CHAPTER

ONE

STRESSES AND STRAINS

1.1 Introduction

The theory of plasticity is the branch of mechanics that deals with the calculation of stresses and strains in a body, made of ductile material, permanently deformed by a set of applied forces. The theory is based on certain experimental observations on the macroscopic behavior of metals in uniform states of combined stresses. The observed results are then idealized into a mathematical formulation to describe the behavior of metals under complex stresses. Unlike elastic solids, in which the state of strain depends only on the final state of stress, the deformation that occurs in a plastic solid is determined by the complete history of the loading. The plasticity problem is, therefore, essentially incremental in nature, the final distortion of the solid being obtained as the sum total of the incremental distortions following the strain path.

A metal may be regarded as macroscopically homogeneous and isotropic when the small crystal grains forming the aggregate are distributed with random orientations. As a result of plastic deformation, the crystallographic directions gradually rotate toward a common axis, producing a *preferred orientation*. An initially isotropic material thereby becomes anisotropic, and its mechanical properties vary with direction. The development of anisotropy with progressive cold work and the resulting strain-hardening are too complex to be successfully incorporated in the theoretical framework. In the mathematical theory of plasticity, it is generally assumed that the material remains isotropic throughout the deformation irrespective of the degree of cold work. Since the strain-hardening characteristic of a metal in a complex state of stress can be related to that in uniaxial tension or compression, it is necessary to examine the uniaxial stress–strain behavior in some detail before considering the general theory of plasticity.

The plastic deformation in a single crystal is generally produced by slip, which is the sliding of adjacent blocks of the crystal along definite crystallographic planes, called slip planes. The boundary line separating the slipped region of a crystal from the neighboring unslipped region is called a *dislocation.* The movement of the dislocation, which is responsible for the slip, is initiated by a line defect causing a local concentration of stress. Slip usually occurs on those planes which are most densely packed with atoms. The magnitude and direction of the relative movement in a slip is specified by a vector known as the *Burgers vector*. A dislocation is said to be one of unit strength when the magnitude of the Burgers vector is equal to one atomic spacing. The terms *edge dislocation* and *screw dislocation* are used to describe the situations where the Burgers vector is normal and parallel respectively to the dislocation line. In general, a dislocation is partly edge and partly screw in character, and the dislocation line forms a curve or a closed loop.†

In a polycrystalline metal, the crystallographic orientation changes from one grain to the next through a narrow transition zone, or grain boundary, which acts as an effective barrier to slip. Dislocations pile up along the active slip planes at the grain boundaries, the effect of which is to oppose the generations of new dislocations. When the applied stress is increased to a critical value, the shear stress developed at the head of the dislocation pile-up becomes large enough to cause dislocation movement across the boundary. The dislocation pile-up is mainly responsible for strain-hardening of the metal in the early stages of plastic deformation. The rate of hardening of the polycrystalline metal is always higher than that of the single crystal, where the increase in yield stress is caused by dislocations interacting with one another and with foreign atoms serving as barriers. The dislocation interactions control the yield strength of a polycrystalline metal only in the later stages of the deformation.

If the temperature of the strain-hardened metal is progressively increased, the cold-worked state becomes more and more unstable, and the material eventually reverts to the unstrained state. The overall process of heat treatment that restores the ductility to the cold-worked metal is known as *annealing.* The temperature at which there is a marked decrease in hardness of the metal is known as the *recrystallization temperature*. The dislocation density decreases considerably on recrystallization, and the cold-worked structure is replaced by a set of new strain-free grains. The greater the degree of cold-work, the lower the temperature necessary for recrystallization, and smaller the resulting grain size.‡

In ductile metals, under favorable conditions, plastic deformation can continue to a very large extent without failure by fracture. Large plastic strains do occur

† For a complete discussion, see A. H. Cottrell, *Dislocations and Plastic Flow in Crystals*, Clarendon Press, Oxford (1953); W. T. Read, *Dislocations in Crystals*, McGraw-Hill Book Company, New York (1953); J. Friedel, *Dislocations*, Addison-Wesley Publishing Company, Reading, Mass. (1964); F. R. N. Nabarro, *Theory of Crystal Dislocations*, Clarendon Press, Oxford (1967); D. Hull, *Introduction to Dislocations*, 2d ed., Pergamon Press, Oxford (1975).

‡ See, for example, G. E. Dieter, *Mechanical Metallurgy*, Chap. 5, 2d ed., McGraw-Hill Book Company, New York (1976). See also R. W. K. Honeycombe, *The Plastic Deformation of Metals*, 2d ed., Edward Arnold, London (1984).

in many metal-working processes, which constitute an important area of application of the theory of plasticity. While elastic strains may be neglected in such problems, the continued change in geometry of the workpiece must be allowed for in the theoretical treatment. Severe plastic strains are produced locally in certain mechanical tests such as the hardness test and the notch tensile test. The significance of these tests cannot be fully appreciated without a knowledge of the extent of the plastic zone and the associated state of stress. Situations in which elastic and plastic strains are comparable in magnitude arise in a number of important structural problems when the loading is continued beyond the elastic limit. Structural designs based on the estimation of collapse loads are more economical than elastic designs, since the plastic method takes full advantage of the available ductility of the material.

1.2 The Stress–Strain Behavior

(i) ***The true stress–strain curve*** The stress–strain curve of an annealed material in simple tension is found to coincide with that in simple compression when the true stress σ is plotted against the true or natural strain ε. The *true stress*, defined as the load divided by the current cross-sectional area of the specimen, can be significantly different from the *nominal stress*, which is the load per unit original area of cross-section. Let l denote the current length of a tensile specimen and dl the increase in length produced by a small increment of the stress. Then the true strain increases by the amount $d\varepsilon = dl/l$. If the initial length is l_0, the total strain is $\varepsilon = \ln(l_0/l)$, called the true or *natural strain.*† For a specimen uniformly compressed from an initial height h_0 to a final height h, the magnitude of the natural strain is $\varepsilon = \ln(h_0/h)$. The conventional or *engineering strain* e, on the other hand, is the amount of extension or contraction per unit original length or height. It follows that $\varepsilon = \ln(l + e)$ in the case of tension, and $\varepsilon = -\ln(l - e)$ in the case of compression. Thus ε becomes progressively lower than e in tension, and higher than e in compression, as the deformation is continued in the plastic range.

Figure 1.1 shows the true stress–strain curve of a typical annealed material in simple tension. So long as the stress is sufficiently small, the material behaves elastically, and the original size of the specimen is regained on removal of the applied load. The initial part of the stress–strain curve is a straight line of slope E, which is known as Young's modulus. The point A represents the *proportional limit* at which the linear relationship between the stress and the strain ceases to hold. The elastic range generally extends slightly beyond the proportional limit. For most metals, the transition from elastic to plastic behavior is gradual, owing to successive yielding of the individual crystal grains. The location of the *yield point* B is, therefore, largely a matter of convention. The corresponding stress Y, known as the yield stress, is generally defined as that for which a specified small amount of permanent deformation is observed. For theoretical purposes, it is often convenient

† The concept of natural strain has been introduced by P. Ludwik, *Elemente der Technologischen Mechanik*, Springer Verlag, Berlin (1909). The natural strains associated with successive deformations are additive, but the engineering strains are not.

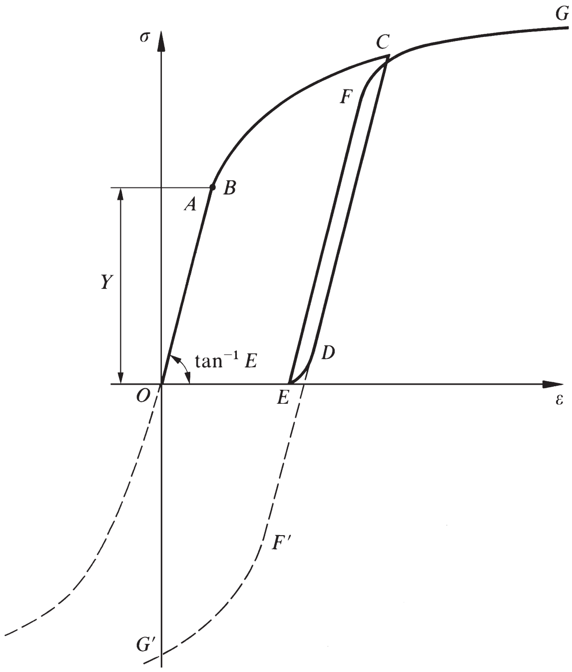


Figure 1.1 True stress–strain curve of metals with effects of unloading and reversed loading.

to assume a sharp yield point defined by the intersection of a pair of straight lines, one of which is a continuation of *OA* and the other a tangent to the stress–strain curve at a point slightly above *B*.

Beyond the yield point, the stress continually increases with further plastic strain, while the slope of the stress–strain curve, representing the rate of strain-hardening, steadily decreases with increasing stress. If the specimen is stressed to some point *C* in the plastic range and the load is subsequently released, there is an elastic recovery following the path *CD* which is very nearly a straight line† of slope *E*. The permanent strain that remains on complete unloading is equal to *OE*. On reapplication of the load, the specimen deforms elasticity until a new yield point *F* is reached. Neglecting the hysteresis loop of narrow width formed during the loading and unloading, *F* may be taken as coincident with *C*. On further loading, the stress–strain curve proceeds along *FG*, virtually as a continuation of the curve *BC*. The curve *EFG* may be regarded as the stress–strain curve of the metal when prestrained by the amount *OE*. The greater the degree of prestrain, the higher the new yield point and the flatter the strain-hardening curve. For a heavily prestrained metal, the rate of strain-hardening is so small that the material may be regarded as approximately nonhardening or *ideally plastic*.

A generic point on the stress–strain curve in the plastic range corresponds to a recoverable elastic strain equal to σ/E, and an irrecoverable plastic strain equal

† L. Prandtl, *Z. angew. Math. Mech.*, **8**: 85 (1928).

to $\varepsilon - \sigma/E$. If the stress is plotted against the plastic strain only, and the material is assumed to have a sharp yield point, the resulting curve will begin at $\sigma = Y$. Let H be the slope of the true stress–strain curve excluding the elastic strain, and T the slope of the curve including the elastic strain, for a given value of the stress σ. The quantities H and T are known as the *plastic modulus* and the *tangent modulus* respectively. A stress increment $d\sigma$ produces an elastic strain increment $d\sigma/E$ and a plastic strain increment $d\sigma/H$, while the total strain increment is $d\sigma/T$. Hence the relationship between H and T is

$$\frac{1}{T} = \frac{1}{E} + \frac{1}{H} \tag{1}$$

In an annealed material, H is considerably greater than T at the initial yielding, but these two moduli rapidly approach one another as the strain is increased. The difference between H and T becomes insignificant when the slope is only a few times the yield stress. At this stage, the elastic strain increment becomes negligible in comparison with the plastic strain increment. When the total strain is sufficiently large, the elastic strain itself is negligible. The stress–strain behavior at sufficiently large strains is identical to that of a hypothetical material in which E is infinitely large. Such a material is regarded as *rigid/plastic*, since it remains undeformed so long as the stress is below the yield point, while the subsequent deformation is entirely plastic.

Suppose that a specimen that has been completely unloaded from a tensile plastic state, represented by the point C, is reloaded in simple compression (Fig. 1.1). The stress–strain curve will then follow the path DF', where the new yield point F' corresponds to a stress that is appreciably smaller in magnitude than that at C. This phenomenon is known as the *Bauschinger effect*,† which occurs in real metals whenever there is a reversal of the stress. The subsequent strain-hardening follows the path $F'G'$, and approaches the stress–strain curve in compression as the loading is continued. The lowering of the yield stress in reversed loading is mainly caused by residual stresses that are left in the specimen on a microscopic scale due to the different stress states in the individual crystals. The Bauschinger effect can, therefore, be largely removed by a mild annealing. In the theory of plasticity, it is generally necessary to neglect the Bauschinger effect, the material being assumed to have identical yield stresses in tension and compression irrespective of the previous cold-work.

Some metals, such as annealed mild steel, exhibit a sharp yield point followed by a sudden drop in the stress, which remains approximately constant during a small amount of further straining. The sharp peak is known as the upper yield point, which is usually 10 to 20 percent higher than the lower yield point represented by the constant stress. At the upper yield point, a lamellar plastic zone, known as *Lüder's band*, inclined at approximately 45° to the tensile axis, appears at a local stress concentration. During the subsequent elongation under constant stress, several Lüder's bands appear and gradually spread over the entire specimen. After a total yield point elongation of about 10 percent, the stress begins to rise again due to

† J. Bauschinger, *Zivilingenieur*, **27**: 289 (1881).

strain-hardening, and the stress–strain curve then continues as before. The yield point drop is suppressed by a light cold-work, but the phenomenon reappears after the metal has been rested for several days at room temperature, or several hours at a relatively high temperature.†

(ii) ***Some consequences of work-hardening*** A longitudinal extension in the tensile test is accompanied by a contraction in the lateral direction. The ratio of the magnitude of the lateral strain increment to that of the longitudinal strain increment is known as the *contraction ratio*, denoted by η. In the elastic range of deformation, the contraction ratio has a constant value equal to Poisson's ratio ν. When the yield point is exceeded, the plastic part of the lateral strain increment for an isotropic material is numerically equal to one-half of the longitudinal plastic strain increment. Since the ratio of the elastic parts of the lateral and longitudinal strain increments is equal to $-\nu$, the total lateral strain increment in uniaxial tension is

$$d\varepsilon' = -\tfrac{1}{2}d\varepsilon + (\tfrac{1}{2} - \nu)d\varepsilon^e$$

where $d\varepsilon^e$ is the elastic part of the longitudinal strain increment $d\varepsilon$. In view of the relationship $d\varepsilon^e = d\sigma/E = (T/E)d\varepsilon$, the contraction ratio becomes

$$\eta = -\frac{d\varepsilon'}{d\varepsilon} = \tfrac{1}{2} - (\tfrac{1}{2} - \nu)\frac{T}{E} \tag{2}$$

Since the slope of the stress–strain curve decreases fairly rapidly in the early stages of strain-hardening, the contraction ratio rapidly approaches the asymptotic value of 0.5 as the strain is increased in the plastic range.‡ For a material having a sharp yield point, the contraction ratio changes discontinuously at this point to a value that depends on the initial rate of strain-hardening. When the tangent modulus becomes of the same order as that of the current yield stress, $\eta \simeq 0.5$, and the incremental change in volume becomes negligible.

The standard tensile test is unsuitable for obtaining the stress–strain curve of metals up to large values of the strain, since the specimen begins to neck when the rate of hardening decreases to a critical value. At this stage, the increase in load due to strain-hardening is exactly balanced by the decrease in load caused by the diminution of the area of cross section. Consequently, the load attains a maximum at the onset of necking. The longitudinal load at any stage is $P = \sigma A$, where A is the current cross-sectional area and σ the current stress, and the corresponding volume of the specimen is lA, where l is the current length. Using the constancy of volume, the maximum load condition $dP = 0$ may be written as

$$\frac{d\sigma}{\sigma} = -\frac{dA}{A} = \frac{dl}{l}$$

† In addition to low-carbon steel, yield point phenomenon has been observed in aluminum, molybdenum, and titanium alloys.

‡ For an experimental investigation on the variation of the contraction ratio, see A. Shelton, *J. Mech. Eng. Sci.*, **3**: 89 (1961).

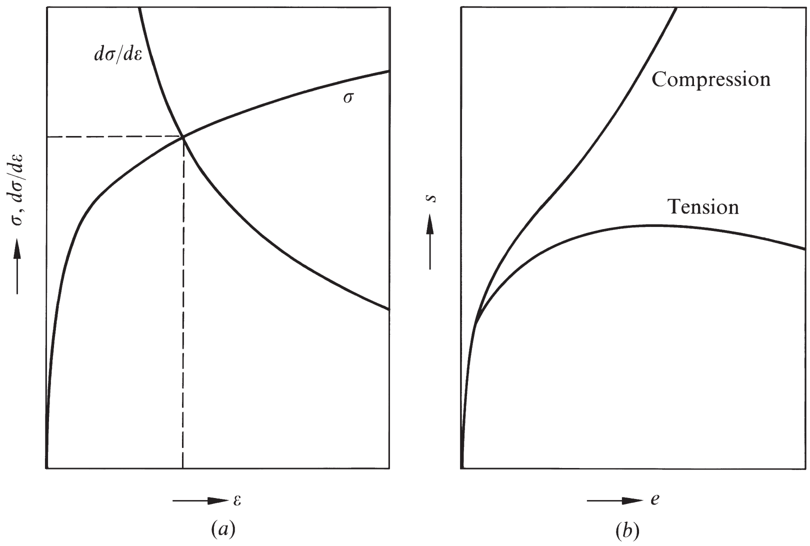


Figure 1.2 Peculiarities in tension and compression. (*a*) Location of point of tensile necking; (*b*) nominal stress versus engineering strain.

Since dl/l is equal to $d\varepsilon$, the condition for the onset of necking becomes

$$\frac{d\sigma}{d\varepsilon} = \sigma \tag{3}$$

When the true stress–strain curve is given, the point on the curve that corresponds to the tensile necking can be located graphically from the fact that the slope at this point is equal to the current stress (Fig. 1.2*a*). A heavily prestrained metal will obviously neck as soon as the yield point is exceeded. Since $d\varepsilon = de/(1+e)$, the condition for necking can be expressed in the alternative form

$$\frac{d\sigma}{de} = \frac{\sigma}{1+e}$$

It follows that the maximum load corresponds to the point of contact of the tangent to the (σ, e) curve from the point $(-1, 0)$ on the negative strain axis.† The tensile test becomes unstable when the load reaches its maximum. The deformation is confined locally in the neck, while the remainder of the specimen recovers elastically under decreasing load until fracture intervenes. The stress distribution in the neck assumes a triaxial state which varies through the cross section of the neck. The test no longer provides a direct measure of the stress–strain behavior. Although the stress–strain curve may be continued by introducing a correction factor that requires

† A Considere, *Ann. ponts et chausses*, **6**: 574 (1885). An interesting discussion has been given by C. R. Calladine, *Engineering Plasticity*, Chap. 2, Pergamon Press, Oxford (1969).

careful measurements of the geometry of the neck,† the experimental difficulties render the method unsuitable for practical purposes.‡

The strain-hardening characteristic of metals at large strains is most conveniently obtained by compressing a solid cylindrical specimen between a pair of parallel platens. In the absence of efficient lubrication, the compression test is complicated by the fact that the friction at the platens restricts the metal flow at the ends of the specimen, causing barreling as the compression proceeds. Since homogeneous compression is thus prevented by friction, the stress–strain curve cannot be derived by the direct measurement of the load and the change in height of the specimen. In actual practice, the difficulty is overcome by using several cylinders with different initial diameter/height ratios, subjecting them to the same load each time on an incremental basis, and then extrapolating the results at each stage to obtain the strain corresponding to zero diameter/height ratio.§ Since the barreling would theoretically disappear for a specimen of infinite height, the extrapolation method eliminates the frictional effect.

