{M}_A = \mathcal{M} \). As already mentioned, \( \mathcal{M}_A \subseteq \mathcal{M} \).

\textbf{Step 3.} If \( A \) is any set in \( \mathcal{M} \), then \( \mathcal{M}_A = \mathcal{M} \). We show that \( \mathcal{C} \subseteq \mathcal{M}_A \), which implies \( \mathcal{M} \subseteq \mathcal{M}_A \) since \( \mathcal{M}_A \) is a monotone class containing \( \mathcal{C} \) and \( \mathcal{M} \) is the minimal monotone class containing \( \mathcal{C} \). Thus \( \mathcal{M}_A = \mathcal{M} \).

\textbf{Theorem 6}

Let \( \mathcal{C} \) be a field and \( \mathcal{M} \) be the minimal monotone class containing \( \mathcal{C} \). Then \( \mathcal{M} = \sigma(\mathcal{C}) \).  

\textbf{Proof}

Evidently, \( \mathcal{M} \subseteq \sigma(\mathcal{C}) \) since every \( \sigma \)-field is a monotone class. By Lemma 1, \( \mathcal{M} \) is a field, and hence a \( \sigma \)-field, by Theorem 3. Thus, \( \mathcal{M} \supseteq \sigma(\mathcal{C}) \).  

\textbf{Remark 10}

Lemma 1 and Theorem 6 just discussed provide an illustration of the intricate relation of fields, monotone classes, and \( \sigma \)-fields in a certain setting. As will also be seen in several places in this book, monotone classes are often used as tools in arguments meant to establish results about \( \sigma \)-fields. In this kind of arguments, the roles of a field and of a monotone class may be substituted by the so-called \( \pi \)-systems and \( \lambda \)-systems, respectively. The definition of these concepts may
be found, for example, in page 41 in Billingsley (1995). A result analogous to Theorem 6 is then Theorem 1.3 in page 5 of the reference just cited, which states that: If $P$ is a $\pi$-system and $G$ is a $\lambda$-system, then $P \subset G$ implies $\sigma(P) \subset G$.

## 2 PRODUCT MEASURABLE SPACES

Consider the measurable spaces $(\Omega_1, A_1)$, $(\Omega_2, A_2)$. Then

**Definition 5**

The product space of $\Omega_1$, $\Omega_2$, denoted by $\Omega_1 \times \Omega_2$, is defined as follows:

$$\Omega_1 \times \Omega_2 = \{ \omega = (\omega_1, \omega_2); \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \}.$$ 

In particular, for $A \in A_1$, $B \in A_2$ the product of $A, B$, denoted by $A \times B$, is defined by:

$$A \times B = \{ \omega = (\omega_1, \omega_2); \omega_1 \in A, \omega_2 \in B \}.$$ 

And the subsets $A \times B$ of $\Omega_1 \times \Omega_2$ for $A \in A_1, B \in A_2$ are called (measurable) rectangles. $A, B$ are called the sides of the rectangle.

From Definition 5, one easily verifies the following lemma.

**Lemma 2**

Consider the rectangle $E = A \times B$. Then

(i) $E^c = (A \times B^c) + (A^c \times \Omega_2) = (A^c \times B) + (\Omega_1 \times B^c)$.

Consider the rectangles $E_1 = A_1 \times B_1$, $E_2 = A_2 \times B_2$. Then

(ii) $E_1 \cap E_2 = (A_1 \cap A_2) \times (B_1 \cap B_2)$. Hence $E_1 \cap E_2 = \emptyset$ if and only if at least one of the sets $A_1 \cap A_2$, $B_1 \cap B_2$ is $\emptyset$.

Consider the rectangles $E_1$, $E_2$ as above, and also the rectangles $F_1 = A'_1 \times B'_1$, $F_2 = A'_2 \times B'_2$. Then

(iii) $(E_1 \cap F_1) \cap (E_2 \cap F_2) = [(A_1 \cap A'_1) \times (B_1 \cap B'_1)] \cap [(A_2 \cap A'_2) \times (B_2 \cap B'_2)]$ (by (ii))

Hence, the left-hand side is $\emptyset$ if and only if at least one of $(A_1 \cap A_2) \cap (A'_1 \cap A'_2)$, $(B_1 \cap B_2) \cap (B'_1 \cap B'_2)$ is $\emptyset$. 


Theorem 7

Let $C$ be the class of all finite sums (i.e., unions of pairwise disjoint) of rectangles $A \times B$ with $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$. Then $C$ is a field (of subsets of $\Omega_1 \times \Omega_2$).

Proof

Clearly, $C \neq \emptyset$. Next, let $E, F \in C$. Then we show that $E \cap F \in C$. In fact, $E, F \in C$ implies that $E = \bigcup_{i=1}^{m} E_i$, $F = \bigcup_{j=1}^{n} F_j$ with $E_i = A_i \times B_i$, $i = 1, \ldots, m$, $F_j = A'_j \times B'_j$, $j = 1, \ldots, n$. Thus $E \cap F = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (E_i \cap F_j)$ and $E_i \cap F_j$, $E'_i \cap F'_j$ are disjoint for $(i, j) \neq (i', j')$ by Lemma 2(ii), (iii). Indeed, in Lemma 2(iii), make the identification: $A_1 = A_i$, $B_1 = B_i$, $A_2 = A_r$, $B_2 = B_r$, $A'_1 = A'_i$, $B'_1 = B'_i$, $A'_2 = A'_r$, $B'_2 = B'_r$ to get $(E_i \cap F_j) \cap (E'_i \cap F'_j) = [(B_i \cap B'_r) \cap (B'_i \cap B_r)] \cap [(B_i \cap B'_r) \cap (B'_i \cap B_r)]$ by the third line on the right-hand side in Lemma 2(iii), and at least one of $(A_i \cap A_r) \cap (A'_i \cap A'_r)$, $(B_i \cap B_r) \cap (B'_i \cap B'_r)$ is equal to $\emptyset$ by Lemma 2(ii). Then, by Lemma 2(iii) again, $(E_i \cap F_j) \cap (E'_i \cap F'_j) = \emptyset$, and therefore $E \cap F = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (E_i \cap F_j)$. However, $E_i \cap F_j = (A_i \cap A'_j) \times (B_i \cap B'_j)$ (by Lemma 2(ii)), and $A_i \cap A'_j \in \mathcal{A}_1$, $B_i \cap B'_j \in \mathcal{A}_2$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Thus $E \cap F$ is the sum of finitely many rectangles and hence $E \cap F \in C$. (By induction it is also true that if $E_k \in C$, $k = 1, \ldots, \ell$, then $\bigcap_{k=1}^{\ell} E_k \in C$.) Finally, $E^c = (\bigcup_{i=1}^{m} E_i)^c = \bigcap_{i=1}^{m} E_i^c = \bigcap_{i=1}^{m} [(A_i \times B_i^c) \cup (A_i^c \times \Omega_2)]$ (by Lemma 2(ii)), and $A_i \times B_i^c$, $A_i^c \times \Omega_2$ are disjoint rectangles so that their sum is in $C$. But then so is their intersection over $i = 1, \ldots, m$ by the induction just mentioned. The proof is completed.