Homogeneous deformation in the simple compression test can be achieved by inserting PTFE (polytetra fluoroethylene) films of suitable thickness between the specimen and the compression platens. As well as producing effective lubrication, the PTFE films are themselves compressed so as to exert radial pressure to the material near the periphery. This inhibits the barreling tendency, except when the film thickness is too small. An excessive film thickness, on the other hand, produces bollarding in which the diameter of the specimen becomes bigger at the ends than at the middle. For a given specimen, there is an optimum film thickness for which neither barreling nor bollarding would occur. The compression should be carried out incrementally, renewing the PTFE films after each load application. Using the constancy of volume, the load required during the homogeneous compression may be written as

$$P = \sigma A = \frac{\sigma A_0 h_0}{h} = \frac{\sigma A_0}{1 - e}$$

where A_0 is the original area of cross section of the specimen. The graph for P against e shows an upward inflection and rises continuously without limit (Fig. 1.2b). Setting $d^2P/de^2 = 0$, and using the fact that $d/d\varepsilon = (1-e)d/de$, the condition for inflection is found as

$$\left(\frac{d}{d\varepsilon} + 2\right)\left(\frac{d\sigma}{d\varepsilon} + \sigma\right) = 0 \tag{4}$$

† P. W. Bridgman, *Trans. A.S.M.E.*, **32**: 553 (1944); N. N. Davidenkov and N. I. Spiridonova, *Proc. Am. Soc. Test. Mat.*, **46**: 1147 (1946). See also E. R. Marshall and M. C. Shaw, *Trans. A.S.M.E.*, **44**: 716 (1952); J. D. Lubahn and R. P. Felgar, *Plasticity and Creep of Metals*, p. 114, Wiley and Sons, New York (1961).

‡ A dynamic analysis for the development of the neck has been given by N. K. Gupta and B. Karunes, *Int. J. Mech. Sci.*, **21**: 387 (1979).

§ The extrapolation method has been developed by G. Sachs, *Zeit. Metallkunde*, **16**: 55 (1924), M. Cook and E. C. Larke, *J. Inst. Metals*, **71**: 371 (1945), A. B. Watts and H. Ford, *Proc. Inst. Mech. Eng.*, **169**: 1141 (1955).

which defines the corresponding point on the true stress–strain curve. This point is most conveniently located if the stress–strain curve is represented by an empirical equation. In view of the incompressibility of the material, the nominal stress is $s = \sigma \exp(\varepsilon)$ in compression and $s = \sigma \exp(-\varepsilon)$ in tension.

The work done in changing the height of a specimen from h to $h + dh$ in simple compression is $-P\,dh$, where P is the current axial load. The incremental work done per unit volume of the specimen is therefore equal to $-P\,dh/Ah$ or $\sigma\,d\varepsilon$. It follows that during the homogeneous compression of a specimen from an initial height h_0 to a current height h, the work done per unit volume is given by the area under the true stress–strain curve up to a total strain of $\ln(h_0/h)$.

(iii) ***Empirical stress–strain equations*** For theoretical computations, it is often necessary to represent an experimentally determined stress–strain curve by an empirical equation of suitable form. When the material is rigid/plastic, it is frequently convenient to employ the Ludwik power law†

$$\sigma = C\varepsilon^n \tag{5}$$

where C is a constant stress, and n is a strain-hardening exponent usually lying between zero and 0.5. The equation predicts a zero initial stress and an infinite initial slope, except for $n = 0$ which represents a nonhardening rigid/plastic material. The higher the value of n, the more pronounced is the strain-hardening characteristic of the material (Fig. 1.3*a*). Since $d\sigma/d\varepsilon = n\sigma/\varepsilon$ in view of (5), it follows from (3) that the magnitude of the true strain at the onset of necking in simple tension is equal to n. The work done per unit volume during a homogeneous extension or contraction is easily shown to be $\sigma\varepsilon/(1 + n)$, where σ and ε are the final values of stress and strain.

The simple power law (5) may be readily modified by including a constant term Y representing the initial yield stress. The stress–strain equation then becomes

$$\sigma = Y(1 + m\varepsilon^n) \tag{6}$$

where m and n are dimensionless constants. Although this formula represents the strict rigid/plastic behavior of metals, it does not give a better fit for an actual stress–strain curve over a wide range of strains. When $n = 1$, the above equation represents a linear strain-hardening, which is a reasonable approximation for heavily prestrained metals. A more successful formula, due to Swift,‡ is the generalized power law

$$\sigma = C(m + \varepsilon)^n \tag{7}$$

where C, m, and n are empirical constants. The stress–strain curve represented by (7) can be obtained from that given by (5) if the stress axis is move along the positive strain axis through a distance m. Hence m may be regarded as the amount of prestrain

† P. Ludwik, *Elem. Technol. Mech.*, Springer Verlag, Berlin (1909).
‡ H. W. Swift, *J. Mech. Phys. Solids*, **1**: 1 (1952).

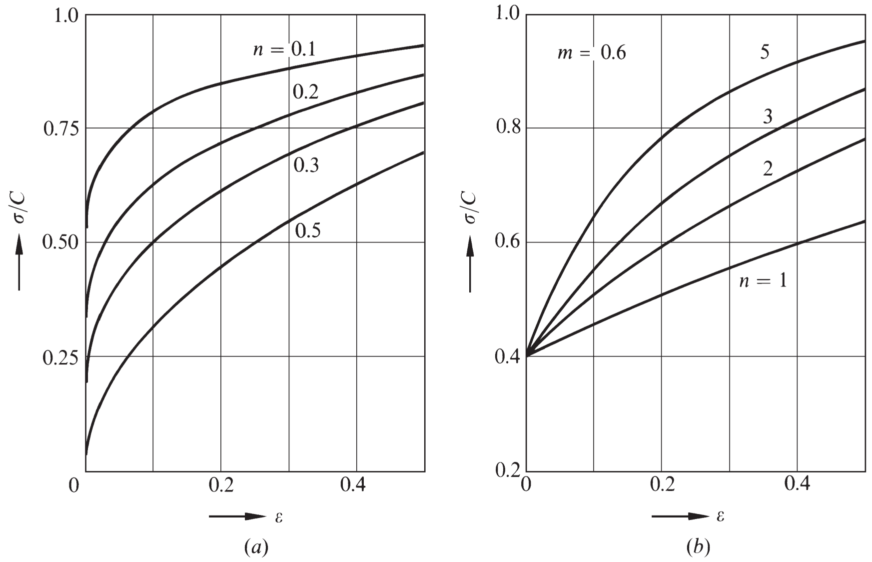


Figure 1.3 Empirical stress–strain curves for rigid/plastic materials. (*a*) Ludwik equation; (*b*) Voce equation.

in a material whose stress–strain curve in the annealed state corresponds to $m = 0$, the value of n remaining the same. If a given prestrained metal is represented by both (5) and (7), the value of n in the two cases will of course be different. The instability strain in simple tension according to the Swift equation is $n - m$ for $m \leqslant n$ and zero for $m \geqslant n$.

For certain applications involving rigid/plastic materials, it is convenient to use an equation suggested by Voce.† In its simplest form, the Voce equation may be written as

$$\sigma = C(1 - me^{-n\varepsilon}) \tag{8}$$

where e is the exponential constant. The curves corresponding to varying m and n approach the asymptote $\sigma = C$ (Fig. 1.3*b*). However, C is unlikely to be the saturation stress of a given metal as the rate of hardening becomes vanishingly small. The rapidity with which the asymptotic value is approached is represented by n. The coefficient m defines the initial state of hardening, the fully hardened material corresponding to $m = 0$. The slope of the stress–strain curve given by (8) is equal to $n(C - \sigma)$, which varies linearly with the stress.

When the elastic and plastic strains are of comparable magnitudes, it is necessary to replace ε in the preceding equations by the plastic strain ε^p. Considering the power law (5), the plastic part of the strain may be assumed to vary as σ^m, where $m = 1/n$, Since the elastic part of the strain is equal to σ/E, the total strain may be expressed

† E. Voce, *J. Inst. Metals*, **74**: 537 (1948). See also J. H. Palm, *Appl. Sci. Res.*, **A-2**: 198 (1948).

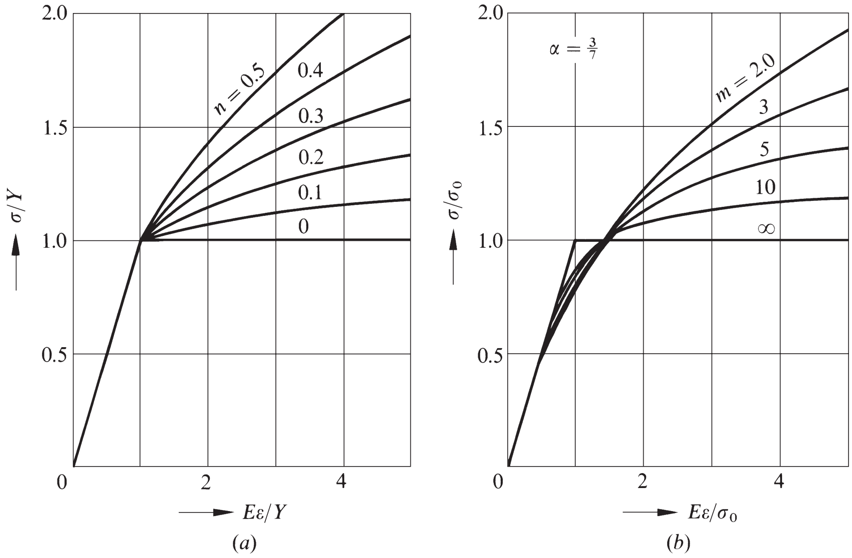


Figure 1.4 Empirical stress–strain curves for elastic/plastic materials. (*a*) Modified Ludwik equation; (*b*) Ramberg-Osgood equation.

by the Ramberg-Osgood equation†

$$\varepsilon = \frac{\sigma}{E}\left\{1 + \alpha\left(\frac{\sigma}{\sigma_0}\right)^{m-1}\right\} \tag{9}$$

where σ_0 is a nominal yield stress and α a dimensionless constant. The slope of the stress–strain curve given by the above equation continuously decreases from the value E at the origin (Fig. 1.4*b*). At the nominal yield point $\sigma = \sigma_0$, the plastic strain is α times the elastic strain, and the secant modulus is $E/(1+\alpha)$. The tangent modulus at any point of the curve is given by

$$\frac{E}{T} = 1 + \alpha m\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \tag{10}$$

The second term on the right-hand side is equal to E/H in view of (1). The stress–strain curve for a range of materials can be reasonably fitted by Equation (9) with $\alpha = 3/7$. For a nonhardening material ($m = \infty$), the equation degenerates into a pair of straight lines meeting at the yield point $\sigma = \sigma_0$.

The contraction ratio η determined from (2) and (10) is plotted against $E\varepsilon/\sigma_0$ in Fig. 1.5, assuming $\alpha = 3/7$. Due to the nature of the Ramberg-Osgood equation, a variation of η is predicted even in the elastic range of straining. The contraction

† W. Ramberg and W. R. Osgood, *NACA Tech. Note*, 902 (1943).

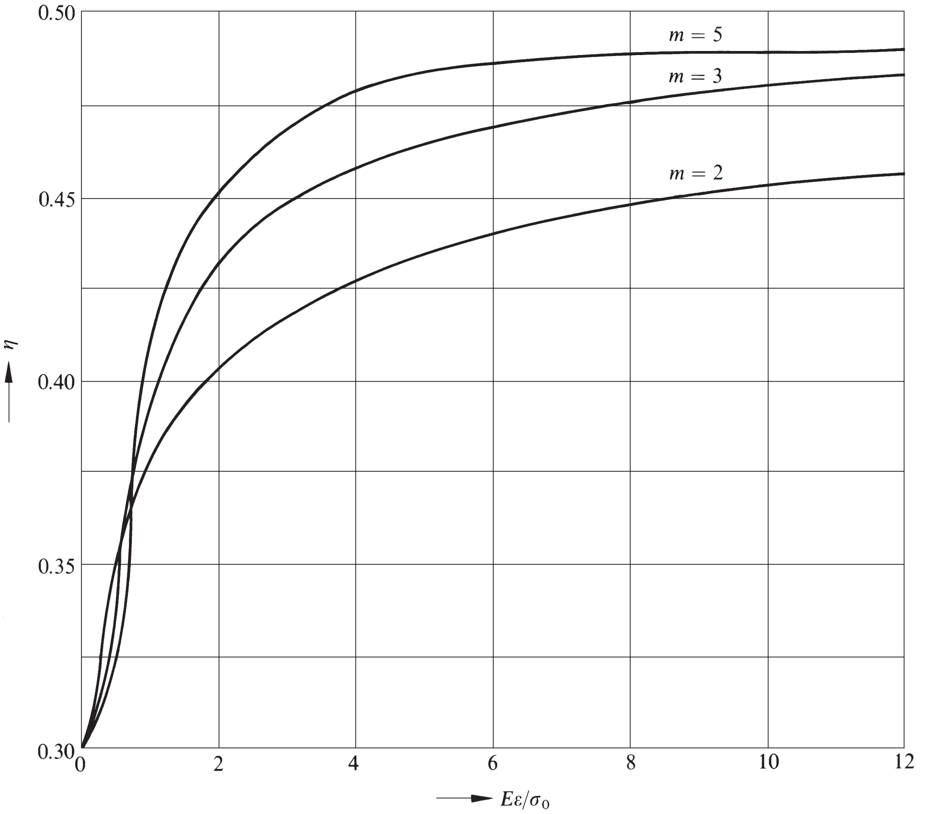


Figure 1.5 Variation of the contraction ratio with longitudinal strain in uniaxial tension according to the Ramberg-Osgood stress–strain equation ($\nu = 0.3$).

ratio increases very rapidly in the neighborhood of the yield point, following which η approaches the value 0.5 in an asymptotic manner. The actual value of η is seen to be reasonably close to 0.5 while the total strain is still of the elastic order of magnitude.

It is sometimes more convenient to employ a stress–strain equation where the curve in the plastic range is expressed by a simple power law, the material being assumed to have a definite yield point at $\sigma = Y$. The empirical representation then becomes

$$\sigma = \begin{cases} E\varepsilon & \varepsilon \leqslant \dfrac{Y}{E} \\ Y\left(\dfrac{E\varepsilon}{Y}\right)^n & \varepsilon \geqslant \dfrac{Y}{E} \end{cases} \tag{11}$$

where n is generally less than 0.5. The slope of the stress–strain curve given by (11) changes discontinuously from E to nE at the yield point (Fig. 1.4a). The tangent modulus at any point in the plastic range is n times the secant modulus. The empirical curve is effectively the Ludwik curve whose initial part is replaced by a chord of slope E.

The Ramberg-Osgood curve represents a continuous transition from the elastic to the plastic behavior expressed by a single equation when the material work-hardens. A similar curve for the ideally plastic material is given by the equation

$$\sigma = Y \tanh\left(\frac{E\varepsilon}{Y}\right)$$

which is due to Prager.† The curve having an initial slope E gradually bends over to approach the yield stress Y in an asymptotic manner. The approach is so rapid that σ is within 1 percent of Y when ε is only $4Y/E$. The tangent modulus at any point on the curve is equal to $E(1-\sigma^2/Y^2)$, and the corresponding plastic modulus is $E(Y^2/\sigma^2-1)$. These moduli soon become negligible while the strain is still quite small.‡

(iv) ***Influence of pressure, strain rate, and temperature*** The tensile test of ductile materials under superimposed hydrostatic pressure has revealed that the yield point and the uniform elongation are unaffected by the applied pressure, but the strain to fracture increases with the intensity of the pressure. The increased ductility of the material is caused by the lateral compressive stresses which inhibit the formation of microcracks that lead to fracture. Test results for both tension and compression of brittle materials under fluid pressure indicate that there is a certain critical pressure above which the material behaves in a ductile manner.§ The stress–strain curves for axially compressed limestone cylinders under uniform fluid pressures acting on the curved surface are shown in Fig. 1.6, where σ denotes the axial compressive stress in excess of the confining pressure p. Each curve corresponds to a particular confining pressure expressed in atmospheres.¶ Some materials are found to suffer a certain amount of permanent volume change when subjected to hydrostatic pressures of exceedingly high magnitude, although the change is negligible in ordinary situations.‖

† W. Prager, *Rev. Fac. Sci.*, *Univ. Istanbul*, **5**: 215 (1941); *Duke Math. J.*, **9**: 228 (1942).

‡ Other forms of stress–strain equation are sometimes used for the derivation of special solutions. See, for example, R. Hill, *Phil. Mag.*, **41**: 1133 (1950), and J. Chakrabarty, *Int. J. Mech. Sci.*, **12**: 315 (1970).

§ The pressure can be accurately measured from the change in resistance of a manganin wire immersed in the pressurized fluid. A detailed account of the experimental investigations regarding the effect of hydrostatic pressure on metals has been presented by P. W. Bridgman, *Studies in Large Plastic Flow and Fracture*, McGraw-Hill Book Company, New York (1952), and by H. Ll. D. Pugh (ed.), *Mechanical Behavior of Materials under Pressure*, Elsevier, Amsterdam (1970).

¶ Experimental results on the compression of marble and limestone cylinders under fluid pressure have been reported by Th. von Karman, *Z. Ver. deut. Ing.*, **55**: 1749 (1911), and by D. T. Griggs, *J. Geol.*, **44**: 541 (1936).

‖ P. W. Bridgman, *J. Appl. Phys.*, **18**: 246 (1947). The effect of hydrolastic pressure on the shear properties of metals has been investigated by B. Crossland, *Proc. Inst. Mech. Eng.*, **169**: 935 (1954); B. Crossland and W. H. Dearden, ibid., **172**, 805 (1958). See also M. C. Shaw, *Int. J. Mech. Sci.*, **22**: 673 (1980).

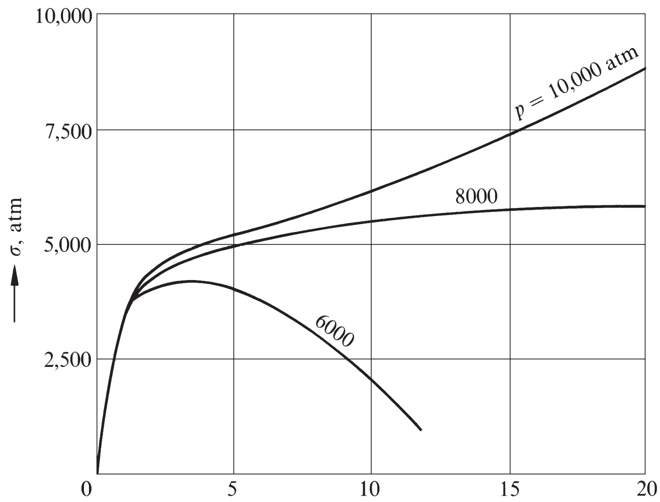


Figure 1.6 Behavior of limestone cylinders under axial thrust and lateral pressure (*after Griggs*).