Remark 11

Clearly, the theorem also holds true if we start out with fields $\mathcal{F}_1$ and $\mathcal{F}_2$ rather than $\sigma$-fields $\mathcal{A}_1$ and $\mathcal{A}_2$.

Definition 6

The $\sigma$-field generated by the field $C$ is called the product $\sigma$-field of $\mathcal{A}_1$, $\mathcal{A}_2$ and is denoted by $\mathcal{A}_1 \times \mathcal{A}_2$. The pair $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2)$ is called the product measurable space of the spaces $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$.
If we have \( n \geq 2 \) measurable spaces \((\Omega_i, \mathcal{A}_i), i = 1, ..., n\), the product measurable space \((\Omega_1 \times ... \times \Omega_n, \mathcal{A}_1 \times ... \times \mathcal{A}_n)\) is defined in an analogous way. In particular, if \( \Omega_1 = ... = \Omega_n = \mathbb{R} \) and \( \mathcal{A}_1 = ... = \mathcal{A}_n = \mathcal{B} \), then the product space \((\mathbb{R}^n, \mathcal{B}^n)\) is the \( n \)-dimensional Borel space, where \( \mathbb{R}^n = \mathbb{R} \times ... \times \mathbb{R} \) (\( n \) factors), and \( \mathcal{B}^n \) is called the \( n \)-dimensional Borel \( \sigma \)-field. The members of \( \mathcal{B}^n \) are called the \( n \)-dimensional Borel sets.

Now we consider the case of infinitely (countably or not) many measurable spaces \((\Omega_i, \mathcal{A}_i), t \in T\), where the \((\neq \emptyset)\) index set \( T \) will usually be the real line or the positive half of it or the unit interval \((0, 1)\) or \([0,1]\).

**Definition 7**

The product space of \( \Omega_t, t \in T \), denoted by \( \prod_{t \in T} \Omega_t \) or \( \Omega_T \), is defined by

\[
\Omega_T = \prod_{t \in T} \Omega_t = \{ \omega = (\omega_t, t \in T) ; \omega_t \in \Omega_t, t \in T \}. \tag{\ref{eq:product_space}}
\]

By forming the point \( \omega = (\omega_t, t \in T) \) with \( \omega_t \in \Omega_t, t \in T \), we tacitly assume, by invoking the axiom of choice, that there exists a function on \( T \) into \( \bigcup_{t \in T} \Omega_t \) with \( \Omega_t \neq \emptyset, t \in T \), whose value at \( t, \omega_t \), belongs in \( \Omega_t \).

Now for \( T = \{1, 2\}, \Omega_1 \times \Omega_2 = \{ \omega = (\omega_1, \omega_2); \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \} \). Also, let \( f : T \to \Omega_1 \cup \Omega_2 \) such that \( f(1) \in \Omega_1, f(2) \in \Omega_2 \). Then \( f(1), f(2) \in (1 \times 2) \). Conversely, any \( (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \) is the (ordered) pair of values of a function \( f \) on \( T \) into \( \Omega_1 \cup \Omega_2 \) with \( f(1) \in \Omega_1, f(2) \in \Omega_2 \); namely, the function for which \( f(1) = \omega_1, f(2) = \omega_2 \). Thus, \( \Omega_1 \times \Omega_2 \) may be looked upon as the collection of all functions \( f \) on \( T \) into \( \Omega_1 \cup \Omega_2 \) with \( f(1) \in \Omega_1, f(2) \in \Omega_2 \). Similar interpretation holds for any finite collection of \( (\neq \emptyset)\Omega_i, i = 1, ..., n \), as well as any collection of \( (\neq \emptyset)\Omega_t, t \in T \) (\( \neq \emptyset \)) (by the axiom of choice). Thus, \( \Omega_T = \prod_{t \in T} \Omega_t = \{ f : T \to \bigcup_{t \in T} \Omega_t; f(t) \in \Omega_t, t \in T \} \). In particular, if \( T = \mathbb{R} \) and \( \Omega_t = \mathbb{R} \), \( t \in T \), then \( \Omega_T = \prod_{t \in T} \Omega_t \) is the set of all real-valued functions defined on \( \mathbb{R} \).

**Remark 12**

In many applications, we take \( T = [0, 1] \), \( \Omega_t = \mathbb{R} \), \( t \in T \) and we consider subsets of \( \prod_{t \in T} \Omega_t \), such as the set of all continuous functions, denoted by \( C([0, 1]) \), or the set of all bounded and right-continuous functions, denoted by \( D([0, 1]) \).
Next, for any positive integer $N$, let $T_N = \{t_1, \ldots, t_N\}$ with $t_i \in T$, $i = 1, \ldots, N$, and let $A_{T_N} = \prod_{t \in T_N} A_t$. Then $A_{T_N}$ is a rectangle in $\Omega_1 \times \cdots \times \Omega_N$.

Furthermore,

**Definition 8**

The subset $A_{T_N} \times \prod_{t \in T_N} \Omega_t = \prod_{t \in T_N} A_t \times \prod_{t \in T_N} \Omega_t$ of $\prod_{t \in T} \Omega_t$ is called a product cylinder in $\Omega_T = \prod_{t \in T} \Omega_t$ with basis $A_{T_N}$ and sides $A_t \in A_t, t \in T_N$. ■

**Theorem 8**

Let $C$ be the class of all finite sums of all product cylinders. Then $C$ is a field (of subsets of $\prod \Omega_t$).

The proof of this theorem is based on the same ideas as those used in proving Theorem 7.

**Definition 9**

The $\sigma$-field generated by $C$ is called the product $\sigma$-field of $A_t, t \in T$, and is denoted by $A_T = \prod_{t \in T} A_t$. The pair $(\Omega_T = \prod_{t \in T} \Omega_t, A_T = \prod_{t \in T} A_t)$ is called the product measurable space of the measurable spaces $(\Omega_t, A_t), t \in T$. The space $(\mathbb{R}^\infty, \mathcal{B}^\infty)$, the (countably) infinite dimensional Borel space, where $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$, and $\mathcal{B}^\infty = \mathcal{B} \times \mathcal{B} \times \cdots$, is often of special interest. $\mathcal{B}^\infty$ is the (countably) infinite Borel $\sigma$-field. The members of $\mathcal{B}^\infty$ are called (countably) infinite dimensional Borel sets. ■

For more information, see also page 62 of Loève (1963).

### 3. Measurable Functions and Random Variables

Let $\Omega, \Omega'$ be two spaces and let $X$ be a mapping such that $X : \Omega \to \Omega'$. Then the set operator $X^{-1}$ associated with the mapping $X$ is defined as follows:

**Definition 10**

$X^{-1} : \mathcal{P}(\Omega') \to \mathcal{P}(\Omega)$ and $X^{-1}(A') = A$, where $A = \{\omega \in \Omega; X(\omega) \in A\}$; $X^{-1}(A')$ is the inverse image of $A'$ under $X$. ■
From Definition 10 it follows that

**Theorem 9**

(i) If $A' \cap B' = \emptyset$, then $[X^{-1}(A')] \cap [X^{-1}(B')] = \emptyset$.
(ii) $X^{-1}(A'^c) = [X^{-1}(A')]^c$.
(iii) $X^{-1}(\bigcup_{j \in I} A'_j) = \bigcup_{j \in I} X^{-1}(A'_j)$ and $X^{-1}(\bigcap_{j \in I} A'_j) = \bigcap_{j \in I} X^{-1}(A'_j)$.
(iv) $X^{-1}(\bigcap_{j \in I} A'_j) = \bigcap_{j \in I} X^{-1}(A'_j)$.
(v) $X^{-1}(A' - B') = X^{-1}(A') - X^{-1}(B')$ (equivalently, $X^{-1}(A' \cap B'^c) = X^{-1}(A') \cap [X^{-1}(B')]^c$).
(vi) If $A' \subseteq B'$, then $X^{-1}(A') \subseteq X^{-1}(B')$.
(vii) $C' \subseteq C''$, then $X^{-1}(C') \subseteq X^{-1}(C'')$, where $X^{-1}(C') = \{ A \subseteq \Omega; A = X^{-1}(A') \text{ for some } A' \in C' \}$; and similarly for $C''$. ■

Now let us assume that $\Omega'$ is supplied with a $\sigma$-field $A'$. Then we have

**Theorem 10**

Define the class $C$ of subsets of $\Omega$ as follows: $C = X^{-1}(A')$. Then $C$ is a $\sigma$-field (i.e., the inverse image of a $\sigma$-field is a $\sigma$-field). ■

**Remark 13**

This $\sigma$-field is called the $\sigma$-field *induced* (in $\Omega$) by $X$.

**Proof of Theorem 10**

This is immediate from (ii) and (iii) of Theorem 9. ■

Next assume that $\Omega$ is supplied with a $\sigma$-field $A$. Then

**Theorem 11**

Define the class $C'$ of subsets of $\Omega'$ as follows: $C' = \{ A' \subseteq \Omega'; X^{-1}(A') \in A \}$. Then $C$ is a $\sigma$-field. ■

**Proof**

Immediate from (ii) and (iii) of Theorem 9. ■
Then we have: $C$ then measurable if no confusion is possible. In particular, if $A$ class generating type, and measurable. Define $f : Ω \rightarrow X$ is measurable, we say that $X$ is measurable, therefore $X$ mapping is a measurable mapping.

**Theorem 12**

Let $C'$ be a class of subsets of $Ω$ and let $A' = σ(C')$. Then $A = σ\{X^{-1}(C')\} = X^{-1}(A')$. ■

**Proof**

We have: $C' \subseteq A'$ implies $X^{-1}(C') \subseteq X^{-1}(A')$, and this implies $A \subseteq X^{-1}(A')$ because $X^{-1}(A')$ is a $σ$-field by Theorem 10. Thus, to show $X^{-1}(A') \subseteq A$. Define $C^*$ as follows: $C^* = \{A' \subseteq Ω; X^{-1}(A') \in A\}$. Then, clearly, $C' \subseteq C^*$, and $C^*$ is a $σ$-field by Theorem 11. Hence $A' \subseteq C^*$ and therefore $X^{-1}(A') \subseteq X^{-1}(C^*) \subseteq A$. Thus $X^{-1}(A') \subseteq A$. ■

Now assume that both $Ω$ and $Ω'$ are supplied with $σ$-fields $A$ and $A'$, respectively. Then

**Definition 11**

If $X^{-1}(A') \subseteq A$ we say that $X$ is measurable with respect to $A$ and $A'$, or just measurable if no confusion is possible. In particular, if $(Ω', A') = (\mathbb{R}^n, B^n)$ and $X$ is measurable, we say that $X$ is an $n$-dimensional random vector and if $n = 1$, a random variable (r.v.). If $(Ω, A) = (\mathbb{R}^n, B^n)$, $(Ω', A') = (\mathbb{R}^m, B^m)$ and $f : Ω \rightarrow Ω'$ is measurable, then $f$ is called a Borel function, and for $m = 1$ a Baire function. ■

The meaning and significance of Theorem 12 are this: if we want to check measurability of $X$, it suffices only to check that $X^{-1}(C') \subseteq A$, where $C'$ is a class generating $A'$. Indeed, if $X : (Ω, A) \rightarrow (Ω', A')$, then $(A, A')$ is measurable. $X$ means $X^{-1}(A') \subseteq A$. Let $X^{-1}(C') \subseteq A$ and let $A' = σ(C')$. Then $σ\{X^{-1}(C')\} \subseteq A$. But $σ\{X^{-1}(C')\} = X^{-1}(A')$. Thus $X^{-1}(A') \subseteq A$. In particular, in the Borel real line, $X$ is a r.v. if only $X^{-1}(C_0)$ or $X^{-1}(C_j)$ or $X^{-1}(C'_j) \subseteq A$, $j = 1, ..., 8$, where $C_0$ is as in Application 1, the classes $C_j$, $j = 1, ..., 8$ are the classes of intervals each consisting of intervals from $C_0$ of one type, and $C'_j$ is the class taken from $C_j$ when the endpoints of the intervals are restricted to be rational numbers $j = 1, ..., 8$.

**Theorem 13**

Let $X : (Ω, A) \rightarrow (Ω', A')$ be measurable and let $f : (Ω', A') \rightarrow (Ω'', A'')$ be measurable. Define $f(X) : Ω \rightarrow Ω'$ as follows: $f(X)(ω) = f[X(ω)]$. Then the mapping $f(X)$ is measurable. That is, a measurable mapping of a measurable mapping is a measurable mapping. ■
Proof
For \( A'' \in \mathcal{A}'' \), \([f(X)]^{-1}(A'') = X^{-1}[f^{-1}(A'')] = X^{-1}(A') \) with \( A' \in \mathcal{A}' \). Thus \( X^{-1}(A') = A \in \mathcal{A} \). ■

Corollary 1

Borel functions of random vectors are random vectors. ■

Proof
Take \((\Omega', \mathcal{A}') = (\mathbb{R}^n, \mathcal{B}^n)\), \((\Omega'', \mathcal{A}'') = (\mathbb{R}^m, \mathcal{B}^m)\). ■

We now consider the measurable spaces \((\Omega, \mathcal{A}), (\Omega', \mathcal{A}')\) and assume that \( \Omega \) and \( \Omega' \) are also provided with topologies \( T \) and \( T' \), respectively. (Recall that \( T \) is a topology for \( \Omega \) if \( T \) is a class of subsets of \( \Omega \) with the following properties: (i) \( \emptyset, \Omega \in T \), (ii) \( T \) is closed under finite intersections of members of \( T \), and (iii) \( T \) is closed under arbitrary unions of members of \( T \).) The pair \((\Omega, T)\) is called a topological space, and the members of \( T \) are called open sets. Also, \( f : (\Omega, T) \to (\Omega', T') \) is said to be continuous (with respect to the topologies \( T \) and \( T' \)), if \( f^{-1}(T') \in T \) for every \( T' \in T' \).