Plastic instability is found to occur in cylindrical bars when subjected to lateral fluid pressures of sufficient magnitude.† The phenomenon is caused by a slight non-uniformity in distortion of the unconstrained surface which is exposed to fluid pressure. When the material is ductile, the longitudinal strain at the onset of necking is exactly the same as that in uniaxial tension, but the cross section of the neck is greatly reduced before fracture. Brittle materials, which normally fracture with no significant plastic strain under simple tension, are found to deform beyond the point of necking when tested under lateral fluid pressure. Moreover, the uniform strain at the onset of necking is found to be identical to that given by (3), with the stress–strain curve obtained in simple compression. For extremely brittle materials, the fracture mode seems to remain brittle even under a fluid pressure acting on the lateral surface.‡

At room temperature, the stress–strain curve of metals is practically independent of the rate of straining attainable in ordinary testing machines. High-speed tensile tests have shown that the yield stress increases with the strain rate, and this effect is more pronounced at elevated temperatures. The true strain rate in simple compression is defined as $\dot{\varepsilon} = -\dot{h}/h$, where h is the current specimen height and $\dot{h}$ its rate of change. To obtain a constant strain rate during a test, it is therefore necessary to decrease the platen speed in proportion to the specimen height. This is achieved by using a cam plastometer in which one of the compression platens is actuated by a cam of logarithmic profile.§ Maintaining a constant temperature during a test is

† J. Chakrabarty. *Proc. 13th Int. M.T.D.R. Conf.*, p. 565, Pergamon Press, Oxford (1972).

‡ P. W. Bridgman, *Phil. Mag.*, July, 63 (1912).

§ The cam plastometer has been devised by E. Orowan, Brit. Iron and Steel Res. Assoc. Rep., MW/F/22 (1950).

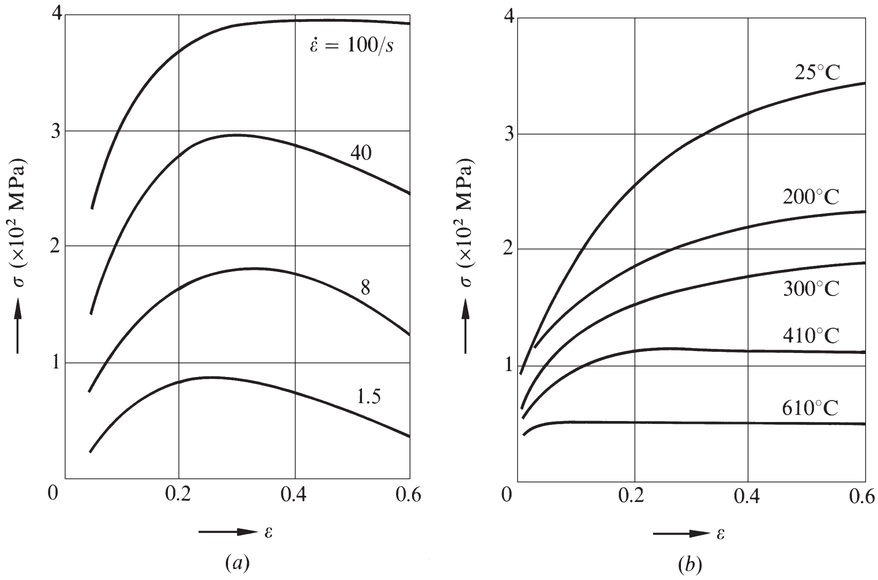


Figure 1.7 Effects of strain rate and temperature on the stress–strain curve of metals. (*a*) EN25 steel at 1000°C (*after Cook*); (*b*) annealed copper at a strain rate of $10^{-3}/s$ (*after Mahtab et al.*).

more difficult, since the heat generated during the test raises the temperature of the specimen adiabatically. Figure 1.7 shows typical stress–strain curves of metals in compression, obtained under constant temperatures and strain rates.†

For a given value of the strain, the combined effect of strain rate and temperature on the yield stress may be expressed by the functional relationship‡

$$\sigma = f\left\{\dot{\varepsilon}\exp\left(\frac{Q}{RT}\right)\right\} \tag{12}$$

where Q is an activation energy for plastic flow, T the absolute testing temperature, and R the universal gas constant equal to 8.314 J/g mol °K. The above relationship has been experimentally confirmed for several metals over wide ranges of strain rate

† For experimental methods and results on the high-speed compression at elevated temperatures, see P. M. Cook, *Proc. Conf. Properties of Materials at High Rates of Strain*, *Inst. Mech. Eng.*, 86 (1957); F. U. Mahtab, W. Johnson, and R. A. C. Slater, *Proc. Inst. Mech. Eng.*, **180**: 285 (1965); S. K. Samanta, *Int. J. Mech. Sci.*, **10**: 613 (1968), *J. Mech. Phys. Solids*, **19**: 117 (1971); T. A. Dean and C. E. N. Sturgess, *Proc. Inst. Mech. Eng.*, **187**: 523 (1973). See also R. A. C. Slater, *Engineering Plasticity*, Chap. 6, Wiley and Sons, London (1977); M. S. J. Hashmi, *J. Strain Anal.*, **15**: 201 (1980).

‡ C. Zener and J. H. Hollomon, *J. Appl. Phys.*, **15**: 22 (1944); T. Trozera, O. D. Sherby, and J. L. Dorn, *Trans. ASME*, **49**: 173 (1957). The expression in the curly bracket of (12) is often called the Zener-Hollomon parameter, which is also useful in the theory of high-temperature creep. A generalized constitutive equation, including the effect of strain, has been discussed by J. M. Alexander, *Plasticity Today* (Ed. H. Sawczick), Elsevier, Amsterdam (1986).

and temperature. When the temperature is held constant, the test results can be fitted by the power law†

$$\sigma = C\varepsilon^n \dot{\varepsilon}^m \tag{13}$$

where C, m and n depend on the operating temperature. The exponent m is known as the *strain-rate sensitivity*, which generally increases with temperature, particularly when it is above the recrystallization temperature. The strain-hardening exponent n, on the other hand, rapidly decreases with increasing values of the elevated temperature.

The dependence of the flow stress on strain rate and temperature for a given strain is sometimes expressed in the alternative form‡

$$\sigma = f\left\{T\left(1 - m\ln\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\right\} \tag{14}$$

where m and $\dot{\varepsilon}_0$ are constants, the quantity in the curly bracket being known as the velocity modified temperature. It is consistent with the fact that an increase in strain rate is in effect equivalent to a decrease in temperature. Equation (14) agrees with test data for a fairly wide range of values of the strain rate and temperature.

Above the recrystallization temperature, the yield stress attains a saturation value after a small amount of strain, as a result of the work-hardening rate being balanced by the rate of thermal softening. The dependence of the saturation stress on strain rate and temperature can be expressed with reasonable accuracy by the empirical equation§

$$\sigma = C\sinh^{-1}\left(m\dot{\varepsilon}^n \exp\frac{b}{T}\right)$$

where b, C, m, and n are material constants. The activation energy Q is then independent of the temperature, and is approximately equal to Rb/n. A distinction between cold- and hot-working of metals is usually made on the basis of the recrystallization temperature, whose absolute value is roughly one-half of the absolute melting temperature. The above equation reduces to a power law when the expression in the parenthesis is sufficiently small.¶

† W. F. Hosford and R. M. Caddell, *Metal Forming Mechanics and Metallurgy*, 2d ed., Chap. 5, Prentice-Hall, Englewood Cliffs, NJ (1993).

‡ C. W. MacGregor and J. C. Fisher, *J. Appl. Mech.*, **13**: 11 (1946).

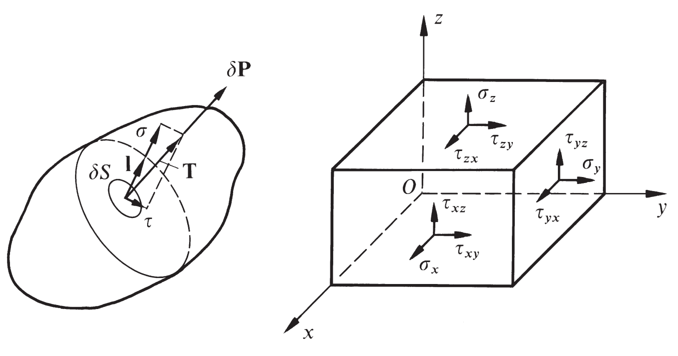
§ C. M. Sellars and W. J. McG. Tegart, *Mem. Sci. Rev. Met.*, **63**: 731 (1966); S. K. Samanta, *Proc. 11th Int. M.T.D.R. Conf.*, Pergamon Press, Oxford (1970).

¶ Large neck-free extensions are possible in certain highly rate-sensitive alloys, called *superplastic alloys*. See W. A. Backofen, I. Turner and H. Avery, *Trans. Q. ASM*, **57**: 981 (1966); J. W. Edington, K. N. Melton, and C. P. Cutler, *Prog. Mater. Sci.*, **21**: 63 (1976); K. A. Padmanabhan and G. J. Davies, *Superplasticity*, Springer-Verlag, Berlin (1980); T. G. Nieh, J. Wadsworth, and O. D. Sherby, *Superplasticity in Metals and Ceramics*, Cambridge University Press, Cambridge (1997).

1.3 Analysis of Stress

(i) ***Stress tensor*** When a body is subjected to a set of external forces, internal forces are produced in different parts of the body so that each element of the body is in a state of statical equilibrium. Through any point O within the body, consider a small surface element δS whose orientation is specified by the unit vector $\mathbf{l}$ along the normal drawn on one side of the element (Fig. 1.8*a*). The material on this side of δS may be regarded as exerting a force $\delta\mathbf{P}$ across the surface element upon the material on the other side. The limit of the ratio $\delta\mathbf{P}/\delta S$ as δS tends to zero is the stress vector $\mathbf{T}$ at O associated with the direction $\mathbf{l}$. For given external loading, the stress acting across any plane passing through a given point O depends on the orientation of the plane. The resolved component of the stress vector along the unit normal $\mathbf{l}$ is called the direct or normal stress denoted by σ, while the component tangential to the plane is known as the shear stress denoted by τ.

Consider now a set of rectangular axes Ox, Oy, and Oz emanating from a typical point O, and imagines a small rectangular parallelepiped at O having its edges parallel to the axes of reference (Fig. 1.8*b*). The normal stresses across the faces of the block are denoted by σ_x, σ_y, and σ_z, where the subscripts denote the directions of the normal to the faces. The shear stress acting on the faces normal to the x axis is resolved into the components τ_{xy} and τ_{xz} parallel to the y and z axes respectively. The first suffix denotes the direction of the normal to the face and the second suffix the direction of the component. In a similar way, the shear stresses on the faces normal to the y axis are denoted by τ_{yx} and τ_{yz}, and those on the faces normal to the z axis by τ_{zx} and τ_{zy}. The stresses are taken as positive if they are directed as shown in the figure, when the outward normals to the faces are in the positive directions of the coordinate axes. The positive directions are all reversed on the remaining faces of the block where the outward normals are in the negative directions of the axes of reference. The nine components of the stress at any point form a second-order tensor σ_{ij}, known as the stress tensor, where i and j take integral



(*a*) (*b*)

Figure 1.8 Definition of stress. (*a*) Normal and shear stresses; (*b*) components of stress tensor.

values 1, 2, and 3. The stress components may be displayed as elements of the square matrix

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

The forces acting on the faces of the parallelepiped are clearly in equilibrium. To examine the couple equilibrium, let $\delta x, \delta y, \delta z$ denote the lengths of these faces along the respective coordinate axes. Then the resultant couple about the z axis is $(\tau_{xy} - \tau_{yx})\delta x\ \delta y\ \delta z$, which must vanish for equilibrium. This gives $\tau_{xy} = \tau_{yx}$. Similarly, the conditions for couple equilibrium about the other two axes give $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$. These identities may be expressed as $\sigma_{ij} = \sigma_{ji}$, implying that the stress tensor is symmetric with respect to its subscripts. Thus there are six independent stress components, three normal components σ_x, σ_y, σ_z, and three shear components τ_{xy}, τ_{yz}, τ_{zx}, which completely specify the state of stress at each point of the body. The matrix representing the stress tensor is evidently symmetrical.

The mean of the three normal stresses, equal to $(\sigma_x + \sigma_y + \sigma_z)/3$, is known as the *hydrostatic stress* denoted by σ_0. A *deviatoric* or *reduced stress* tensor s_{ij} is defined as that which is obtained from σ_{ij} by reducing the normal stress components by σ_0. This gives the deviatoric stresses as

$$s_{ij} = \begin{bmatrix} s_x & s_{xy} & s_{xz} \\ s_{yx} & s_y & s_{yz} \\ s_{zx} & s_{zy} & s_z \end{bmatrix} = \begin{bmatrix} (\sigma_x - \sigma_0) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_0) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma_0) \end{bmatrix}$$

The deviatoric normal stresses are therefore given by

$$3s_x = 2\sigma_x - \sigma_y - \sigma_z, \qquad 3s_y = 2\sigma_y - \sigma_z - \sigma_x, \qquad 3s_z = 2\sigma_z - \sigma_x - \sigma_y$$

The deviatoric shear stresses are the same as the actual shear stresses. Since $s_x + s_y + s_z = 0$, the deviatoric normal stresses cannot all have the same sign. The difference between any two normal components of the deviatoric stress is the same as that between the corresponding components of the actual stress. Expressed in suffix notation, the relationship between s_{ij} and σ_{ij} is

$$s_{ij} = \sigma_{ij} - \sigma_0\delta_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{15}$$

where δ_{ij} is the *Kronecker delta* whose value is unity when $i = j$ and zero when $i \neq j$. Evidently, $\delta_{ij} = \delta_{ji}$. Any repeated or dummy suffix indicates a summation of all terms obtainable by assigning the values 1, 2, and 3 to this suffix in succession. Thus $\sigma_{kk} = \sigma_x + \sigma_y + \sigma_z$. It follows from the definition of the delta symbol that $\sigma_{ij}\delta_{jk} = \sigma_{ik}$, where j is a dummy suffix and i, k are free suffixes. Each term of a tensor equation must have the same free suffixes, but a dummy suffix can be replaced by any other letter different from the free suffixes.

(ii) ***Stresses on an oblique plane*** Consider the equilibrium of a small tetrahedron *OABC* of which the edges *OA*, *OB*, and *OC* are along the coordinate axes (Fig. 1.9). Let (l, m, n) be the directions cosines of a straight line drawn along the exterior normal to the oblique plane *ABC*. These are the components of the unit normal **1** with respect to *Ox*, *Oy*, and *Oz*. If the area of the face *ABC* is denoted by δS, the faces *OAB*, *OBC*, and *OCA* have areas $n\,\delta S, l\,\delta S$, and $m\,\delta S$ respectively. The stress vector **T** acting across the face *ABC* has components T_x, T_y, and T_z along the axes of reference. Resolving the forces in the directions *Ox*, *Oy*, and *Oz*, we get

$$\begin{aligned} T_x &= l\sigma_x + m\tau_{xy} + n\tau_{zx} \\ T_y &= l\tau_{xy} + m\sigma_y + n\tau_{yz} \\ T_z &= l\tau_{zx} + m\tau_{yz} + n\sigma_z \end{aligned} \tag{16}$$

on cancelling out δS from each equation of force equilibrium. When δS tends to zero, these equations give the components of the stress vector at *O*, associated with the direction (l, m, n), in terms of the components of the stress tensor. Using the suffix notation and the summation convention, (16) can be expressed as

$$T_j = l_i \sigma_{ij}$$

where $l_1 = l,\ l_2 = m,\ l_3 = n$. The above equation is equivalent to three equations corresponding to the three possible values of the free suffix j. A single free suffix therefore characterizes a vector. The normal stress across the plane specified by its

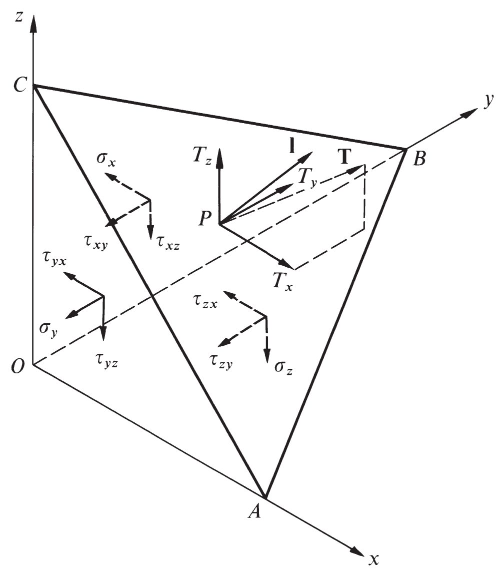


Figure 1.9 Stresses across an oblique plane in a three-dimensional state of stress.

normal (l, m, n) is

$$\begin{aligned}\sigma &= lT_x + mT_y + nT_z = l_jT_j = l_il_j\sigma_{ij} \\ &= l^2\sigma_x + m^2\sigma_y + n^2\sigma_z + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx} \qquad (17)\end{aligned}$$

The shear stress across the plane can be resolved into two components in a pair of mutually perpendicular directions in the plane. Denoting one of these directions by (l', m', n'), the corresponding shear component is obtained as

$$\begin{aligned}\tau' &= l'T_x + m'T_y + n'T_z = l'_jT_j = l_il'_j\sigma_{ij} \\ &= ll'\sigma_x + mm'\sigma_y + nn'\sigma_z + (lm' + ml')\tau_{xy} + (mn' + nm')\tau_{yz} + (nl' + ln')\tau_{zx} \qquad (18)\end{aligned}$$

This evidently is the resolved component of the resultant stress in the direction (l', m', n'). The direction cosines satisfy the well-known geometrical relations

$$l^2 + m^2 + n^2 = 1 \qquad l'^2 + m'^2 + n'^2 = 1 \qquad ll' + mm' + nn' = 0 \qquad (19)$$

The first two equations express the fact (l, m, n) and (l', m', n') represent unit vectors, while the last relation expresses the orthogonality of these vectors. The shear stress is most conveniently found from the fact that its magnitude is $\sqrt{T^2 - \sigma^2}$, and its direction cosines are proportional to its rectangular components

$$T_x - l\sigma \qquad T_y - m\sigma \qquad T_z - n\sigma$$

Let x_i and x'_i represent two sets of rectangular axes through a common origin O, and a_{ij} denote the direction cosine of the x'_i axis with respect to the x_j axis. The direction cosine of the x_i axis with respect to the x'_j axis is then equal to a_{ji}. It follows from geometry that the coordinates of any point in space referred to the two sets of axes are related by the equations

$$x'_i = a_{ij}x_j \qquad x_j = a_{ij}x'_i \qquad (20)$$

The components of any vector transform† according to the same law as (20). Let σ'_{ij} denote the components of the stress tensor when referred to the set of axes x'_i. A defining property of tensors is the transformation law

$$\sigma'_{ij} = a_{ik}a_{jl}\sigma_{kl} \qquad (21)$$

Let us suppose that $a_{11} = l$, $a_{12} = m$, $a_{13} = n$, and $a_{21} = l'$, $a_{22} = m'$, $a_{23} = n'$. The normal stress across the plane (l, m, n) is then equal to σ'_{11}, and the corresponding expression (17) can be readily verified from (21). Similarly, the component of the shear stress across the plane resolved in the direction (l', m', n') is equal to σ'_{12} which can be shown to be that given by (18).

† It follows from (20) that $x'_i = a_{ik}x_k = a_{ik}a_{jk}x'_j$, indicating that $a_{ik}a_{jk} = \delta_{ij}$, which furnishes six independent relations of types (19).

(iii) ***Principal stresses*** The normal stress σ has maximum and minimum values for varying orientations of the oblique plane. Regarding l and m as the independent direction cosines, the conditions for stationary σ may be written as $\partial\sigma/\partial l = 0$, $\partial\sigma/\partial m = 0$. Differentiating the first equation of (19) partially with respect to l and m, we get $\partial n/\partial l = -l/n$ and $\partial n/\partial m = -m/n$. Inserting these results into the partial derivatives of (17), and using (16), the stationary condition can be expressed as

$$\frac{T_x}{l} = \frac{T_y}{m} = \frac{T_z}{n}$$

This shows that the resultant stress across the plane acts in the direction of the normal when the normal stress has a stationary value. Each of the above ratios is therefore equal to the normal stress σ. The substitution into (16) gives

$$\begin{aligned} l(\sigma_x - \sigma) + m\tau_{xy} + n\tau_{zx} &= 0 \\ l\tau_{xy} + m(\sigma_y - \sigma) + n\tau_{yz} &= 0 \\ l\tau_{zx} + m\tau_{yz} + n(\sigma_z - \sigma) &= 0 \end{aligned} \tag{22}$$

In suffix notation, these relations are equivalent to $l_i(\sigma_{ij} - \sigma\delta_{ij}) = 0$, which follows directly from the fact that $T_j = \sigma l_j$ across a principal plane. The set of linear homogeneous equations (22) would have a nonzero solution for l, m, n if the determinant of their coefficients vanishes. Thus

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

Expanding this determinant, we obtain a cubic equation in σ having three real roots σ_1, σ_2, σ_3, which are known as the *principal stresses*. These stresses act across planes on which the shear stresses are zero. The cubic may be expressed in the form

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0 \tag{23}$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{ii} \tag{24}$$

$$\begin{aligned} I_2 &= -(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \\ &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \tfrac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) \end{aligned} \tag{25}$$

$$\begin{aligned} I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \\ &= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix} = \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{26}$$

The expressions for I_1, I_2, I_3 in terms of the principal stresses follow from the fact that (23) is equivalent to the equation $(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0$. Since the stationary values of the normal stress do not depend on the orientation of the coordinate axes, the coefficients of (23) must also be independent of the choice of the axes of references. The quantities I_1, I_2, I_3 are therefore known as the *invariants* of the stress tensor.†

The direction cosines corresponding to each principal stress can be found from the first equation of (19) and any two equations of (22) with the appropriate value of σ. Let (l_1, m_1, n_1) and (l_2, m_2, n_2) represent the directions of σ_1 and σ_2 respectively. If we express (22) in terms of l_1, m_1, n_1, and σ_1, multiply these equations by l_2, m_2, n_2 in order and add them together, and then subtract the resulting equation from that obtained by interchanging the subscripts, we arrive at the result

$$(\sigma_1 - \sigma_2)(l_1 l_2 + m_1 m_2 + n_1 n_2) = 0$$

If $\sigma_1 \neq \sigma_2$, the above equation indicates that the directions (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular to one another. It follows, therefore, that the principal directions corresponding to distinct values of the principal stresses are mutually orthogonal. These directions are known as the *principal axes* of the stress. When two of the principal stresses are equal to one another, the direction of the third principal stress is uniquely determined, but all directions perpendicular to this principal axis are principal directions. When $\sigma_1 = \sigma_2 = \sigma_3$, representing a hydrostatic state of stress, any direction in space is a principal direction.

The invariants of the deviatoric stress tensor are obtained by replacing the actual stress components in (24) to (26) by the corresponding deviatoric components. The first deviatoric stress invariant is

$$J_1 = s_x + s_y + s_z = s_1 + s_2 + s_3 = s_{ii} = 0$$

where s_1, s_2, s_3 are the principal deviatoric stresses. These principal values are the roots of the cubic equation

$$s^3 - J_2 s - J_3 = 0 \tag{27}$$

where

$$\begin{aligned} J_2 &= -(s_x s_y + s_y s_z + s_z s_x) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \\ &= \tfrac{1}{2}(s_x^2 + s_y^2 + s_z^2) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \\ &= \tfrac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \end{aligned} \tag{28}$$

$$\begin{aligned} J_3 &= s_x s_y s_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - s_x \tau_{yz}^2 - s_y \tau_{zx}^2 - s_z \tau_{xy}^2 \\ &= \begin{vmatrix} s_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & s_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & s_z \end{vmatrix} = s_1 s_2 s_3 = \tfrac{1}{3}(s_1^3 + s_2^3 + s_3^3) \end{aligned} \tag{29}$$

† Any symmetric tensor of second order has three real principal values, the basic invariants of the tensor being identical in form to those for the stress.

The last two expressions for J_2 are obtained from the first expression by adding the identically zero terms $\frac{1}{2}(s_x + s_y + s_z)^2$ and $\frac{1}{3}(s_x + s_y + s_z)^2$ respectively, and noting the fact that $s_x - s_y = \sigma_x - \sigma_y$ etc. Similarly, the last expression for J_3 follows from the preceding one on adding the term $\frac{1}{3}(s_1 + s_2 + s_3)^3$. In suffix notation, these invariants can be written as

$$J_2 = \tfrac{1}{2} s_{ij} s_{ij} \qquad J_3 = \tfrac{1}{3} s_{ij} s_{jk} s_{ki} \tag{30}$$

The repetition of all suffixes is a characteristic of invariants, which are scalars. Substituting $\sigma = s + I_1/3$ in (23) and comparing the coefficients of the resulting equation with those of (27), we obtain

$$J_2 = I_2 + \tfrac{1}{3} I_1^2 \qquad J_3 = I_3 + \tfrac{1}{3} I_1 I_2 + \tfrac{2}{27} I_1^3$$

When J_2 and J_3 have been found, equation (27) may be solved by means of the substitution $s = 2\sqrt{J_2/3}\cos\phi$, which reduces the cubic to

$$\cos 3\phi = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{3/2} \tag{31}$$

Since $4J_2^3 \geqslant 27J_3^2$, the right-hand side† of (31) lies between -1 and 1, and one value of ϕ lies between 0 and $\pi/3$. The principal deviatoric stresses may therefore be written as

$$s_1 = 2\sqrt{\frac{J_2}{3}}\cos\phi \qquad s_2, s_3 = -2\sqrt{\frac{J_2}{3}}\cos\left(\frac{\pi}{3} \pm \phi\right) \tag{32}$$

where $0 \leqslant \phi \leqslant \pi/3$. Any function of these principal components is also a function of the invariants, which play an important part in the mathematical development of the theory of plasticity.

(iv) ***Principal shear stresses*** When the principal stresses and their directions are known, it is convenient to take the principal axes as the axes of reference. If Ox, Oy, Oz denote the coordinate axes associated with the principal stresses σ_1, σ_2, σ_3 respectively, the components of the stress vector across a plane whose normal is in the direction (l, m, n) are

$$T_x = l\sigma_1 \qquad T_y = m\sigma_2 \qquad T_z = n\sigma_3$$

The normal stress across the oblique plane therefore becomes

$$\sigma = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 \tag{33}$$

† Using (32) and (31), it can be shown that $4J_2^3 - 27J_3^2 = (\sigma_1 - \sigma_2)^2(\sigma_2 - \sigma_3)^2(\sigma_3 - \sigma_1)^2$, which is a positive quantity for distinct values of the principal stresses.

If the magnitude of the shear stress across the plane is denoted by τ, then

$$\begin{aligned}\tau^2 = T^2 - \sigma^2 &= (l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2) - (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2 \\ &= (\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 + (\sigma_3 - \sigma_1)^2 n^2 l^2 \end{aligned} \tag{34}$$

in view of the relation $l^2 + m^2 + n^2 = 1$. Since the components of the normal stress along the coordinate axes are $(l\sigma, m\sigma, n\sigma)$, the components of the shear stress are $l(\sigma_1 - \sigma), m(\sigma_2 - \sigma), n(\sigma_3 - \sigma)$. Hence the direction cosines of the shear stress are

$$l_s = l\left(\frac{\sigma_1 - \sigma}{\tau}\right) \qquad m_s = m\left(\frac{\sigma_2 - \sigma}{\tau}\right) \qquad n_s = n\left(\frac{\sigma_3 - \sigma}{\tau}\right) \tag{35}$$

A plane which is equally inclined to the three principal axes is known as the *octahedral plane*, the direction cosines of its normal being given by $l^2 = m^2 = n^2 = 1/3$. These relations are satisfied by four pairs of parallel planes forming a regular octahedron having its vertices on the principal axes. By (33) and (34), the octahedral normal stress is equal to the hydrostatic stress σ_0, and the octahedral shear stress is of the magnitude

$$\tau_0 = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\tfrac{2}{3}J_2}$$

The components of the octahedral shear stress along the principal axes are numerically equal to $1/\sqrt{3}$ times the deviatoric principal stresses.

We now proceed to determine the stationary values of the shear stress for varying orientations of the oblique plane. To this end, we put $n^2 = 1 - l^2 - m^2$ in (34), and express it in the form

$$\tau^2 = l^2(\sigma_1^2 - \sigma_3^2) + m^2(\sigma_2^2 - \sigma_3^2) + \sigma_3^2 - \{l^2(\sigma_1 - \sigma_3) + m^2(\sigma_2 - \sigma_3) + \sigma_3\}^2$$

where l and m are treated as the independent variables. We shall follow the convention $\sigma_1 > \sigma_2 > \sigma_3$. Equating to zero the derivatives of τ^2 with respect to l and m, we obtain

$$\begin{aligned} l(\sigma_1 - \sigma_3)[(1 - 2l^2)(\sigma_1 - \sigma_3) - 2m^2(\sigma_2 - \sigma_3)] = 0 \\ m(\sigma_2 - \sigma_3)[(1 - 2m^2)(\sigma_2 - \sigma_3) - 2l^2(\sigma_1 - \sigma_3)] = 0 \end{aligned} \tag{36}$$

These equations are obviously satisfied for $l = m = 0$, and hence $n = 1$, which corresponds to a principal stress direction for which the shear stress has a minimum value of zero. To obtain a maximum value of the shear stress, we set $l = 0$ satisfying the first equation of (36), and use this value in the second equation to get $l - 2m^2 = 0$. This gives $l = 0, m^2 = n^2 = 1/2$ corresponding to maximum shear stress equal to $\frac{1}{2}(\sigma_2 - \sigma_3)$ according to (34). Similarly, the direction represented by $m = 0$, $n^2 = l^2 = 1/2$ satisfies (36), and furnishes a maximum value of $\frac{1}{2}(\sigma_1 - \sigma_3)$ for the shear stress. Finally, setting $n = 0$ and hence $l^2 + m^2 = 1$, we find that τ is a maximum for $l^2 = m^2 = 1/2$, giving a stationary value equal to $\frac{1}{2}(\sigma_1 - \sigma_2)$. The three

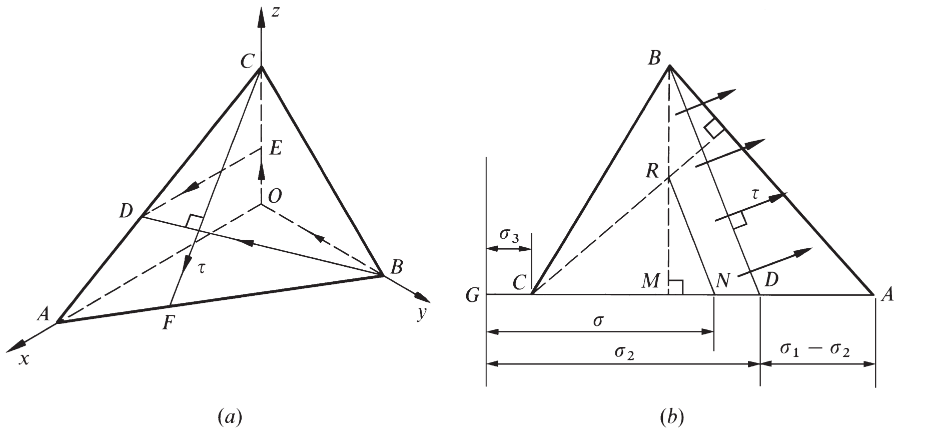


Figure 1.10 Construction for the normal stress and the direction of the shear stress.

stationary shear stresses, known as the *principal shear stresses*, may therefore be written as

$$\tau_1 = \tfrac{1}{2}(\sigma_2 - \sigma_3) \qquad \tau_2 = \tfrac{1}{2}(\sigma_1 - \sigma_3) \qquad \tau_3 = \tfrac{1}{2}(\sigma_1 - \sigma_2) \tag{37}$$

These stresses act in directions which bisect the angles between the principal axes. By (33), the normal stresses acting on the planes of τ_1, τ_2, τ_3 are immediately found to be, respectively,

$$\tfrac{1}{2}(\sigma_2 + \sigma_3) \qquad \tfrac{1}{2}(\sigma_1 + \sigma_3) \qquad \tfrac{1}{2}(\sigma_1 + \sigma_2)$$

In view of the assumption $\sigma_1 > \sigma_2 > \sigma_3$, the greatest shear stress is of magnitude $\frac{1}{2}(\sigma_1 - \sigma_3)$, and it acts across a plane whose normal bisects the angle between the directions of σ_1 and σ_3. It follows from (32) that the greatest shear stress is equal to $\sqrt{J_2}\cos(\pi/6 - \phi)$, where ϕ lies between zero and $\pi/3$ satisfying (31).

(v) ***Shear stress and the oblique triangle*** Consider now the direction of the shear stress on an inclined plane in relation to the true shape of the oblique triangle. It is assumed for simplicity that the direction cosines (l, m, n) are all positive.† Let δh denote the perpendicular distance from the origin O to the oblique plane ABC (Fig. 1.10*a*). Then the distances of the vertices A, B, C from O are $\delta h/l$, $\delta h/m$, $\delta h/n$ respectively, their ratios being

$$OA{:}OB{:}OC = mn{:}nl{:}lm \tag{38}$$

The sides of the triangle are readily found from the right-angled triangles AOB, BOC, and COA. The true shape of the oblique triangle ABC is therefore defined by the ratios

$$AB{:}BC{:}CA = n\sqrt{1 - n^2}{:}l\sqrt{1 - l^2}{:}m\sqrt{1 - m^2} \tag{39}$$

† No generality is lost in this assumption, since the positive directions of the axes of reference can be arbitrarily chosen, and the expressions for σ and τ involve only the squares of the direction cosines.

The vertical angles of the triangle follow from (39) and the well-known cosine law. The results can be conveniently put in the form

$$\tan A = \frac{l}{mn} \qquad \tan B = \frac{m}{nl} \qquad \tan C = \frac{n}{lm} \tag{40}$$

The coordinate axes in Fig. 1.10*a* are in the directions of the principal stresses. A line BD is drawn from the apex B to meet the opposite side of AC at D, such that BD is perpendicular to the direction of the shear stress across the plane. The components of the vector **BD** along the axes Ox, Oy, Oz are equal to ED, $-OB$, OE respectively. Since BD is orthogonal to both the directions (l, m, n) and (l_s, m_s, n_s), the scalar products of **BD** with the unit vectors representing these directions must vanish. Using (35) and (33), it is easily shown that

$$ED{:}OB{:}OE = mn(\sigma_2 - \sigma_3){:}nl(\sigma_1 - \sigma_3){:}lm(\sigma_1 - \sigma_2) \tag{41}$$

If $\sigma_1 > \sigma_2 > \sigma_3$, the line BD must meet AC internally as shown. Indeed, from the similar triangles CDE and CAO, we have

$$\frac{CD}{CA} = \frac{ED}{OA} = \frac{ED}{OB}\frac{OB}{OA} = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} \tag{42}$$

in view of (38) and (41). If points A, D, C, and G are located along a straight line, such that $GA = \sigma_1$, $GD = \sigma_2$, and $GC = \sigma_3$, and the true shape triangle ABC is constructed on CA as base (Fig. 1.10*b*), then in view of (42), the shear stress is directed at right angles to the line joining B and D. Since $n_s < 0$ by (35), the direction of the shear stress vector is obtained by a 90° counterclockwise rotation from the direction BD. If R is the orthocenter of the triangle ABC, and BM is drawn perpendicular to CA, then by Eqs. (40),

$$\frac{CM}{AM} = \frac{\cot C}{\cot A} = \frac{l^2}{n^2} \qquad \frac{MR}{MB} = \frac{\cot A}{\tan C} = m^2 \tag{43}$$

since angle MRC is equal to the vertical angle A. If RN is drawn parallel to BD, meeting CA at N, then $MN/MD = MR/MB = m^2$, which gives

$$\begin{aligned} GN &= GM + MN = (l^2 + m^2 + n^2)GM + m^2 MD \\ &= l^2(GA - MA) + m^2 GD + n^2(GC + CM) = l^2 GA + m^2 GD + n^2 GC \end{aligned}$$

The expression on the right-hand side is equal to σ in view of (33). Hence GN represents the magnitude of the normal stress transmitted across the plane.† It follows from (34) and (41) that if OB represents the quantity $nl(\sigma_1 - \sigma_3)$ to a certain scale, then BD will represent the shear stress τ to the same scale. Hence

$$\frac{OB}{BD} = nl\left(\frac{\sigma_1 - \sigma_3}{\tau}\right) = \frac{nl}{\tau}CA$$

† The constructions for the normal stress and the direction of the shear stress are due to H. W. Swift, *Engineering*, **162**: 381 (1946).