Theorem 14

Let \( f : \Omega \to \Omega' \) be continuous and let that \( T \subseteq A, A' = \sigma(T') \). Then \( f \) is measurable. ■

Proof
Continuity of \( f \) implies \( f^{-1}(T') \in T, T' \in T' \). Hence \( f^{-1}(T') \subseteq A. \) Since \( T' \) generates \( A' \), we have \( f^{-1}(A') = \sigma(f^{-1}(T')) \subseteq A \) by Theorem 12. ■

Application 2

Recall that a class of sets in \( T \) is a base for \( T \) if every \( T \) in \( T \) is the union of members of this class. A topology \( T \) and the corresponding topological space are called separable if there exists a countable base for \( T \). In the spaces \((\mathbb{R}^k, \mathcal{B}^k), k \geq 1\), the “usual” topology \( T_k \) is the one with base the class of all finite open intervals.
(rectangles) or only the class of all open intervals (rectangles) with rational endpoints. This second base is countable and the topology \( T_k \) and the space \((\mathbb{R}^k, T_k)\) are separable. Then, clearly, \( B^k \) is generated by \( T_k \) (see Theorem 7, Definition 6, and the paragraph following it). Thus we have

**Corollary 2**

Let \( X : (\Omega, \mathcal{A}) \to (\mathbb{R}^n, B^n) \) be measurable and let \( f : (\mathbb{R}^m, B^m) \to (\mathbb{R}^n, B^n) \) be continuous. Then \( f(X) : \Omega \to \mathbb{R}^m \) is measurable (i.e., continuous functions of a random vector are random vectors).

**Proof**

Follows by the fact that \( T_m \) and \( T_n \) generate \( B^m \) and \( B^n \), respectively. ■

This corollary implies that the usual operations applied on r.v.s, such as forming sums, products, or quotients, will give r.v.s.

Now if \( X : \Omega \to \mathbb{R}^n \), then \( X \) can be written as \( X = (X_1, ..., X_n) \). In connection with this we have

**Theorem 15**

Let \( X = (X_1, ..., X_n) : (\Omega, \mathcal{A}) \to (\mathbb{R}^n, B^n) \). Then \( X \) is a random vector (measurable function) if and only if \( X_j, j = 1, ..., n \) are r.v.s.

**Proof**

Let \( B_i \in B, i = 1, ..., n \). Then \( X^{-1}(B_1 \times ... \times B_n) = (X_1, ..., X_n)^{-1}(B_1 \times ... \times B_n) = (X_1 \in B_1) \cap ... \cap (X_n \in B_n) = [X_1^{-1}(B_1)] \cap ... \cap [X_n^{-1}(B_n)] \). Thus, if \( X_j, j = 1, ..., n \) are r.v.s, then \( X_j^{-1}(B_j) \in \mathcal{A} \) for every \( j \) and hence so is \( \bigcap_{j=1}^n X^{-1}(B_j) \). So, if \( X_j, j = 1, ..., n \), are measurable, so is \( X \) (by the definition of the product \( \sigma \)-field \( B^n \)). Next, consider the projection functions \( f_j : \mathbb{R}^n \to \mathbb{R} \) such that \( f_j(x_1, ..., x_n) = x_j, j = 1, ..., n \). It is known that \( f_j, j = 1, ..., n \), are continuous, hence measurable. Then \( X_j = f_j(X), j = 1, ..., n \), and the measurability of \( X \) implies the measurability of \( X_j, j = 1, ..., n \). ■
Let $X$ be a r.v. Then the positive part of $X$, denoted by $X^+$, and the negative part of $X$ denoted by $X^-$, are defined as follows:

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases},$$

$$X^- = \begin{cases} 0 & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}.$$

Then, clearly, $X = X^+ - X^-$ and $|X| = X^+ + X^-$. Now as a simple application of the Corollary to Theorem 14, we show that both $X^+$ and $X^-$ are r.v.s and then, of course, so is $|X|$. To this end, take $n = m = 1$ and define $f$ by:

$$f(x) = x^+, \text{ which is a continuous function of } x,$$

and similarly for $f(x) = x^-$. Directly, the measurability of $X^+$ is established as follows. In order to prove the measurability of $X^+$, it suffices to show that $(X^+) = 1((\infty, x]) \in \mathcal{A}$ for $x \in \mathbb{R}$. For $x < 0$, $(X^+ \leq x) = \emptyset$. For $x = 0$, $(X^+ \leq 0) = (X \leq 0) \in \mathcal{A}$. For $x > 0$, $(X^+ \leq x) = (X^+ = 0) \cup (0 < X^+ \leq x) = (X \leq 0) \cup (0 < X \leq x) = (X \leq x) \in \mathcal{A}$.

(Recall that for a sequence $\{x_n\}$ of real numbers, and as $n \to \infty$:

1. $\limsup x_n \text{ or } \liminf x_n = x$ if for every $\varepsilon > 0$ there exists $n(\varepsilon) > 0$ integer such that $n \geq n(\varepsilon)$ implies $x_n \leq x + \varepsilon$, and $x_n > x - \varepsilon$ for at least one $n \geq n(\varepsilon)$.
2. $\limsup x_n \text{ or } \liminf x_n = x$ if for every $\varepsilon > 0$ there exists $n(\varepsilon) > 0$ integer such that $n \geq n(\varepsilon)$ implies $x_n \geq x - \varepsilon$ and $x_n < x + \varepsilon$ for at least one $n \geq n(\varepsilon)$.

Also,

3. $\lim x_n = \inf_n \sup_{i \geq n} x_i = \inf_n y_n, y_n \overset{def}{=} \sup_{i \geq n} x_i,$

so that $y_n \downarrow$ and set $\inf_n y_n = \lim_n y_n \overset{def}{=} \bar{x}.$

4. $\lim x_n = \sup_n \inf_{i \geq n} x_i = \sup_n z_n, z_n \overset{def}{=} \inf_{i \geq n} x_i,$

so that $z_n \uparrow$ and set $\sup_n z_n = \lim_n z_n \overset{def}{=} \underline{x}.$

For every $n \geq 1$, $z_n \leq y_n$ so that $\sup_n z_n \leq \inf_n y_n$ or equivalently $\underline{x} \leq \bar{x}$. If $\underline{x} \geq \bar{x}$, then the common value $x = \bar{x} = \underline{x}$ is the $\lim_n$ of $x_n.$)
Next let $X_n, n \geq 1,$ be r.v.s. Then define the following mappings (which are assumed to be finite). The sup and inf are taken over $n \geq 1$ and all limits are taken as $n \to \infty.$

$$\sup_n X_n : \left( \sup_n X_n \right)(\omega) = \sup_n X_n(\omega),$$

$$\inf_n X_n : \left( \inf_n X_n \right)(\omega) = \inf_n X_n(\omega),$$

$$\limsup_n X_n \text{ or } \lim_n X_n : \left( \limsup_n X_n \right)(\omega) = \limsup_n X_n(\omega), \quad \omega \in \Omega$$

$$\liminf_n X_n \text{ or } \lim_n X_n : \left( \liminf_n X_n \right)(\omega) = \liminf_n X_n(\omega).$$

Then $\liminf_n X_n \leq \limsup_n X_n$ and if $\liminf_n X_n = \limsup_n X_n,$ this defines the mapping $\lim_n X_n.$ Then we have the following theorem:

**Theorem 16**

If $X_n, n \geq 1,$ are r.v.s, then the mappings just defined are also r.v.s. 