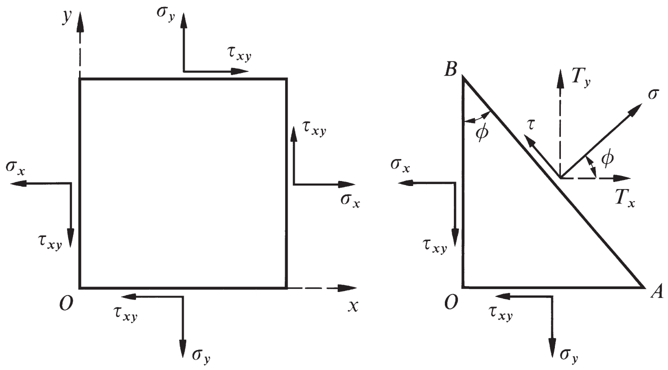


Figure 1.11 An element in a state of plane stress.

with reference to Fig. 1.10. Since $RN/BD = MR/MB = m^2$ by (43), and $CA = \sqrt{OC^2 + OA^2}$, we have

$$RN = m^2 \cdot BD = \frac{m^2\tau}{nl}\frac{OB}{CA} = \frac{m\tau}{\sqrt{1-m^2}}$$

in view of (38). It follows that the magnitude of the shear stress on the plane is $\tau = RN \tan\beta$, where β is the angle made by the normal to the plane with the direction of the intermediate principal stress σ_2.

(vi) ***Plane stress*** A state of plane stress is defined by $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. The z axis then coincides with a principal axis, and the corresponding principal stress vanishes.† The orientation of Ox and Oy with respect to the other two principal axes is, however, arbitrary. Consider a plane AB perpendicular to the xy plane, and let ϕ be the counterclockwise angle made by the normal to the plane with the x axis (Fig. 1.11). The shear stress τ will be reckoned positive when it is directed to the left of the exterior normal. Setting $l = \cos\phi$, $m = \sin\phi$, and $n = 0$ in (16), the components of the stress vector across AB are obtained as

$$T_x = \sigma_x \cos\phi + \tau_{xy}\sin\phi \qquad T_y = \tau_{xy}\cos\phi + \sigma_y \sin\phi \tag{44}$$

The resolved components of the resultant stress along the normal and the tangent to the plane are

$$\sigma = T_x\cos\phi + T_y\sin\phi \qquad \tau = -T_x\sin\phi + T_y\cos\phi$$

† The results for plane stress are directly applicable to the more general situation where the z axis coincides with the direction of any nonzero principal stress.

Substituting for T_x and T_y in the above equations, the normal and shear stresses across the plane are obtained as

$$\begin{aligned}\sigma &= \sigma_x \cos^2\phi + \sigma_y \sin^2\phi + 2\tau_{xy} \sin\phi\cos\phi \\ &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\phi + \tau_{xy}\sin 2\phi \qquad (45)\\ \tau &= -(\sigma_x - \sigma_y)\sin\phi\cos\phi + \tau_{xy}(\cos^2\phi - \sin^2\phi) \\ &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\phi + \tau_{xy}\cos 2\phi \qquad (46)\end{aligned}$$

These results may be directly obtained from (16) and (17) by setting $l = m' = \cos\phi$, $m = -l' = \sin\phi$ and $n = n' = 0$. Since $d\sigma/d\phi = 2\tau$, which is readily verified from above, the shear stress vanishes on the plane for which the normal stress has a stationary value. This corresponds to $\phi = \alpha$, where

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (47)$$

which defines two directions at right angles to one another, giving the principal axes in the plane of Ox and Oy. The principal stresses σ_1, σ_2 are the roots of the equation

$$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau_{xy}^2$$

which is obtained by writing $T_x = \sigma\cos\phi$ and $T_y = \sigma\sin\phi$ in (44), and then eliminating ϕ between the two equations. The solution is

$$\sigma_1, \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad (48)$$

The acute angle made by the direction of the algebraically greater principal stress σ_1 with the x axis is measured in the counterclockwise sense when τ_{xy} is positive, and in the clockwise sense when τ_{xy} is negative. It follows from (48) that

$$\sigma_x + \sigma_y = \sigma_1 + \sigma_2 \qquad \sigma_x\sigma_y - \tau_{xy}^2 = \sigma_1\sigma_2 \qquad (49)$$

These are the basic invariants of the stress tensor in a state of plane stress. Evidently, any function of these invariants is also an invariant.

Let $O\xi$, and $O\eta$ represent a new pair of rectangular axes in the (x, y) plane, and let ϕ be the angle of inclination of the ξ axis to the x axis measured in the counterclockwise sense. Then the stress components σ_ξ and $\tau_{\xi\eta}$, referred to the new axes, are directly given by the right-hand sides of (45) and (46) respectively. The remaining stress component σ_η is obtained by writing $\pi/2 + \phi$ for ϕ in (45), resulting in

$$\begin{aligned}\sigma_\eta &= \sigma_x \sin^2\phi + \sigma_y \cos^2\phi - 2\tau_{xy}\sin\phi\cos\phi \\ &= \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\phi - \tau_{xy}\sin 2\phi \qquad (50)\end{aligned}$$

It immediately follows that $\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y$, which shows the invariance of the first expression of (49). The invariance of the second expression may be similarly verified.

Considering the principal axes as the axes of reference, the shear stress across an inclined plane can be written as $\tau = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\phi$, which indicates that the shear stress is directed to the right of the outward normal to the plane when $\sigma_1 > \sigma_2$ and $0 < \phi < \pi/2$. The shear stress has its greatest magnitude when $\phi = \pm\pi/4$, the maximum value of the shear stress being

$$\tau_{\max} = \frac{1}{2}|\sigma_1 - \sigma_2| = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{51}$$

There are two other principal shear stresses, having magnitudes $\frac{1}{2}|\sigma_1|$ and $\frac{1}{2}|\sigma_2|$, and bisecting the angles between the z axis and the directions of σ_1 and σ_2 respectively. A little examination of the three principal values reveals that the numerically greatest shear stress occurs in the plane of the applied stresses when σ_1 and σ_2 have opposite signs, and out of the plane of the applied stresses when they are of the same sign. In view of (49), the former corresponds to $\sigma_x\sigma_y < \tau_{xy}^2$ and the latter to $\sigma_x\sigma_y > \tau_{xy}^2$. A state of pure shear is given by $\sigma_1 = -\sigma_2$, since the normal stress then vanishes on the planes of maximum shear.

1.4 Mohr's Representation of Stress

(i) ***Two-dimensional stress state*** A useful graphical method of analyzing the state of stress has been developed by Mohr.† In this method, the normal and shear stresses across any plane are represented by a point on a plane diagram in which σ and τ are taken as rectangular coordinates. For the present purpose, it is necessary to regard the shear stress as positive when it has a clockwise moment about a point within the element. In Fig. 1.12, the stresses acting on planes perpendicular to the x and y axes are represented by the points X and Y on the (σ, τ) plane. The circle drawn on XY as diameter, and having its center C on the σ axis, is called the Mohr circle for the considered state of stress. The points A and B, where the circle is intersected by the σ axis, define the principal stresses, since $OA = \sigma_1$ and $OB = \sigma_2$ in view of (48) and the geometry of Mohr's diagram. By (47), the angle made by CA with CX is twice the angle α which the direction of σ_1 makes with the x axis in the physical plane. The normal and shear stresses transmitted across a plane, whose normal is inclined at a counterclockwise angle ϕ to the x axis, correspond to the point L on the Mohr circle, where CL is inclined to CX at an angle 2ϕ measured in the same sense. The proof of the construction follows from the fact that $CD = CL\cos 2\alpha$ and $XD = CL$ $\sin 2\alpha$, where XD is perpendicular to OA. Then from the geometry of the figure,

$$ON = OC + CL\cos 2(\alpha - \phi) = OC + CD\cos 2\phi + XD\sin 2\phi$$
$$LN = CL\sin 2(\alpha - \phi) = -CD\sin 2\phi + XD\cos 2\phi$$

These expressions are equivalent to (45) and (46) in view of the present sign convention. If LC is produced to meet the circle again at M, then the coordinates of M give

† O. Mohr, *Zivilingenieur*, **28**: 112 (1882). See also his book, *Technische Mechanik*, Berlin (1906).

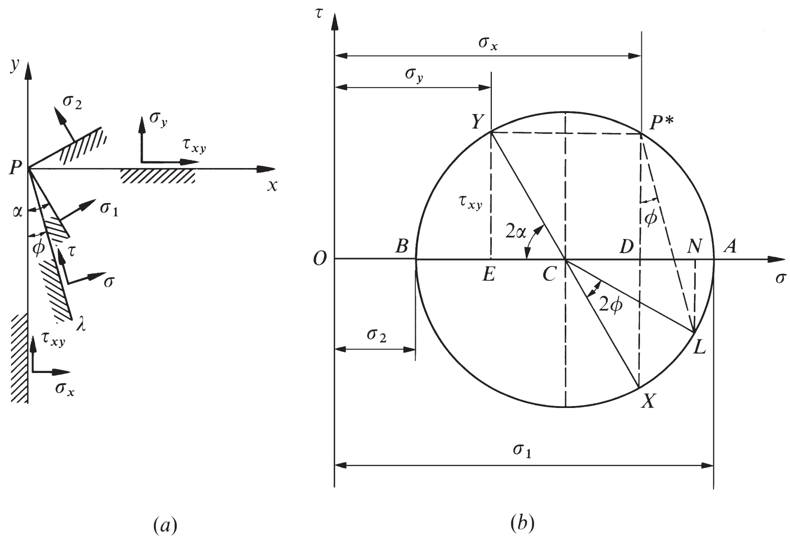


Figure 1.12 Mohr's construction for a two-dimensional state of stress. (*a*) Physical plane; (*b*) stress plane.

the stresses across a plane perpendicular to that corresponding to L. The maximum shear stress is evidently equal to the radius of the Mohr circle, and acts on planes that correspond to the extremities of the vertical diameter. The normal stress across these planes is equal to the distance of the center of the circle from the origin of the stress plane.

It is instructive to consider the following alternative construction, also due to Mohr. Let a generic point P, the state of stress at which is being discussed, be taken as the origin of coordinates in the physical plane (Fig. 1.12*a*). All planes passing through P and containing the z axis are denoted by their traces in the xy plane. The normal and shear stresses corresponding to the points X and Y on the Mohr circle are transmitted across the planes Py and Px respectively. The lines through X and Y drawn parallel to these planes intersect the circle at a common point P^*, which is called the pole of the Mohr circle. When the stress circle and the pole are given, the stresses acting across any plane $P\lambda$ through P are found by locating the point L on the circle such that P^*L is parallel to $P\lambda$, the angle XCL at the center being twice the peripheral angle XP^*L over the arc XL. The planes corresponding to the principal stresses are parallel to P^*A and P^*B, and those corresponding to the maximum shear stress are parallel to P^*S and P^*T. It may be noted that the magnitude of the resultant stress across any plane is equal to the distance of the corresponding stress point on the Mohr circle from the origin of the stress plane.

(ii) ***Three-dimensional stress state*** Suppose that the principal stresses σ_1, σ_2, σ_3 are known in magnitude and direction for a three-dimensional state of stress. These principal values are assumed as distinct, and so labeled that $\sigma_1 > \sigma_2 > \sigma_3$. A graphical method developed by Mohr can be used to find the variation of normal and shear stresses with the direction (l, m, n). We begin with the relations

$$\begin{aligned} l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 &= \sigma \\ l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2 &= \sigma^2 + \tau^2 \\ l^2 + m^2 + n^2 &= 1 \end{aligned} \tag{52}$$

This is a set of three linear equations for the squares of the direction cosines. The solution is most conveniently obtained by eliminating n^2 from the first two equations by means of the third, resulting in

$$l^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \tag{53}$$

$$m^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \tag{54}$$

$$n^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \tag{55}$$

Let one of the direction cosines, say n, be held constant while the other two are varied. By (55), the normal and shear stresses then vary according to the equation

$$\tau^2 + \{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_2)\}^2 = \tfrac{1}{4}(\sigma_1 - \sigma_2)^2 + n^2(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) \tag{56}$$

In the stress plane, σ and τ therefore lie on a circle whose center is on the σ axis at a distance $\frac{1}{2}(\sigma_1 + \sigma_2)$ from the origin. The square of the radius of the circle is given by the right-hand side of (56). The radius varies from $\frac{1}{2}(\sigma_1 - \sigma_2)$ for $n = 0$ to $\frac{1}{2}(\sigma_1 + \sigma_2) - \sigma_3$ for $n = 1$.

In Fig. 1.13, the points A, B, C with coordinates $(\sigma_1, 0)$, $(\sigma_2, 0)$, $(\sigma_3, 0)$ are the principal points of the Mohr diagram. The centers of the segments AB, BC, and CA are denoted by the points P, Q, and R. The upper semicircle drawn on the diameter AB corresponds to $n = 0$. As n increases from 0 to 1, the radius of the semicircle varies from PB to PC. Similarly, the upper semicircles with BC and CA as diameters correspond to $l = 0$ and $m = 0$ respectively. For constant values of l, (53) defines a family of circles having the equation

$$\tau^2 + \{\sigma - \tfrac{1}{2}(\sigma_2 + \sigma_3)\}^2 = \tfrac{1}{4}(\sigma_2 - \sigma_3)^2 + l^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) \tag{57}$$

The center of these circles is at Q, while the radius varies from QB for $l = 0$ to QA for $l = 1$. Finally, considering constant values of m, we have the family of circles

$$\tau^2 + \{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_3)\}^2 = \tfrac{1}{4}(\sigma_1 - \sigma_3)^2 + m^2(\sigma_1 - \sigma_2)(\sigma_3 - \sigma_2) \tag{58}$$

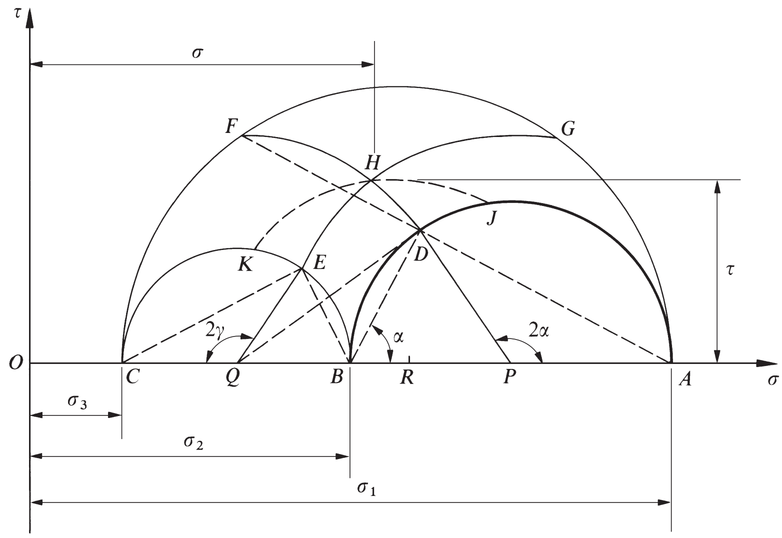


Figure 1.13 Mohr's representation of stress in three dimensions.

with the center at R, and the radius decreasing from RC for $m = 0$ to RB for $m = 1$. For arbitrary values of (l, m, n), the state of stress will correspond to a point in the space between the three semicircles drawn on the diameters AB, BC, and CA.

To find the values of σ and τ across any given plane, let $\alpha = \cos^{-1} l$ and $\gamma = \cos^{-1} n$ be the angles made by the normal to the plane with the directions of σ_1 and σ_3 respectively. Set off angles APD and CQE equal to 2α and 2γ respectively, by drawing the radii PD and QE to the appropriate semicircles. The circular arcs DHF and EHG, drawn with centers Q and P respectively, intersect one another at H giving the required stress point.† If the lines AD and CE are produced, they will meet the outermost semicircle at F and G respectively. Since the angle ABD is equal to α, and $BD = (\sigma_1 - \sigma_2)\cos\alpha$, the triangle BDQ furnishes

$$\begin{aligned} QD^2 &= QB^2 + BD^2 + 2QB \cdot BD \cos\alpha \\ &= \tfrac{1}{4}(\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)\cos^2\alpha \end{aligned}$$

Hence QD is identical to the radius of the circle (57) corresponding to the given value of l. Similarly, the radius PE is equal to that of the circle (56) corresponding to the given value of n. This completes the proof of the construction for the stress point H. It can be shown that the circular arc drawn through H with center at R cuts the semicircles on AB and BC at J and K respectively, where BJ and BK are each inclined at an angle $\beta = \cos^{-1} m$ to the vertical through B.

† Numerical examples have been given by J. M. Alexander, *Strength of Materials*, Chap. 4, Ellis Horwood Limited, Chichester, U.K. (1981).

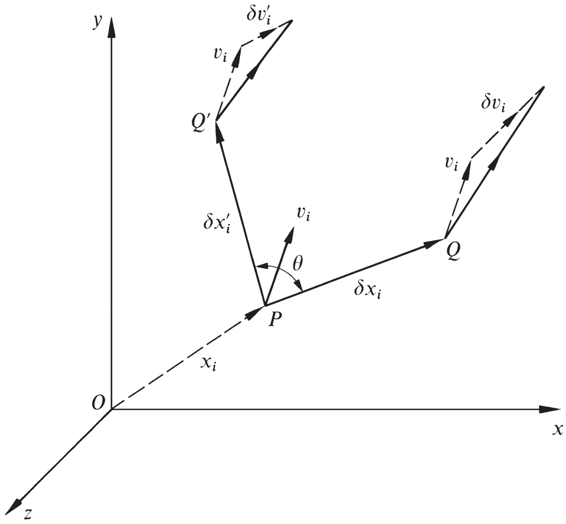


Figure 1.14 Instantaneous velocities of three neighboring particles in a deforming region.

The semicircles with centers P, Q, R are in fact one-half of the two-dimensional Mohr circles for the planes perpendicular to the directions of σ_3, σ_1, σ_2 respectively. Considering the first semicircle, the coordinates of any point such as D are easily shown to be those given by (45) and (46) with the principal axes taken as the axes of reference. For three-dimensional stress states, there is no graphical construction for finding the principal stresses and their directions from given components of the stress. When one of the axes of reference coincides with a principal axis, the problem of finding the remaining principal stresses and their directions is essentially two-dimensional in character.

1.5 Analysis of Strain Rate

(i) ***Rates of deformation and rotation*** A body is said to be deformed or strained when changes occur in the relative positions of the particles forming the body. The instantaneous rate of straining at any point of the body is specified by the velocity field in the neighborhood of this point. Let v_i denote the components (u, v, w) of the velocity of a typical particle P whose instantaneous coordinates are denoted by x_i (Fig. 1.14). Consider a neighboring particle Q situated at an infinitesimal distance from P, the coordinates of Q being $x_i + \delta x_i$. Then the relative velocity of Q with respect to P is given by

$$\delta v_i = \frac{\partial v_i}{\partial x_j} \delta x_j$$

which is equivalent to three equations corresponding to the three components (δu, δv, δw) of the relative velocity. The velocity gradient tensor $\partial v_i / \partial x_j$ may be regarded as the sum of its symmetric part $\dot{\varepsilon}_{ij}$ and antisymmetric part ω_{ij}, where

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{59}$$

Evidently, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ji}$ and $\omega_{ij} = -\omega_{ji}$, indicating the properties of symmetry and antisymmetry of the respective tensors. The expression for the relative velocity therefore becomes

$$\delta v_i = \dot{\varepsilon}_{ij}\,\delta x_j + \omega_{ij}\,\delta x_j \tag{60}$$

To obtain the physical significance of the decomposed parts of the relative velocity, let $\omega_{21} = -\omega_{12} = \omega_z$, $\omega_{32} = -\omega_{23} = \omega_x$, and $\omega_{13} = -\omega_{31} = \omega_y$, the remaining components of ω_{ij} being identically zero. The second equation of (59) then gives

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{61}$$

It follows from above that the quantities ω_x, ω_y, ω_z form the components of the vector†

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\mathrm{curl}\,\mathbf{v}$$

where $\mathbf{v}$ is the velocity vector of the particle P. The components of the relative velocity given by the second term on the right-hand side of (60) are $\omega_y\,\delta z - \omega_z\,\delta y$, $\omega_z\,\delta x - \omega_x\,\delta z$, $\omega_x\,\delta y - \omega_y\,\delta x$. They form the components of the vector product $\boldsymbol{\omega} \times \delta\mathbf{s}$, where $\delta\mathbf{s}$ denotes the infinitesimal vector PQ. The second part of the relative velocity therefore corresponds to an instantaneous rigid body rotation of the neighborhood of P with an angular velocity $\boldsymbol{\omega}$. The antisymmetric tensor ω_{ij} is known as the *spin tensor*. The relationship between the tensor ω_{ij} and the associated *spin vector* ω_k may be written as

$$\omega_{ij} = -e_{ijk}\omega_k \qquad 2\omega_k = -e_{kij}\omega_{ij} = e_{kij}\frac{\partial v_j}{\partial x_i} \tag{62}$$

where e_{ijk} is the *permutation symbol* whose value is $+1$ or -1 according to whether i, j, k form an even or odd permutation‡ of 1, 2, 3. When two of the suffixes i, j, k are equal, e_{ijk} is identically zero. It follows from the definition that

$$e_{ijk} = e_{jki} = e_{kij} = -e_{ikj} = -e_{kji} = -e_{jik}$$

If the neighborhood of P undergoes an instantaneous deformation, the first term on the right-hand side of (60) must be nonzero. The symmetric tensor $\dot{\varepsilon}_{ij}$ is therefore called the *rate of deformation* or the true strain rate at P at the instant under

† The direction of $\boldsymbol{\omega}$ is parallel to the direction of advancement of a right-handed screw turning in the same sense as that of the rigid body rotation.

‡ This means that $e_{123} = e_{231} = e_{312} = 1$ and $e_{213} = e_{132} = e_{321} = -1$, while all other components are zero. The permutation tensor has the important property $e_{mij}e_{mkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, which may be obtained by eliminating ω_k between the two relations (62), and comparing the result for ω_{ij} with that given by (59).

consideration. The rectangular components of the strain rate are

$$\begin{aligned}
&\dot{\varepsilon}_x = \frac{\partial u}{\partial x} && \dot{\varepsilon}_y = \frac{\partial v}{\partial y} && \dot{\varepsilon}_z = \frac{\partial w}{\partial z} \\
&\dot{\gamma}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) && \dot{\gamma}_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) && \dot{\gamma}_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned} \tag{63}$$

The first three are the normal components and the last three are the shear components of the strain rate. When the total deformation is small, the expressions on the right-hand sides of (63) give the components of the strain itself with u, v, w regarded as the components of the displacement of the particle.†

For the mechanical interpretation of the components of the tensor $\dot{\varepsilon}_{ij}$, consider first the rate of change of the instantaneous length δs of the material line element PQ. The square of this line element is $\delta s^2 = \delta x_i \, \delta x_i$, which gives

$$\delta s(\delta s)^{\cdot} = \delta v_i \, \delta x_i = \frac{\partial v_i}{\partial x_j} \delta x_i \, \delta x_j = \frac{\partial v_j}{\partial x_i} \delta x_i \, \delta x_j$$

where the dot denotes the material derivative, specifying the rate of change following the motion of the particles. Using the expression for $\dot{\varepsilon}_{ij}$ given by (59), the above relation may be written as

$$\delta s(\delta s)^{\cdot} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \delta x_i \, \delta x_j = \dot{\varepsilon}_{ij} \, \delta x_i \, \delta x_j$$

If the unit vector in the direction PQ is denoted by l_i, then $\delta x_i = l_i \, \delta s$. The ratio $(\delta s)^{\cdot}/\delta s$, called the *rate of extension* $\dot{\varepsilon}$ in the direction PQ, is then obtained as

$$\begin{aligned}
\dot{\varepsilon} &= l_i l_j \dot{\varepsilon}_{ij} \\
&= l^2 \dot{\varepsilon}_x + m^2 \dot{\varepsilon}_y + n^2 \dot{\varepsilon}_z + 2lm\dot{\gamma}_{xy} + 2mn\dot{\gamma}_{yz} + 2nl\dot{\gamma}_{zx}
\end{aligned} \tag{64}$$

where (l, m, n) are the direction cosines of PQ. It follows that the components $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$ are the rates of extension at the particle P in the coordinate directions.

Consider, now, a second material line element PQ' emanating from P, the instantaneous coordinates of Q' being $x_i + \delta x_i'$. Then the velocity of Q' relative to that of P is

$$\delta v_i' = \frac{\partial v_i}{\partial x_j} \, \delta x_j'$$

The scalar product of the infinitesimal vectors PQ and PQ' is $\delta x_i \, \delta x_i'$, and its material rate of change is

$$(\delta x_i \, \delta x_i')^{\cdot} = \delta x_i \, \delta v_i' + \delta x_i' \, \delta v_i = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \delta x_i \, \delta x_j' = 2\dot{\varepsilon}_{ij} \, \delta x_i \, \delta x_j'$$

† The expressions for the strain rates and the components of spin in cylindrical and spherical coordinates are given in App. B.

Let the instantaneous length of the line element PQ' be denoted by $\delta s'$, and the rate of extension in this direction by $\dot{\varepsilon}'$. If the included angle between PQ and PQ' is denoted by θ, the scalar product $\delta x_i\,\delta x_i'$ is equal to $\delta s\,\delta s'\cos\theta$. The above equation therefore becomes

$$[(\dot{\varepsilon}+\dot{\varepsilon}')\cos\theta-\dot{\theta}\sin\theta]\delta s\,\delta s'=2\dot{\varepsilon}_{ij}\,\delta x_i\,\delta x_j' \qquad (65)$$

If the neighborhood of P undergoes an instantaneous rigid body motion, the material triangle PQQ' retains its shape following the motion, giving $\dot{\varepsilon}=\dot{\varepsilon}'=\dot{\theta}=0$. It follows from (65) that $\dot{\varepsilon}_{ij}$ then vanishes identically as expected. The rate at which an instantaneous right angle between a pair of material line elements decreases is twice the rate of shear, denoted by $\dot{\gamma}$. Setting $\delta x_i=l_i\,\delta s$, $\delta x_j'=l_j'\,\delta s'$, and $\theta=\pi/2$ in (65), the *rate of shear* associated with the directions l_i and l_i' is obtained as

$$\begin{aligned}\dot{\gamma}=l_il_j'\dot{\varepsilon}_{ij}&=ll'\dot{\varepsilon}_x+mm'\dot{\varepsilon}_y+nn'\dot{\varepsilon}_z+(lm'+ml')\dot{\gamma}_{xy}\\&\quad+(mn'+m'n)\dot{\gamma}_{yz}+(nl'+ln')\dot{\gamma}_{zx}\end{aligned} \qquad (66)$$

where (l', m', n') are the direction cosines of PQ'. It follows from (66) that $\dot{\gamma}_{xy}$, $\dot{\gamma}_{yz}$, and $\dot{\gamma}_{zx}$ are the rates of shear associated with the appropriate coordinate directions. In the engineering literature, the shear rate is taken as equal to the rate of decrease of the angle formed by an instantaneous pair of orthogonal material line elements. The engineering components of the rate of shear are therefore twice the corresponding tensor components. During a finite deformation, the engineering shear strain associated with a pair of orthogonal line elements in the unstrained state is the tangent of the angle by which the right angle decreases.

(ii) ***Principal strain rates*** The relative velocity of Q with respect to P, corresponding to pure deformation in the neighborhood of P, may be resolved into a component along PQ and a component perpendicular to PQ. These resolved components are equal to $\dot{\varepsilon}\,\delta s$ and $\dot{\gamma}\,\delta s$ respectively, as may be seen from (60), (64), and (66), the unit vector l_i' being considered in the appropriate perpendicular direction. The direction PQ represents a principal direction of the rate of deformation, if the relative velocity of pure deformation is directed along PQ. In this case $\dot{\gamma}=0$, and $\dot{\varepsilon}_{ij}\,\delta x_j$ is equal to $\dot{\varepsilon}\,\delta x_i$, where $\delta x_i=l_i\,\delta s$. Hence

$$\dot{\varepsilon}_{ij}l_j=\dot{\varepsilon}l_i \qquad \text{or} \qquad (\dot{\varepsilon}_{ij}-\dot{\varepsilon}\,\delta_{ij})l_j=0 \qquad (67)$$

This consists of three scalar equations, analogous to (22), for the components of the unit vector l_j. Equating to zero the determinant of the coefficients formed by the expression in the parenthesis of (67), we obtain the cubic equation

$$\dot{\varepsilon}^3-N_1\dot{\varepsilon}^2-N_2\dot{\varepsilon}-N_3=0$$

whose roots are the *principal strain rates* $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$, $\dot{\varepsilon}_3$. The coefficients N_1, N_2, N_3 are the basic invariants of the strain rate tensor, their expressions in terms of the components of $\dot{\varepsilon}_{ij}$ being

$$N_1=\dot{\varepsilon}_{ii} \qquad N_2=\tfrac{1}{2}(\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}-\dot{\varepsilon}_{ii}\dot{\varepsilon}_{jj}) \qquad N_3=|\dot{\varepsilon}_{ij}| \qquad (68)$$

where the last expression denotes the determinant of the matrix of the tensor $\dot{\varepsilon}_{ij}$. When the principal values of the strain rate are distinct, each principal strain rate is associated with a unique principal direction. The three principal directions are mutually orthogonal and are known as the *principal axes* of the strain rate. In analogy with (15), a deviatoric strain rate $\dot{e}_{ij}$ is defined as

$$\dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_0 \delta_{ij} = \dot{\varepsilon}_{ij} - \tfrac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij} \tag{69}$$

The principal axes of $\dot{e}_{ij}$ are the same as those of $\dot{\varepsilon}_{ij}$. The principal components of the deviatoric strain rate are obtained by subtracting the mean extension rate $\dot{\varepsilon}_0$ from the corresponding principal strain rates. The principal shear rates have the values

$$\tfrac{1}{2}|\dot{\varepsilon}_1 - \dot{\varepsilon}_2| \qquad \tfrac{1}{2}|\dot{\varepsilon}_2 - \dot{\varepsilon}_3| \qquad \tfrac{1}{2}|\dot{\varepsilon}_3 - \dot{\varepsilon}_1|$$

These are the maximum values of the magnitude of $\dot{\gamma}$ at the considered particle. Each principal shear rate is associated with directions which bisect the angles between the corresponding pair of principal axes of the rate of deformation.

The first invariant N_1 is equal to the rate of change of volume per unit volume in the neighborhood of a typical particle P. This may be shown by considering a small rectangular parallelepiped at P with its edges parallel to the principal axes. If the instantaneous lengths of the edges are denoted by δa, δb, δc, the rates at which these lengths change following the motion are $\dot{\varepsilon}_1\,\delta a$, $\dot{\varepsilon}_2\,\delta b$, $\dot{\varepsilon}_3\,\delta c$ respectively. The instantaneous volume $\delta a\;\delta b\;\delta c$ of the parallelepiped therefore changes at the rate $(\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3)\delta a\,\delta b\,\delta c$. If the local density of the material is denoted by ρ, the mass of the parallelepiped is $\rho\,\delta a\,\delta b\,\delta c$, which remains constant following the motion. Setting the rate of change of this mass to zero, we have

$$\dot{\rho}/\rho = -(\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) = -(\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z) = -\dot{\varepsilon}_{ii}$$

Expressing the rates of extension in terms of the velocity gradients, the above relation can be written as

$$\dot{\rho} + \rho\frac{\partial v_i}{\partial x_i} = \dot{\rho} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{70}$$

For an incompressible material, the density remains constant, and consequently $\dot{\varepsilon}_{ii}$ must vanish. In this case, the components of the deviatoric strain rate are identical to those of the actual strain rate.

Consider the situation where the principal axes of the strain rate remain fixed with respect to an element as it continues to deform. The axes of reference are assumed to take part in the rotation of the element so that they are parallel to the principal axes at each stage. Let x, y, z denote the coordinates of the center of the element at any instant t, measured in the directions of $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$, $\dot{\varepsilon}_3$ respectively. If the initial coordinates x_0, y_0, z_0 are taken as independent space variables, which do not change following the motion, the material rate of change of each variable is given by its partial derivative with respect to t. The first principal strain rate may therefore be written as

$$\frac{\partial \varepsilon_1}{\partial t} = \frac{\partial u/\partial x_0}{\partial x/\partial x_0} = \frac{(\partial/\partial t)(\partial x/\partial x_0)}{\partial x/\partial x_0}$$

Similar expressions may be written down for the other two principal strain rates. These equations are immediately integrated to give the total principal strains

$$\varepsilon_1 = \ln\left(\frac{\partial x}{\partial x_0}\right) \qquad \varepsilon_2 = \ln\left(\frac{\partial y}{\partial y_0}\right) \qquad \varepsilon_3 = \ln\left(\frac{\partial z}{\partial z_0}\right)$$

which are the logarithms of the ratios of the final and initial lengths of the material line elements along the principal axes. When the principal axes of the strain rate rotate with respect to the element, the principal components of the successive strain increments cannot be interpreted as increments of principal strains.†

(iii) ***Instantaneous plane strain*** The instantaneous state of strain is called plane if one of the principal strain rates vanishes at each point of the deforming body. If the z axis is taken along this principal axis, we have $\dot{\varepsilon}_z = \dot{\gamma}_{yz} = \dot{\gamma}_{zx} = 0$ for an instantaneous plane strain condition. The velocity field therefore has the form

$$u = u(x, y) \qquad v = v(x, y) \qquad w = 0$$

and the nonzero strain rates $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, and $\dot{\gamma}_{xy}$ are all independent of z. A typical material line element PQ in the plane $z = \text{const}$ instantaneously extends and rotates in the same plane. A part of the instantaneous rotation corresponds to a local rigid body spin of the material about an axis through P with an angular velocity $\omega_z = \omega$, which is reckoned positive when the rotation is counterclockwise. The condition of compatibility of the components of strain rate is

$$\frac{\partial^2 \dot{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \dot{\varepsilon}_y}{\partial x^2} = 2\frac{\partial^2 \dot{\gamma}_{xy}}{\partial x\, \partial y} \tag{71}$$

which is readily verified on direct substitution from (63). It is a consequence of the fact that three strain-rate components are defined by two velocity components.‡

Let ϕ denote the counterclockwise orientation of a line element PQ with respect to the x axis. Then the instantaneous coordinate differences between the particles P and Q are $\delta x = \delta s \cos\phi$ and $\delta y = \delta s \sin\phi$, where δs is the length of the element PQ. Let (δu^*, δv^*) denote the relative velocity of Q with respect to P corresponding to pure deformation. Then

$$\delta u^* = (\dot{\varepsilon}_x \cos\phi + \dot{\gamma}_{xy} \sin\phi)\delta s$$
$$\delta v^* = (\dot{\gamma}_{xy} \cos\phi + \dot{\varepsilon}_y \sin\phi)\delta s$$

† For a discussion of finite homogeneous strains, in which all straight lines remain straight during straining, see J. C. Jaeger, *Elasticity, Fracture, and Flow*, Chap. 1, Methuen and Company, London (1969).

‡ For a three-dimensional velocity field, there are six equations of compatibility, three of which are of type (71). They are necessary and sufficient conditions for the existence of single-valued velocities. See, for example, L. E. Malvern, *Introduction to Mechanics of a Continuous Medium*, p. 189, Prentice-Hall, Englewood Cliffs, N.J. (1969).

in view of the first term on the right-hand side of (60). The resolved component of this relative velocity in the direction PQ is equal to $\dot{\varepsilon}\,\delta s$, where $\dot{\varepsilon}$ is the rate of extension along PQ. Hence

$$\begin{aligned}\dot{\varepsilon} &= \frac{\delta u^* \cos\phi + \delta v^* \sin\phi}{\delta s} \\ &= \dot{\varepsilon}\cos^2\!\phi + \dot{\varepsilon}_y \sin^2\!\phi + 2\dot{\gamma}_{xy}\sin\phi\cos\phi \end{aligned} \tag{72}$$

The expression for $\dot{\varepsilon}$ is also obtained from (65) by setting $l = \cos\phi$, $m = \sin\phi$, $n = 0$. The shear strain rate $\dot{\gamma}$ associated with the directions ϕ and $\pi/2 + \phi$ is given by the resolved component of the relative velocity $(\delta u^*, \delta v^*)$ in the direction perpendicular to PQ. Thus

$$\begin{aligned}\dot{\gamma} &= \frac{-\delta u^* \sin\phi + \delta v^* \cos\phi}{\delta s} \\ &= -(\dot{\varepsilon}_x - \dot{\varepsilon}_y)\sin\phi\cos\phi + \dot{\gamma}_{xy}(\cos^2\!\phi - \sin^2\theta) \end{aligned} \tag{73}$$

It follows from the nature of the derivation that $\dot{\gamma}$ is the counterclockwise angular velocity of PQ corresponding to pure deformation of the neighborhood of P. An element PR inclined at $\pi/2 + \phi$ to the x axis has a clockwise angular velocity equal to $\dot{\gamma}$. The right angle between PQ and PR therefore decreases at the rate $2\dot{\gamma}$, which is the engineering shear rate at P associated with these directions. It follows that the total angular velocities of PQ and PR are $\omega + \dot{\gamma}$ and $\omega - \dot{\gamma}$ respectively measured in the counterclockwise sense.