**Proof**

We have $(\sup_n X_n \leq x) = (X_n \leq x, n \geq 1) = \bigcap_{n=1}^\infty (X_n \leq x) \in \mathcal{A}.$ Thus $\sup_n X_n$ is a r.v. Now $\inf_n X_n = -\sup(-X_n)$ and then the measurability of $\sup_n X_n$ implies the measurability of $\inf_n X_n.$

Next, $\limsup_n X_n = \inf_n (\sup_{j \geq n} X_j).$ Thus, if $Y_n = \sup_{j \geq n} X_j,$ then $Y_n, n \geq 1,$ are r.v.s and then so is the $\inf_n Y_n.$ Finally, $\liminf_n X_n = -\limsup_n (-X_n),$ and then the previous result implies the measurability of $\liminf_n X_n.$ The measurability of $\lim_n X_n,$ if the limit exists, is an immediate consequence of the last two results.

A measurable mapping $X$ on $(\Omega, \mathcal{A})$ into $(\bar{\mathbb{R}}, \mathcal{B})$, the extended Borel real line, is an extended r.v. Then Theorem 16 still holds true if the operations applied on $X_n, n \geq 1,$ produce extended r.v.s.
Definition 12
Consider the measurable space \((\Omega, \mathcal{A})\) and let \(\{A_j, j \in I\}\) be a collection of sets in \(\mathcal{A}\) such that \(A_i \cap A_j = \emptyset, i, j \in I, i \neq j,\) and \(\sum_j A_j = \Omega.\) Then this collection is called a (measurable) partition of \(\Omega.\) The partition is finite if \(I\) is a finite set and infinite otherwise. ■

Definition 13
Let \(\{A_j, j = 1, ..., n\}\) be a (finite, measurable) partition of \(\Omega,\) and define the mapping \(X : \Omega \rightarrow \mathbb{R}\) as follows: \(X = \sum_{j=1}^{n} \alpha_j I_{A_j},\) such that \(\alpha_j \in \mathbb{R}, j = 1, ..., n\) (which may be assumed to be distinct). Then \(X\) is called a simple r.v.

Remark 14
By \(I_A\) we denote the indicator of the set \(A;\) i.e.,

\[
I_A(\omega) = \begin{cases} 
1 & \text{if } \omega \in A \\
0 & \text{if } \omega \in A^c.
\end{cases}
\]

It is evident that simple and elementary r.v.s are indeed r.v.s. What is more important, however, is that some kind of an inverse of this statement also holds true. More precisely,

Theorem 17
Every r.v. is the pointwise limit of a sequence of simple r.v.s. ■

Proof
Consider the r.v. \(X,\) the interval \([-n, n),\) and define the sets:

\[
A_{n,j} = (\frac{j-1}{2^n} \leq X < \frac{j}{2^n}), \quad j = -n2^n + 1, -n2^n + 2, ..., n2^n, \]

\[
A''_{n} = (X < -n), \quad A'_{n} = (X \geq n), \quad n = 1, 2, ...
\]

Then, clearly, \(\{A_{n,j}, j = -n2^n + 1, ..., n2^n, A'_n, A''_n\}\) is a (measurable) partition of \(\Omega.\) Thus, if we define \(X_n\) by \(X_n = \sum_{j=-n2^n+1}^{n2^n} \frac{j-1}{2^n} I_{A_{n,j}} + (-n)I_{A'_n} + nI_{A''_n},\) then \(X_n\) is a simple r.v.
We are going to show next that \( X_n(\omega) \to X(\omega) \) for every \( \omega \in \Omega \).

Let \( \omega \in \Omega \). Then there exists \( n_o = n_o(\omega) \) such that \( |X(\omega)| < n_o \).

It is asserted that \( \omega \in A_{nj} \) for \( n \geq n_o \), some \( j = -n^{2^n} + 1, \ldots, n^{2^n} \).

This is so because for \( n \geq n_o \), \([j-1/2^n, j/2^n)\), \( j = -n^{2^n} + 1, \ldots, n^{2^n} \) form a partition of \([-n, n)\). Let that \( \omega \in A_{nj(n)} \).

Then \( j(n) - 1/2^n \leq X(\omega) < j(n) \) \( 2^n \). But then \( X_n(\omega) = \frac{j(n) - 1}{2^n} \) so that

\[ |X_n(\omega) - X(\omega)| < \frac{1}{2^n} \]. Thus \( X_n(\omega) \to X(\omega) \) \( n \to \infty \).

To this theorem we have the following:

**Corollary 3**

If the r.v. \( X \geq 0 \), then there exists a sequence of simple r.v.s \( X_n \) such that \( 0 \leq X_n \uparrow X \) as \( n \to \infty \).

**Proof**

If \( X \geq 0 \), then \( X_n \) of the theorem becomes as follows:

\[ X_n = \sum_{j=1}^{2^n} \frac{j-1}{2^n} I_{A_{nj}} + nA_n^a, \]

so that \( 0 \leq X_n \to X \). We will next show that \( X_n \uparrow \). For each \( n \), we have that \([0, n)\) is divided into the \( n2^n \) subintervals \([j-1/2^n, j/2^n)\), \( j = 1, 2, \ldots, n2^n \), and for \( n+1 \), \([0, n+1)\) is divided into \((n+1)2^{n+1}\) subintervals \([j-1/2^{n+1}, j/2^{n+1})\), \( j = 1, 2, \ldots, (n+1)2^{n+1} \), and each one of the intervals in the first class of intervals is split into two intervals in the second class of intervals. Thus \( X_n(\omega) \leq X_{n+1}(\omega) \) for every \( \omega \in \Omega \) (see following picture).
Remark 15

The significance of the corollary is that the nondecreasing simple r.v.s \(X_n\) are also \(\geq 0\). This point will be exploited later on in the so-called Lebesgue Monotone Convergence Theorem.

Remark 16

Theorem 17 and its corollary are, clearly, true even if \(X\) is an extended r.v.

Exercises

1. Consider the measurable space \((\Omega, \mathcal{A})\) and let \(A_n \in \mathcal{A}, n = 1, 2, \ldots\). Then recall that

\[
\liminf_{n \to \infty} A_n = \lim_{n \to \infty} \bigcap_{j=n}^{\infty} A_j \quad \text{and} \quad \limsup_{n \to \infty} A_n = \lim_{n \to \infty} \bigcup_{j=n}^{\infty} A_j.
\]

(i) Show that \(\lim_{n \to \infty} A_n \subseteq \liminf_{n \to \infty} A_n\). (If also \(\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n\), then this set is denoted by \(\lim_{n \to \infty} A_n\) and is called the limit of the sequence \(\{A_n\}, n \geq 1\).