The direction ϕ corresponds to a principal direction of the strain rate if the corresponding shear rate vanishes. Since $d\dot{\varepsilon}/d\phi = 2\dot{\gamma}$ in view of (72) and (73), the longitudinal strain rate has a stationary value in the principal direction. The condition $\dot{\gamma} = 0$ gives

$$\tan 2\phi = \frac{2\dot{\gamma}_{xy}}{\dot{\varepsilon}_x - \dot{\varepsilon}_y} \tag{74}$$

which defines two mutually perpendicular directions representing the principal axes in the xy plane. The principal axes of stress and strain rate coincide if the ratios on the right-hand sides of (47) and (74) are equal to one another. The principal strain rates are expressed as

$$\dot{\varepsilon}_1, \dot{\varepsilon}_2 = \tfrac{1}{2}(\dot{\varepsilon}_x + \dot{\varepsilon}_y) \pm \tfrac{1}{2}\sqrt{(\dot{\varepsilon}_x - \dot{\varepsilon}_y)^2 + 4\dot{\gamma}_{xy}^2} \tag{75}$$

The second term on the right-hand side represents the maximum rate of shear in the plane of the instantaneous motion. Choosing the principal axes in this plane as the new axes of reference, the rate of extension $\dot{\varepsilon}$ and the total angular velocity $\dot{\phi}$ of a material line element PQ may be written from (72) and (73) as

$$\begin{aligned}\dot{\varepsilon} &= \tfrac{1}{2}(\dot{\varepsilon}_1 + \dot{\varepsilon}_2) + \tfrac{1}{2}(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)\cos 2\phi \\ \phi &= \qquad \omega \qquad - \tfrac{1}{2}(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)\sin 2\phi \end{aligned} \tag{76}$$

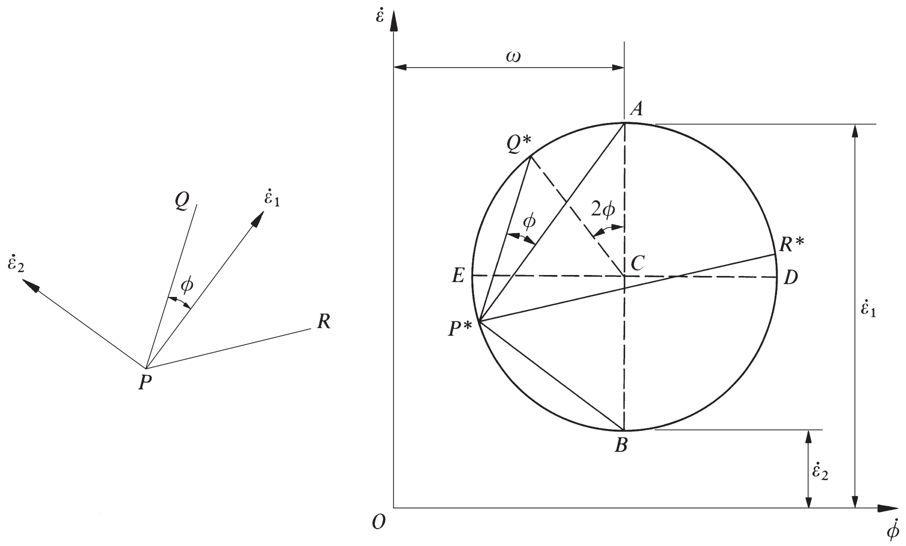


Figure 1.15 Geometrical representation of the rates of extension and rotation in plane strain.

where ω is the component of spin at P. It follows that $\dot{\varepsilon}$ and $\dot{\phi}$ can be represented by a point whose locus is a circle with parametric equations (76). If $\dot{\varepsilon}$ is taken as the ordinate and $\dot{\phi}$ as the abscissa (Fig. 1.15), the coordinates of the center C of the circle are ω, $\frac{1}{2}(\dot{\varepsilon}_1 + \dot{\varepsilon}_2)$, and the radius of the circle is $\frac{1}{2}(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)$, where $\dot{\varepsilon}_1 > \dot{\varepsilon}_2$. The highest and lowest points of the circle, denoted by A and B, represent the maximum and minimum rates of extension, together with an angular velocity equal to ω.

Let a point Q^* on the circle correspond to the direction ϕ with respect to the first principal axis. Then in view of (76), the angle ACQ^* is equal to 2ϕ. The line Q^*P^* drawn parallel to the direction PQ meets the circle again at P^* which may be regarded as the pole of the circle. Since the peripheral angle AP^*Q^* is equal to ϕ, the line P^*A is parallel to the first principal direction at P. To find the point on the circle corresponding to any given direction in the plane of motion, it is only necessary to draw a line in this direction through P^* and locate its second intersection with the circle. Let P^*R^* be drawn parallel to some given direction PR through P. Then the rate of change of the material angle QPR is equal to the difference between the abscissas of the corresponding points R^* and Q^* on the circle. For the considered orientation of the line elements PQ and PR, the angle between them instantaneously decreases. The difference between the abscissas of the points D and E, which correspond to the maximum shear directions at P, is greater than that of any other pair of points on the circle. It follows, therefore, that the right angle formed by the material line elements in the maximum shear directions changes at a rate which is numerically greater than that for any other material angle in the plane of motion.†

† The above construction is due to W. Prager, *Introduction to Mechanics of Continua*, p. 69, Ginn and Company, Boston (1961). Mohr's construction for the rates of extension and shear associated with any angle ϕ is identical to that for the normal and shear stresses.

(iv) ***Equilibrium and virtual work*** Consider a mass of material occupying a finite volume V and bounded by a surface S at a generic instant t. The material is in equilibrium under surface forces distributed over S, and body forces (such as gravitational and centrifugal forces) distributed throughout V. The body force acting on a typical volume element dV is equal to $\rho g_j\, dV$, where g_j denotes the body force per unit mass and ρ the current density. The force T_j acting on a typical surface element dS, specified by its exterior unit normal l_i, is equal to $\sigma_{ij} l_i\, dS$. The condition of force equilibrium requires the resultant of these forces to vanish, leading to

$$\int \sigma_{ij} l_i\, dS + \int \rho g_j\, dV = 0$$

where the surface integral extends over S and the volume integral over V. Using Green's theorem, the surface integral can be transformed into a volume integral, reducing the above expression to

$$\int \left(\frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j \right) dV = 0$$

The vanishing of the above integral requires that the expression in the parenthesis must vanish identically. The equilibrium condition therefore becomes

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j = 0 \tag{77}$$

which is equivalent to three equations corresponding to the three coordinate directions. In view of the symmetry of the stress tensor, (77) also ensures that the resultant moment of the surface and body forces is identically zero.†

Equation (77) must be satisfied throughout the interior of the body. At the boundary of the body, the force T_j per unit area acting on a typical surface element must be equal to the stress vector across this element. The boundary condition may therefore be written as

$$T_j = l_i \sigma_{ij} \tag{78}$$

where l_i is the unit vector along the exterior normal to the surface at the considered point. In general, (77) and (78) must be supplemented by other equations to determine the stress components uniquely.

Consider, now, a continuous velocity field v_j, which is chosen independently of an equilibrium distribution of stress σ_{ij}. The rate of work done by the distribution of surface traction T_j (in the absence of body forces) is

$$\int T_j v_j\, dS = \int l_i \sigma_{ij} v_j\, dS = \int \frac{\partial}{\partial x_i}(\sigma_{ij} v_j) dV = \int \sigma_{ij} \frac{\partial v_j}{\partial x_i} dV$$

† See, for example, W. Prager, *Introduction to Mechanics of Continua*, p. 46, Ginn and Company, Boston (1961).

in view of (77). The transformation of the surface integral into the volume integral follows from Gauss' divergence theorem (or Green's theorem) applied to the vector $\sigma_{ij}v_j$. Since

$$\sigma_{ij}\frac{\partial v_j}{\partial x_i} = \sigma_{ji}\frac{\partial v_j}{\partial x_i} = \sigma_{ij}\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\sigma_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \sigma_{ij}\dot{\varepsilon}_{ij}$$

by the symmetry of the stress tensor and the interchangeability of dummy suffixes, we obtain the principle of virtual work in the form

$$\int T_j v_j \, dS = \int \sigma_{ij}\dot{\varepsilon}_{ij} \, dV \tag{79}$$

where $\dot{\varepsilon}_{ij}$ is the rate of deformation associated with the velocity v_j. Equation (79) states that the rate of work done by the external forces on any virtual velocity field is equal to the rate of dissipation of internal energy. If the velocity field is discontinuous, the energy dissipated due to shearing across the discontinuities must be included on the right-hand side of (79).

1.6 Concepts of Stress Rate

(i) ***Objective stress rates*** The rate of deformation of a solid, for a given state of stress, is generally a function of the instantaneous rate of change of the stress. The stress rate tensor used in this relation, known as the constitutive relation, must be defined in such a way that it vanishes in the event of an instantaneous rigid body rotation. Such a stress rate is called an *objective stress rate.* From the physical standpoint, it is natural to consider the rate of change of the stress referred to a set of axes that participates in the instantaneous rotation of a typical element. Although the stress components with respect to a fixed coordinate system are changed by the rotation of the element, the components with respect to the rotating system remain unaffected.

Consider two sets of rectangular axes x_i and x_i', which have a common origin O, and which are coincident at an instant t. During a small interval of time dt, the first set of axes is assumed to remain fixed, while the second set of axes takes part in the rigid-body rotation of the given element. An infinitesimal vector PQ drawn in the element from its center P is denoted by δx_j and $\delta x_j'$ with respect to the two coordinate systems at the instant $t + dt$. The difference $\delta x_j - \delta x_j'$ is equal to $\delta v_j \, dt$, where δv_j denotes the relative velocity of rotation of Q with respect to P at the instant t. Recalling that $\delta v_j = \omega_{ji} \, \delta x_i'$, where ω_{ij} denotes the rate of rotation of the neighborhood of P, we have

$$\delta x_j = \delta x_j' + (\omega_{ji} \, dt)\delta x_i' = (\delta_{ij} - \omega_{ij} \, dt)\delta x_i' \tag{80}$$

The angle which the x_j axis makes with the x_i' axis instantaneously changes at the rate ω_{ij} when $i \neq j$. Let σ_{ij} denote the true (or Cauchy) stress at the particle P when $dt = 0$. The material rates of change of the stress referred to the fixed and the rotating axes are denoted by $\dot{\sigma}_{ij}$ and $\overset{\circ}{\sigma}_{ij}$ respectively. At the time $t + dt$, the primed and the unprimed stress components become $\sigma_{ij} + \overset{\circ}{\sigma}_{ij} \, dt$ and $\sigma_{ij} + \dot{\sigma}_{ij} \, dt$ respectively. In view of (20), expressed in the infinitesimal form, a_{ij} is given by the expression in the parenthesis of (80). The transformed stress tensor may therefore be written from (21) as

$$\sigma_{ij} + \mathring{\sigma}_{ij}\,dt = (\delta_{ik} - \omega_{ik}\,dt)(\delta_{jl} - \omega_{jl}\,dt)(\sigma_{kl} + \dot{\sigma}_{kl}\,dt)$$

Neglecting the terms containing squares and cubes of dt, and using the symmetry of the stress tensor, the relationship between $\dot{\sigma}_{ij}$ and $\mathring{\sigma}_{ij}$, which is due to Jaumann,† is obtained as

$$\mathring{\sigma}_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\omega_{jk} - \sigma_{jk}\omega_{ik} \tag{81}$$

The quantity $\mathring{\sigma}_{ij}$ may be regarded as the rigid body derivative of the true stress at the instant under consideration. It can be significantly different from the material derivative $\dot{\sigma}_{ij}$ whenever the rate of rotation is important.

Consider now the scalar triple product $p_{ij}\sigma_{jk}\omega_{ik}$ where p_{ij} is an arbitrary tensor having the same principal axes as those of σ_{ij}. Since this expression is an invariant, it is convenient to take the common principal axes as the axes of reference. The tensor $p_{ij}\sigma_{jk}$ then corresponds to a diagonal matrix, while the diagonal components of ω_{ik} are always zero. Consequently, the scalar product of these two tensors is identically zero. It is similarly shown that the triple product $p_{ij}\sigma_{ik}\omega_{jk}$ also vanishes. It follows, therefore, from (81) that

$$p_{ij}\mathring{\sigma}_{ij} = p_{ij}\dot{\sigma}_{ij}$$

Thus $\mathring{\sigma}_{ij}$ and $\dot{\sigma}_{ij}$ have the same scalar product with any second-order tensor whose principal axes coincide with those of σ_{ij}. This property has an important consequence in the theory of plasticity.

Various other definitions of the stress rate, vanishing for an instantaneous rigid-body rotation, have been proposed in the literature. An objective stress rate sometimes used in the literature to replace $\mathring{\sigma}_{ij}$ is the material rate of change of the modified stress tensor

$$\tau_{ij} = \frac{\partial a_i}{\partial x_k}\frac{\partial a_j}{\partial x_l}\sigma_{kl}$$

where a_i are the initial coordinates of the particle which is currently at x_i. The material derivative of τ_{ij}, when the initial state coincides with that at the generic instant t, is easily shown to be‡

$$\dot{\tau}_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\frac{\partial v_j}{\partial x_k} - \sigma_{jk}\frac{\partial v_i}{\partial x_k} \tag{82}$$

† G. Jaumann, *Sitz. Akad. Wiss. Wien* **120**: 385 (1911). The result has been rederived by H. Fromm, *Ing.-Arch.*, **4**: 452 (1933); S. Zaremba, *Mem. Sci. Math.,* No. 82, Paris (1937); W. Noll, *J. Rat. Mech. Anal.*, **4**: 3 (1955); R. Hill, *J. Mech. Phys. Solids*, **7**: 209 (1959). See also W. Prager, *Q. Appl. Math.*, **18**: 403 (1961); A. J. M. Spencer, *Continuum Mechanics*, Longman, London (1980).

‡ The derivation of (82) is very similar to that of (87). The tensor $(\rho^{\mathrm{o}}/\rho)\tau_{ij}$, where ρ^{o} and ρ are the initial and final densities of the material, is called the Kirchhoff stress. The material rate of change of the Kirchhoff stress at the initial state is $\dot{\tau}_{ij} + \dot{\varepsilon}_{kk}\sigma_{ij}$.

It follows from (81) and (82) that $\dot{\tau}_{ij}$ differs from $\mathring{\sigma}_{ij}$ by the quantity $\sigma_{ik}\dot{\varepsilon}_{jk} + \sigma_{jk}\dot{\varepsilon}_{ik}$, which is appreciable when the rate of deformation becomes significant. It is important to note that $\dot{\tau}_{ij}$ and $\dot{\sigma}_{ij}$ do not have the same scalar product with any tensor p_{ij} which is coaxial with σ_{ij}.

(ii) ***Nominal stress rate*** Through a typical particle P in the deforming material, consider a small surface element represented by the vector δS_i at any instant t. The magnitude of this vector is the current area δS, and the direction of this vector is that of the normal to the surface in the current state. The coordinates of P are denoted by x_i in the instantaneous state, and by a_i in some initial reference state with respect to a fixed set of rectangular axes. The initial area of the surface element is δS_0, the corresponding vector being denoted by δS_i^{o}. Consider now a material line element PQ emanating from the particle P. If the instantaneous components of the vector PQ are denoted by δx_i, the corresponding components in the initial state are given by

$$\delta a_i = \frac{\partial a_i}{\partial x_j}\delta x_j$$

The volume of the material cylinder, specified by the axial vector PQ and having the given surface element as its base, changes from $\delta S_i^{\mathrm{o}}\,\delta a_i$ in the initial state to $\delta S_i\,\delta x_i$ in the current state. The conservation of mass requires

$$\rho\,\delta S_j\,\delta x_j = \rho_0\,\delta S_i^{\mathrm{o}}\,\delta a_i$$

where ρ_0 and ρ are the initial and current densities of the material at the particle P. Substituting for δa_i, we have

$$\left(\frac{\rho}{\rho_0}\delta S_j - \frac{\partial a_i}{\partial x_j}\delta S_i^{\mathrm{o}}\right)\delta x_j = 0$$

Since this equation must be satisfied for any arbitrary vector δx_j, the expression in the parenthesis must vanish. Hence

$$\frac{\rho}{\rho_0}\delta S_j = \frac{\partial a_i}{\partial x_j}\delta S_i^{\mathrm{o}} \qquad (83)$$

The infinitesimal force δP_j transmitted in the current state may be referred to the surface element in the initial state through a nominal stress tensor t_{ij}. The true stress tensor σ_{ij}, on the other hand, is associated with the surface element in the current state to give the same infinitesimal force. Expressed mathematically,

$$\delta P_j = t_{ij}\,\delta S_i^{\mathrm{o}} = \sigma_{kj}\,\delta S_k \qquad (84)$$

Thus $t_{ij}\,\delta S_0$ is the jth component of the force currently acting on a surface element which was initially perpendicular to the ith axis. Substitution for δS_k from (83) leads to the relationship between t_{ij} and σ_{ij} as

$$t_{ij} = \frac{\rho_0}{\rho}\frac{\partial a_i}{\partial x_k}\sigma_{kj} \qquad (85)$$

This relation shows that the nominal stress tensor t_{ij} is not symmetric. Nevertheless, it is convenient to introduce this tensor for treating the problems of uniqueness and stability. The material derivative of t_{ij} is obtained by applying the operator.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_m \frac{\partial}{x_m}$$

where v_m is the instantaneous velocity of the considered particle. The first term on the right-hand side represents the local part and the second term the convective part of the derivative. Since the initial coordinates do not change following the particle, $da_i/dt = 0$, in view of which the material rate of change of the tensor $\partial a_i/\partial x_k$ is obtained as

$$\frac{d}{dt}\left(\frac{\partial a_i}{\partial x_k}\right) = \frac{\partial}{\partial x_k}\left(\frac{\partial a_i}{\partial t}\right) + v_m \frac{\partial}{\partial x_k}\left(\frac{\partial a_i}{\partial x_m}\right) = -\frac{\partial v_m}{\partial x_k}\frac{\partial a_i}{\partial x_m} \tag{86}$$

Considering the material derivative of (85), and using (70) and (86), it is easily shown that

$$\frac{dt_{ij}}{dt} = \frac{\rho_0}{\rho}\left(\frac{d\sigma_{mj}}{dt} + \sigma_{mj}\frac{\partial v_k}{\partial x_k} - \sigma_{kj}\frac{\partial v_m}{\partial x_k}\right)\frac{\partial a_i}{\partial x_m}$$

where a dummy suffix has been replaced by another. If the initial state is now assumed to coincide with the instantaneous state, $\rho_0 = \rho$, $a_i = x_i$, and consequently $\partial a_i/\partial x_m = \delta_{im}$. Denoting the instantaneous rate of change by a dot as usual, we finally obtain†

$$\dot{t}_{ij} = \dot{\sigma}_{ij} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} - \sigma_{jk}\frac{\partial v_i}{\partial x_k} \tag{87}$$

which relates the *nominal stress rate* $\dot{t}_{ij}$ to the true stress rate $\dot{\sigma}_{ij}$ with respect to a fixed set of rectangular axes. It follows from (84) that when $\dot{t}_{ij}$ vanishes, the force transmitted across the surface element instantaneously remains constant, despite the deformation and the rotation of the element.