(ii) Show that \((\liminf_{n \to \infty} A_n) = \lim_{n \to \infty} A_n^c\), \((\limsup_{n \to \infty} A_n) = \lim_{n \to \infty} A_n^c\). Conclude that if \(\lim_{n \to \infty} A_n = A\), then \(\lim_{n \to \infty} A_n^c = A^c\).

(iii) Show that \(\lim_{n \to \infty} (A_n \cap B_n) = \left(\lim_{n \to \infty} A_n\right) \cap \left(\lim_{n \to \infty} B_n\right)\)

and \(\lim_{n \to \infty} (A_n \cup B_n) = \left(\lim_{n \to \infty} A_n\right) \cup \left(\lim_{n \to \infty} B_n\right)\).

(iv) Show that \(\lim_{n \to \infty} (A_n \cap B_n) \subseteq \left(\liminf_{n \to \infty} A_n\right) \cap \left(\liminf_{n \to \infty} B_n\right)\)

and \(\lim_{n \to \infty} (A_n \cup B_n) \supseteq \left(\limsup_{n \to \infty} A_n\right) \cup \left(\limsup_{n \to \infty} B_n\right)\).

(v) By a counterexample, show that the inverse inclusions in part (iv) do not hold, so that

\(\liminf_{n \to \infty} (A_n \cap B_n)\) need not be equal to \(\left(\liminf_{n \to \infty} A_n\right) \cap \left(\liminf_{n \to \infty} B_n\right)\)

and

\(\lim_{n \to \infty} (A_n \cup B_n)\) need not be equal to \(\left(\lim_{n \to \infty} A_n\right) \cup \left(\lim_{n \to \infty} B_n\right)\).
(vi) If \( \lim_{n \to \infty} A_n = A \) and \( \lim_{n \to \infty} B_n = B \), then show that \( \lim_{n \to \infty} (A_n \cap B_n) = A \cap B \) and \( \lim_{n \to \infty} (A_n \cup B_n) = A \cup B \).

(vii) If \( \lim_{n \to \infty} A_n = A \), then show that for any set \( B \), \( \lim_{n \to \infty} (A_n \Delta B) = A \Delta B \), where \( A \Delta B \) is the symmetric difference of \( A \) and \( B \).

(viii) If \( A_{2j-1} = B \) and \( A_{2j} = C \), \( j = 1, 2, \ldots \), determine \( \lim_{n \to \infty} A_n \) and \( \lim_{n \to \infty} B_n \). Under what condition on \( B \) and \( C \) does the limit exist, and what is it equal to?

**Hint:**

(i) Use the definition of \( \lim_{n \to \infty} A_n \) and \( \lim_{n \to \infty} B_n \), and show that each side is contained in the other.

(ii) Use the definition of \( \lim_{n \to \infty} A_n \), \( \lim_{n \to \infty} B_n \), and DeMorgan’s laws.

(iii), (iv) Use the definition of \( \lim_{n \to \infty} A_n \) and \( \lim_{n \to \infty} B_n \), and then show that each side is included in the other.

(v) A choice of the \( A_n \)s and \( B_n \)s that does this is if one takes \( A_{2j-1} = A \), \( A_{2j} = A_0 \), \( B_{2j-1} = B \), \( B_{2j} = B_0 \), \( j \geq 1 \), then take \( \Omega = \mathbb{R} \), and finally select \( A, A_0, B \) and \( B_0 \) suitably.

(vi) Use parts (iii) and (iv).

(vii) It follows from parts (vi) and (ii).

(viii) It follows from part (v).

2. (i) Setting \( \overline{A} = \lim \inf_{n \to \infty} A_n \), \( \overline{A} = \lim \sup_{n \to \infty} A_n \), and \( A = \lim_{n \to \infty} A_n \) if it exists, show that all \( A, \overline{A}, \) and \( \overline{A} \) are in \( A \).

(ii) If \( A_n \uparrow \) as \( n \to \infty \), show that \( \lim_{n \to \infty} A_n \) exists and is equal to \( \bigcup_{n=1}^{\infty} A_n \), and if \( A_n \downarrow \) as \( n \to \infty \), then \( \lim_{n \to \infty} A_n \) exists and is equal to \( \bigcap_{n=1}^{\infty} A_n \).

3. Carry out the details of the proof of Theorem 1.

4. By means of an example, show that \( A_j \in \mathcal{F}, j \geq 1 \), need not imply that \( \bigcup_{j=1}^{\infty} A_j \in \mathcal{F} \), and similarly for \( \bigcap_{j=1}^{\infty} A_j \).

5. Let \( \mathcal{P} = \{ A_n, n = 1, 2, \ldots \} \) be a partition of \( \Omega \) where \( A_n \neq \emptyset, n \geq 1 \), and let \( \mathcal{C} \) be the class of all sums of members in \( \mathcal{P} \). Then show that \( \mathcal{C} \) is the \( \sigma \)-field generated by the class \( \mathcal{P} \).
6. Let $C_0$ be the class of all intervals in $\mathbb{R}$, and consider the eight classes $C_j, j = 1, \ldots, 8$, each of which consists of all intervals in $C_0$ of one type. Then $B = \sigma(C_j), j = 1, \ldots, 8$. Also, if $C'_j$ denotes the class we get from $C_j$ by considering intervals with rational endpoints, then $\sigma(C'_j) = B, j = 1, \ldots, 8$.

*Hint:* One may choose to carry out the detailed proof for just one of these classes, e.g., the class $C_1 = \{(x, y); x, y \in \mathbb{R}, x < y\}$ or the class $C'_1 = \{(x, y); x, y \text{ rationals in } \mathbb{R} \text{ with } x < y\}$.

7. (i) If $C$ is the class of all finite sums of intervals in $\mathbb{R}$ of the form: $(\alpha, \beta], \alpha, \beta \in \mathbb{R}, \alpha < \beta; (-\infty, \alpha], \alpha \in \mathbb{R}; (\beta, \infty), \beta \in \mathbb{R}$, then $C$ is a field and $\sigma(C) = B$.

(ii) The same is true if $C$ is the class of all finite sums of all kinds of intervals in $\mathbb{R}$.

8. Consider the space $(\Omega, \mathcal{F})$ and for an arbitrary but fixed set $A$ with $\emptyset \subset A \subset \Omega$, define $\mathcal{F}_A$ by: $\mathcal{F}_A = \{B \subseteq \Omega; B = A \cap C, C \in \mathcal{F}\}$. Then $\mathcal{F}_A$ is a $\sigma$-field (of subsets of $A$).

*Hint:* Notice that the complement of a set in $\mathcal{F}_A$ is with respect to the set $A$ rather than $\Omega$.

9. Consider the space $(\Omega, A)$ and let $A$ be as in Exercise 8. Define $\mathcal{A}_A$ by $\mathcal{A}_A = \{B \subseteq \Omega; B = A \cap C, C \in A\}$. Then $\mathcal{A}_A$ is a $\sigma$-field (of subsets of $A$). Furthermore, $\mathcal{A}_A = \sigma(\mathcal{F}_A)$, where $\mathcal{F}_A$ is as in Exercise 8 and $A = \sigma(\mathcal{F})$.

*Hint:* First, show that $\mathcal{A}_A$ is a $\sigma$-field and $\sigma(\mathcal{F}_A) \subseteq \mathcal{A}_A$. Next, show that $\sigma(\mathcal{F}_A)\supseteq\mathcal{A}_A$ by showing that, for any $\sigma$-field $A^*$ of subsets of $A$, it holds that $A^*\supseteq\mathcal{F}_A$. This is done by defining $\mathcal{M} = \{C \in A; A \cap C \in A^*\}$ and showing that $\mathcal{M}$ is a monotone class.

10. Show that, if $\{A_n\}, n \geq 1$, is a nondecreasing sequence of $\sigma$-fields, then $\bigcup_{n=1}^{\infty} A_n$ is always a field, but it may fail to be a $\sigma$-field.

11. Carry out the details of the proof of Theorem 4.

12. By means of an example, show that a monotone class need not be a field.

13. Carry out the details of the proof of Lemma 2.

14. Let $\Omega_1, \Omega_2$ be two spaces and let $A, A_i \subseteq \Omega_1, B, B_i \subseteq \Omega_2, i = 1, 2$. Then show that
Exercises 23

16. (i) Let $A$ and $B$ be the class of all countable sums of rectangles in the product space $\Omega_1 \times \Omega_2$. Then show that:

$$E \cap F = (A_1 \times \ldots \times A_n) \cap (B_1 \times \ldots \times B_n) = (A_1 \cap B_1) \times \ldots \times (A_n \cap B_n).$$

(ii) If also $B_i \subseteq \Omega$, $i = 1, 2, \ldots, n$, and $F = B_1 \times \ldots \times B_n$, then show that:

$$E \cap F = (A_1 \times \ldots \times A_n) \cap (B_1 \times \ldots \times B_n).$$

15. (i) With $A \subseteq \Omega_1$, and $B \subseteq \Omega_2$, show that $A \times B = \emptyset$ if and only if at least one of $A$ or $B$ is equal to $\emptyset$.

(ii) With $A_1, A_2 \subseteq \Omega_1$ and $B_1, B_2 \subseteq \Omega_2$, set $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ and assume that $E_1$ and $E_2$ are $\neq \emptyset$. Then $E_1 \subseteq E_2$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Explain why the assumption that $E_1$ and $E_2$ are $\neq \emptyset$ is essential.

16. (i) Let $A_i \subseteq \Omega$, $i = 1, 2, \ldots, n$ and set $E = A_1 \times \ldots \times A_n$. Then $E = \emptyset$ if and only if at least one of $A_i$, $i = 1, 2, \ldots, n$, is $\emptyset$.

(ii) If also $B_i \subseteq \Omega$, $i = 1, 2, \ldots, n$, and $F = B_1 \times \ldots \times B_n$, then show that:

$$E \cap F = (A_1 \times \ldots \times A_n) \cap (B_1 \times \ldots \times B_n) = (A_1 \cap B_1) \times \ldots \times (A_n \cap B_n).$$

17. For $i = 1, 2, \ldots, n$, let $A_i, B_i, C_i \subseteq \Omega_i$ and set $E = A_1 \times \ldots \times A_n$, $F = B_1 \times \ldots \times B_n$, $G = C_1 \times \ldots \times C_n$. Suppose that $E, F$, and $G$ are all $\neq \emptyset$ and that $E = F + G$. Then show that there exists a $j$ with $1 \leq j \leq n$ such that $A_j = B_j + C_j$ while $A_i = B_i = C_i$ for all $i \neq j$.

18. In reference to Theorem 7, show that $C$ is still a field, if $A_i$ is replaced by a field $F_i$, $i = 1, 2$.

19. Consider the measurable spaces $(\Omega_i, A_i)$, $i = 1, 2$, and let $C$ be the class of all countable sums of rectangles in the product space $\Omega_1 \times \Omega_2$. Then by an example, show that $C$ need not be a $\sigma$-field.

Remark: Compare it to Theorem 7 in this chapter.

Hint: Take $\Omega_1 = [0, 1]$ and show that the main diagonal $D$ of the rectangle $[0, 1] \times [0, 1]$ belongs in the $\sigma$-field generated by the field of all finite rectangles, but it is not in $C$. 


20. Carry out the details of the proof of Theorem 10.

21. Carry out the details of the proof of Theorem 11.

22. Consider the mapping $X$ defined on $(\Omega, A)$ onto $\Omega' = X(\Omega)$, the image of $\Omega$ under $X$, and let $C' \subseteq \mathcal{P}(\Omega')$ be defined as follows:

$$C' = \{ B \subseteq \Omega' ; B = X(A) , A \in A \}.$$ 

Then, by means of an example, show that $C'$ need not be a $\sigma$-field.

*Remark:* Compare this result with Theorem 11 in this chapter.

23. Consider the measurable space $(\Omega, A)$ and let $X$ be defined by $\sum_{i=1}^{\infty} \alpha_i I_{A_i}$ or $X = \sum_{i=1}^{\infty} \alpha_i I_{A_i}$, where $\alpha_i \in \mathbb{R}$ for all $i$ and $\{A_1, ..., A_n\}$ or $\{A_i, i \geq 1\}$ are partitions of $\Omega$. Then show that $X$ is a r.v. (a simple r.v. and an elementary r.v., respectively) if and only if the partitions are measurable (i.e., $A_i \in A$ for all $i$).

24. If $X$ and $Y$ are mappings on $\Omega$ into $\mathbb{R}$, show that:

$$\{ \omega \in \Omega ; X(\omega) + Y(\omega) < x \} = \bigcup_{r \in Q} \{ \omega \in \Omega ; X(\omega) < r \} \cap \{ \omega \in \Omega ; Y(\omega) < x - r \},$$

where $Q$ is the set of rationals in $\mathbb{R}$.

25. If $X$ is a r.v. defined on the measurable space $(\Omega, A)$, then $|X|$ is also a r.v. By an example, show that the converse need not be true.

26. By a direct argument (that is, by using the definition of measurability), show that, if $X$ and $Y$ are r.v.s, then so are the mappings $X \pm Y$, $XY$, and $X/Y$ ($Y \neq 0$ a.s.).

27. Carry out the details of the proof of the Corollary to Theorem 14.

28. If $X$ and $Y$ are r.v.s defined on $(\Omega, A)$, show that: $$(X + Y)^+ \leq X^+ + Y^+$$ and $$(X + Y)^- \leq X^- + Y^-.$$ 

29. Let $A_1, A_2, ...$ be arbitrary events in $(\Omega, A)$, and define $B_m$ by: $B_m$ is the first event which occurs among the events $A_1, A_2, ...$. Then:

(i) Express $B_m$ in terms of $A_n$s, $m \geq 1$.

(ii) Show that $B_1, B_2, ...$ are pairwise disjoint.

(iii) Show that $\sum_{m=1}^{\infty} B_m = \bigcup_{n=1}^{\infty} A_n$. 
30. For a sequence of events \( \{A_n\}, n \geq 1 \), show that:

\[(i) \quad \lim_{n \to \infty} A_n = \{\omega \in \Omega; \omega \in A_n \text{ for all but finitely many } n\}, \]
\[(ii) \quad \limsup_{n \to \infty} A_n = \{\omega \in \Omega; \omega \in A_n \text{ for infinitely many } n\} \]
(to be denoted by \( (A_n \text{ i.o.}) \) and read \( A_n \text{s occur infinitely often} \)).

31. If \( A_n \) and \( B_n \) are events such that \( A_n \subseteq B_n, n \geq 1 \), then show that \( (A_n \text{ i.o.}) \subseteq (B_n \text{ i.o.}) \).

32. In \( \mathbb{R} \), let \( Q \) be the set of rational numbers, and for \( n = 1, 2, \ldots \), let \( A_n \) be defined by
\[ A_n = \{ r \in (1 - \frac{1}{n+1}, 1 + \frac{1}{n}); r \in Q \}. \]
Examine whether or not the \( \lim_{n \to \infty} A_n \) exists.

33. In \( \mathbb{R} \), define the sets \( A_n, n = 1, 2, \ldots \) as follows:
\[ A_{2n-1} = [-1, -\frac{1}{2n-1}], A_{2n} = [0, \frac{1}{2n}]. \]
Examine whether or not the \( \lim_{n \to \infty} A_n \) exists.

34. Take \( \Omega = \mathbb{R} \), and let \( A_n \) be the \( \sigma \)-field generated by the class \( \{[0, 1), [1, 2), \ldots, [n-1, n)\}, n \geq 1 \). Then show that:

\[(i) \quad A_n \subseteq A_{n+1}, n \geq 1. \]
\[(ii) \quad \text{The class } \bigcup_{n=1}^{\infty} A_n \text{ is not a } \sigma \text{-field.} \]
\[(iii) \quad \text{Describe explicitly } A_1 \text{ and } A_2. \]

35. Let \( A_1, \ldots, A_n \) be arbitrary subsets of an abstract set \( \Omega \), and let \( A'_i \) be either \( A_i \) or \( A_i^c \), \( i = 1, \ldots, n \). Define the class \( C \) of subsets of \( \Omega \) as follows:
\[ C = \{\text{all unions of the intersections } A'_1 \cap \ldots \cap A'_n\}. \]
Then show that:

(i) The class $\mathcal{C}$ is a field (generated by the sets $A_1, \ldots, A_n$).

(ii) Compute the number of elements of $\mathcal{C}$.

36. If $f : \Omega \to \Omega'$, then show that:

(i) $f^{-1}[f(A)] \subseteq A, A \subseteq \Omega$.

(ii) $f[f^{-1}(B)] \subseteq B, B \subseteq \Omega'$.

By concrete examples, show that the relations in (i) and (ii) may be strict.

37. (i) On the measurable space $(\Omega, \mathcal{A})$, define the function $X$ as follows:

$$X(\omega) = \begin{cases} -1 & \text{on } A_1 \\ 1 & \text{on } A_1^c \cap A_2 \\ 0 & \text{on } A_1^c \cap A_2^c, \end{cases}$$

where $A_1, A_2 \in \mathcal{A}$. Examine whether or not $X$ is a r.v.

(ii) On the measurable space $(\Omega, \mathcal{A})$ with $\Omega = \{a, b, c, d\}$ and $\mathcal{A} = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}$, define the function $X$ as follows:

$X(a) = X(b) = -1, X(c) = 1, X(d) = 2$. Examine whether or not $X$ is a r.v.

(iii) If $\Omega = \{-2, -1, 0, 1, 2\}$ and $X$ is defined on $\Omega$ by $X(\omega) = \omega$, determine the field induced by $X$ and that induced by $X^2$. Verify that the latter is contained in the former.

38. For a sequence of r.v.s $\{X_n\}, n \geq 1$, set $\mathcal{B}_k = \sigma(X_k, X_{k+1}, \ldots), k \geq 1$.

Then show that for every $k$ and $l$ with $k < l$, it holds that $\mathcal{B}_k \subseteq \sigma(\mathcal{B}_k, \mathcal{B}_{k+1}, \ldots, \mathcal{B}_l)$.

39. For the r.v.s $X_1, X_2, \ldots, X_n$, set $S_k = \sum_{j=1}^{k} X_j, k = 1, \ldots, n$, and show that $\sigma(X_1, X_2, \ldots, X_n) = \sigma(S_1, S_2, \ldots, S_n)$.

40. For any set $B \subseteq \mathbb{R}$, the set $B + c \overset{def}{=} B_c$ is defined by: $B_c = \{y \in \mathbb{R}; y = x + c, x \in B\}$. Then show that if $B$ is measurable, so is $B_c$.

41. Let $\Omega$ be an abstract set, and let $\mathcal{C}$ be an arbitrary nonempty class of subsets of $\Omega$. Define the class $\mathcal{F}_1$ to consists of all members of $\mathcal{C}$ as well as all of their complements; i.e.,
\[ F_1 = \{ A \subseteq \Omega; A \in C \text{ or } A = C^c \}\]
\[ = \{ A \subseteq \Omega; A \in C \text{ or } A^c \in C \} = C \cup \{ C^c; C \in C \}, \]
so that \( F_1 \) is closed under complementation.

Next, define the class \( F_2 \) as follows:
\[ F_2 = \text{all finite intersections of members of } F_1 \]
\[ = \{ A \subseteq \Omega; A = A_1 \cap \ldots \cap A_m, A_i \in F_1, i = 1, \ldots, m, m \geq 1 \}. \]

Also, define the class \( F_3 \) by
\[ F_3 = \text{all finite unions of members of } F_2 \]
\[ = \{ A \subseteq \Omega; A = \bigcup_{i=1}^{n} A_i \text{ with } A_i \in F_2, i = 1, \ldots, n, n \geq 1 \} \]
\[ = \{ A \subseteq \Omega; A = \bigcup_{i=1}^{n} A_i \text{ with } A_i = A_{1i}^1 \cap \ldots \cap A_{mi}^m, A_{1i}^1, \ldots, A_{mi}^m \in F_1, m_i \geq 1 \text{ integers}, i = 1, \ldots, n, n \geq 1 \}. \]

Set \( F_3 = F \) and show that
(i) \( F \) is a field.
(ii) \( F \) is the field generated by \( C \); i.e., \( F = F(C) \).

42. Refer to Exercise 41, and set \( A_1 = F_1 \). Then define the classes \( A_2 \) and \( A_3 \) instead of \( F_2 \) and \( F_3 \), respectively, by replacing finite intersections and finite unions by countable intersections and countable unions, respectively. Set \( A_3 = A \) and examine whether or not \( A \) is a \( \sigma \)-field.

Hint: For \( A \in A \), check whether you can declare that \( A^c \in A \).