(iii) ***Equilibrium equations and boundary conditions*** Let V_0 be the initial volume and S_0 the initial surface of the material which instantaneously fills the volume V with surface S. Denote by l_i° the unit vector along the exterior normal to an initial surface element of area dS_0. The forces currently acting on typical surface and volume elements of the material may be expressed as $t_{ij}l_i^{\circ}\, dS_0$ and $\rho_0 g_j\, dV_0$ respectively. Equating the resultant of these forces over the entire body to zero, we get

$$\int t_{ij}l_i^{\circ}\, dS_0 + \int \rho_0 g_j\, dV_0 = 0$$

† The expression (87) is due to R. Hill, *J. Mech. Phys. Solids*, **6**: 236 (1958).

where the integrands are considered as functions of the initial coordinates a_i and the time t. The transformation by Green's theorem furnishes the result

$$\int \left(\frac{\partial t_{ij}}{\partial a_i} + \rho_0 g_j \right) dV_0 = 0$$

Since this equation holds for any arbitrary region V_0, the integrand must vanish. The equation of equilibrium in terms of the nominal stress therefore becomes

$$\frac{\partial t_{ij}}{\partial a_i} + \rho_0 g_j = 0$$

The material derivative of this equation is simply the partial derivative with respect to t, since the initial coordinates are taken as the space variables. Denoting the material rate of change of the nominal stress by $\dot{t}_{ij}$ when the initial state is assumed as that at the instant t, we obtain

$$\frac{\partial \dot{t}_{ij}}{\partial x_i} + \rho \dot{g}_j = 0 \tag{88}$$

This is the rate equation of equilibrium expressed in its simplest form. Inserting from (87), the equation may be written down in terms of the true stress rate $\dot{\sigma}_{ij}$. When body forces are neglected (as is usual with gravitational forces), the rate equation becomes

$$\frac{\partial \dot{\sigma}_{ij}}{\partial x_i} - \frac{\partial v_i}{\partial x_k} \frac{\partial \sigma_{jk}}{\partial x_i} = 0 \tag{89}$$

This expression is also obtained if we apply the operator d/dt on equation (77). The second term on the left-hand side of (89) represents the effect of the instantaneous motion of the element.

The components of the stress rate must be in equilibrium with the instantaneous rate of change of boundary tractions. Since the future position of a typical surface element is not known in advance, when positional changes are taken into account, it is convenient to express the boundary condition in terms of the traction rate based on the initial configuration. If δP_j denotes the current load vector acting on a surface element of initial area δS_0, then the ratio $\delta P_j / \delta S_0$ as δS_0 tends to zero is the nominal traction F_j. It follows from (84) that

$$F_j = l_i^{\mathrm{o}} t_{ij}$$

If the material rate of change of the nominal traction is denoted by $\dot{F}_j$ when the initial state is taken as that at the instant considered ($l_i^{\mathrm{o}} = l_i$), then

$$\dot{F}_j = l_i \dot{t}_{ij} \tag{90}$$

A different situation arises when a part of the boundary surface is subjected to a uniform normal pressure p through an inviscid fluid. In this case, the infinitesimal load vector on the surface element is

$$\delta P_j = -p\, \delta S_j = -p \left(\frac{\rho_0}{\rho} \frac{\partial a_i}{\partial x_j} \right) l_i^{\mathrm{o}}\, \delta S_0$$

in view of (83). The load per unit initial area of the element therefore becomes

$$F_j = -p\left(\frac{\rho_0}{\rho}\frac{\partial a_i}{\partial x_j}\right) l_i^{\text{o}} \tag{91}$$

Taking the material derivative of the nominal traction (91), and using the relations (70) and (86), we obtain

$$\frac{dF_j}{dt} = \frac{\rho_0}{\rho}\left\{-\left(\dot{p} + p\frac{\partial v_k}{\partial x_k}\right)\frac{\partial a_i}{\partial x_j} + p\frac{\partial v_k}{\partial x_j}\frac{\partial a_i}{\partial x_k}\right\} l_i^{\text{o}}$$

since the unit vector l_i^{o} does not change during the motion of the surface. If the initial state is regarded as identical to the instantaneous state, $\rho_0 = \rho, a_i = x_i$, $l_i^{\text{o}} = l_i$, and the nominal traction rate becomes†

$$\dot{F}_j = -\dot{p}l_j + p\left(l_k\frac{\partial v_k}{\partial x_j} - l_j\frac{\partial v_k}{\partial x_k}\right) \tag{92}$$

It follows that even when the pressure remains constant, the nominal traction changes as a result of the instantaneous distortion of the unconstrained surface. The equilibrium equation and the boundary condition, expressed in the rate form, must be supplemented by the constitutive equation for the particular solid in formulating the boundary value problem of the incremental type.

Problems

1.1 In a certain annealed material, the yield point is taken as that for which the permanent strain is one-quarter of the recoverable elastic strain. The true stress–strain curve for the material in the plastic range may be represented by the empirical equation

$$\sigma = \frac{E}{180}\varepsilon^{0.25}$$

where E is Young's modulus. Determine the stress Y at the yield point as a fraction of E, and compute the true and nominal values of the uniaxial instability stress in terms of Y.

Answer: $Y = E/943$, $\sigma = 3.71\,Y$, $s = 2.89\,Y$.

1.2 The true stress/engineering strain curve of a material in simple tension may be represented by the equation $\sigma = Ce^n$, where C and n are empirical constants. Show that the value of e at the onset of necking in uniaxial tension is $n/(1-n)$. Suppose that a bar of material is axially compressed to a strain of $e < n$, and is subsequently extended to the point of necking. Assuming no buckling, and neglecting Bauschinger effect, show that the ratio of the final and initial lengths of the bar is $(1-e)^2/(1-n)$.

1.3 Prove that according to the Voce equation for the stress–strain curve, the true stress and the natural strain at the onset of instability in uniaxial tension are

$$\sigma = \frac{Cn}{1+n} \qquad \varepsilon = \frac{\ln[m(1+n)]}{n}$$

† J. Chakrabarty, *Z. angew. Math. Phys.*, **24**: 270 (1973). A more general type of loading has been examined by R. Hill, *J. Mech. Phys. Solids*, **10**: 185 (1962).

What would be the instability strain when $m(1+n)$ is less than unity? Show that the stress–strain curve can be linearized by introducing a new strain measure ε^* defined as

$$\varepsilon^* = \frac{1}{n}\left\{1 - \left(\frac{l_0}{l}\right)^n\right\}$$

where l_0 and l are the initial and final lengths of a tension bar.

1.4 In the simple compression of a short cylinder, the curve representing the variation of the load with the amount of compression shows a point of inflection. If the true stress–strain curve of the material is expressed by the empirical equation $\sigma = C\varepsilon^n$, show that the natural strain corresponding to the point of inflection is

$$\varepsilon = \tfrac{1}{4}\left[\sqrt{n(8+n)} - 3n\right]$$

For what range of values of n will this strain exceed the instability strain in simple tension?

Answer: $0 < n < 1/6$.

1.5 In the homogeneous compression of a cylindrical specimen, the curve for the nominal stress against the natural strain has a point of inflection. Show that the corresponding point on the true stress–strain curve is given by

$$\left(\frac{d}{d\varepsilon} + 1\right)^2 \sigma = 0$$

Assuming the empirical equation $\sigma = C\varepsilon^n$, show that the true strain is $\sqrt{n} - n$ at the point of inflection. For what values of n will this strain exceed the uniaxial instability strain?

Answer: $0 < n < 0.25$.

1.6 The effect of elastic deformation of the material on the instability strain may be estimated by considering the stress–strain equation in the Ramberg-Osgood form

$$\varepsilon = \frac{\sigma}{E} + \frac{3\sigma_0}{7E}\left(\frac{\sigma}{\sigma_0}\right)^{1/n}$$

where σ_0 is the nominal yield stress and n is the strain-hardening exponent. Show that the true strain at the onset of necking in simple tension becomes

$$\varepsilon \simeq n + \left(\frac{7n}{3}\right)^n \left(\frac{\sigma_0}{E}\right)^{1-n}$$

to a close approximation. Assuming $n = 0.05$ and $\sigma_0/E = 0.002$, compute the percentage error involved in using the simple power law $\sigma = C\varepsilon^n$.

Answer: 4.76%.

1.7 The plane structure shown in Fig. A consists of three bars pin-jointed at their ends. The central bar OB is made of a material whose stress–strain curve is represented by $\sigma = C_1\varepsilon^{n_1}$. The inclined bars OA and OC are made of a different material, having its stress–strain law expressed by $\sigma = C_2\varepsilon^{n_2}$, where $n_2 < n_1$. If the initial angle of inclinations ψ is such that plastic instability occurs simultaneously in the three bars on the application of a vertical load at O, show that

$$\cos\psi = \sqrt{\frac{\exp(2n_2) - 1}{\exp(2n_1) - 1}}$$

1.8 Suppose that the bars of Fig. A have the same cross-sectional area A, and are made of a material that strain-hardens according to the law $\sigma/Y = (E\varepsilon/Y)^n$. Show that the relationship between the applied load P and the deflection δ of point O, for sufficiently small strains in the fully plastic range, is given by

$$\frac{P}{AY} = (1 + 2\cos^{2n+1}\psi)\left(\frac{E\delta}{Yl}\right)^n \qquad \frac{E\delta}{Yl} \geqslant \sec^2\psi$$

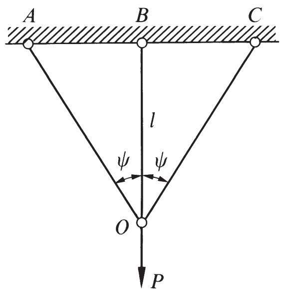


Figure A

How is this equation modified when OB is plastic while OA and OC are still elastic? Obtain a graphical plot of P/AY against $E\delta/Yl$ over the range $0 < E\delta/Yl < 5$, assuming $n = 0.25$ and $\psi = 45°$.

1.9 The stress–strain curve of a rigid/plastic metal can be accurately fitted (except for very small strains) by the Ludwik equation $\sigma = C\varepsilon^n$. It is required to approximate this curve by the straight line $\sigma = Y + H\varepsilon$, giving the same plastic work over a total strain of ε_0 (Fig. B). If the difference between the stresses predicted by the two equations at $\varepsilon = \frac{1}{2}\varepsilon_0$ is exactly one-half of that at $\varepsilon = \varepsilon_0$, show that

$$\frac{Y}{\sigma_0} = \frac{3-n}{1+n} - 2^{1-n} \qquad \frac{H\varepsilon_0}{2\sigma_0} = 2^{1-n} - \frac{2-n}{1+n}$$

where $\sigma_0 = C\varepsilon_0^n$. Assuming $n = 0.3$, estimate the maximum percentage error in the linear approximation where the straight line falls below the curve.

Answer: 7.8%.

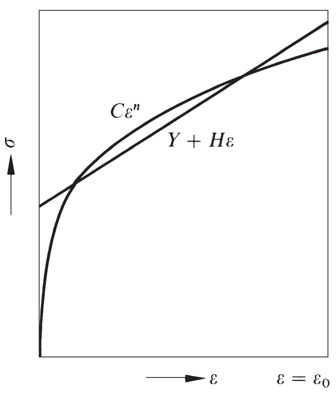


Figure B

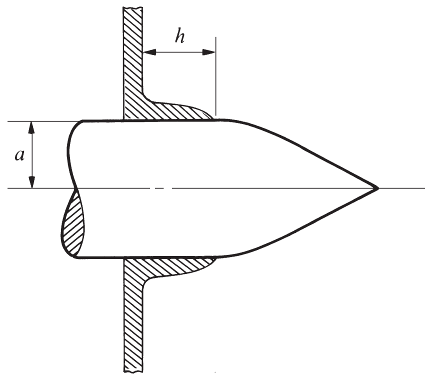


Figure C

1.10 Derive an expression for the hoop stress that exists in a thin circular ring of mean radius r, thickness t, and density ρ, rotating about its own axis with an angular velocity ω. If the deformation is continued in the plastic range, tensile instability would occur when the angular velocity attains a maximum. Representing the true stress–strain curve by the empirical equation $\sigma = C\varepsilon^n$, show that the instability or bursting speed is given by

$$\rho\omega^2 r_0^2 = C\left(\frac{n}{2}\right)^n \exp(-n)$$

where r_0 is the mean radius of the undeformed ring.

1.11 In a thin-walled spherical shell under a uniform internal pressure p, the state of stress is a balanced biaxial tension σ, which is related to the compressive thickness strain ε by the uniaxial stress–strain law for the material. If plastic instability occurs in the shell when the internal pressure attains a maximum, show that $d\sigma/d\varepsilon = \frac{3}{2}\sigma$ at the onset of instability. Assuming the empirical stress–strain equation $\sigma = C\varepsilon^n$, obtain the dimensionless bursting pressure

$$\frac{p}{C} = \frac{2t_0}{r_0}\left(\frac{2}{3}n\right)^n \exp(-n)$$

where t_0 and r_0 are the initial wall thickness and mean radius respectively.

1.12 A compound bar is made up of a solid cylinder which just fits into a hollow one, the two cylinders being firmly bonded at their common interface. The true stress–strain curve is given by $\sigma = C_1\varepsilon^{n_1}$ for the inner cylinder and by $\sigma = C_2\varepsilon^{n_2}$ for the outer cylinder. If the two cylinders carry equal loads at the onset of instability, when the compound bar is subjected to longitudinal tension, show that the ratio of the cross-sectional areas of the outer and inner cylinders is

$$\frac{A_2}{A_1} = \frac{C_1}{C_2}\left(\frac{n_1 + n_2}{2}\right)^{n_1 - n_2}$$

1.13 Fig. C illustrates the perforation of a uniform plate of thickness t_0 by a smooth cylindrical drift of radius a having a conical end. Each element of the raised lip may be assumed to form under a uniaxial tensile hoop stress of varying intensity. Show that the height of the lip is $h = \frac{2}{3}a$, and that its thickness varies as the cube root of the distance from the outer edge. If the material strain-hardens according to the law $\sigma = C\varepsilon^n$, show that the plastic work done during the process is

$$W = \pi t_0 a^2 C \frac{\Gamma(1+n)}{2^{1+n}}$$

where $\Gamma(x)$ is the gamma function of any positive variable x. Find the numerical value of W/t_0a^2C when $n = 0.5$.

Answer: 0.984.

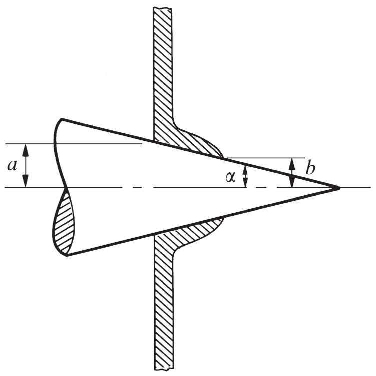


Figure D

1.14 A plate of uniform thickness t_0 is perforated by a smooth conical drift of semiangle α as shown in Fig. D. The axis of the drift moves perpendicular to the plane of the plate and develops a conical lip of base radius a. Assuming a uniaxial state of stress to exist in each element, show that the radius of the outer cross section of the lip is $b = a(1 - \sin\alpha)^{2/3}$. Show also that the thickness t of an element that was situated at a radius r_0 in the undeformed state is given by

$$\frac{t}{t_0} = \left(\frac{r_0}{b}\right)^{1/2}\left\{1 + \left(\frac{r_0}{b}\right)^{3/2}\sin\alpha\right\}^{-1/3}$$

1.15 Suppose that the structure shown in Fig. A is loaded in the fully plastic range to produce a small vertical deflection δ of the joint O. The bars are identical in material and cross section, and the strain-hardening of each bar is given by $\sigma/Y = (E\varepsilon/Y)^n$. If the residual stresses left in the vertical and the inclined bars are σ_1' and σ_2' respectively on complete unloading from the plastic state, show that

$$\frac{\sigma_2'}{Y} = -\frac{\sigma_1'}{2Y}\sec\psi = \frac{\cos^{2n}\psi - \cos^2\psi}{1 + 2\cos^3\psi}\left(\frac{E\delta}{Yl}\right)^n$$

Assuming δ to be three times that at the initial yielding, calculate σ_1' and the residual deflection δ' when $n = 0.25$ and $\psi = 45°$.

Answer: $\sigma_1'/Y = -0.372$, $E\delta'/Yl = 1.312$.

1.16 Two uniform vertical wires AB and CD, shown in Fig. E, support a load W acting at the free end of an initially horizontal rigid bar hinged at O. The lower ends of the wires are attached to blocks which can slide along a frictionless groove in the rigid bar. The strain-hardening exponents for the wires AB and CD are n and $2n$ respectively. If plastic instability occurs simultaneously in them when the load is increased to a critical value, show that

$$\frac{b}{a} = e^n + \frac{(e^n - 1)(e^{2n} - 1)}{1 - e^n\sqrt{2 - e^{2n}}}$$

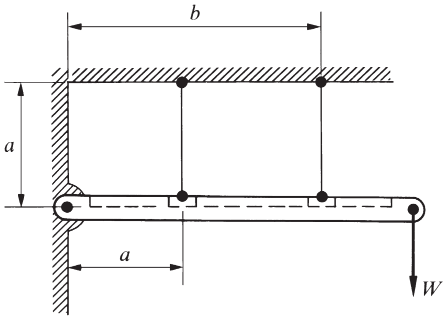


Figure E

1.17 Let $\sigma_1 > \sigma_2 > \sigma_3$ be the principal stresses at any point P in a stressed body, and consider a straight line through P having direction cosines

$$\sqrt{\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}},\quad 0,\quad \sqrt{\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}}$$

with respect to the principal axes. Show that the resultant shear stress at P across any plane containing the given straight line is in the direction of this line.

1.18 At a typical point O in a stressed body, the normal stress across a certain plane is equal to the intermediate principal stress, while the shear stress is the geometric mean of the principal shear stresses other than the absolute maximum. Assuming $\sigma_1 > \sigma_2 > \sigma_3$, show that the direction cosines of the normal to the plane with respect to the principal axes are

$$\frac{1}{2}\sqrt{\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}},\quad \frac{\sqrt{3}}{2},\quad \frac{1}{2}\sqrt{\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}}$$

Find the direction of the shear stress across the plane, and show that it coincides with that of the greatest shear stress at O when $\sigma_1 + \sigma_3 = 2\sigma_2$.

1.19 Referring to the oblique triangle of Fig. 1.10*a*, where *CF* is drawn along the shear stress vector to meet the side *AB* at *F*, show that

$$\frac{AF}{AB} = \left(\frac{\sigma_2 - \sigma}{\sigma - \sigma_3}\right)\frac{m^2}{n^2}$$

Show also that *AB* is divided internally or externally at *F* according as the ratio $(\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_2)$ is greater or less than l^2/n^2.

1.20 A typical point *O* in a stressed body is taken as the origin of coordinates with rectangular axes in the directions of the principal stresses σ_1, σ_2, and σ_3. If *P* is any point on the surface of the quadric

$$\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = \pm c^2$$

where *c* is a constant, show that the normal stress at *O* acting on the plane perpendicular to *OP* has the magnitude c^2/r^2, where $OP = r$. Show also that the resultant stress across this plane is directed along the normal to the quadric surface at *P*, and is of magnitude c^2/hr, where *h* is the perpendicular distance of *O* from the tangent plane through *P*.

1.21 The rectangular components of the stress tensor at a certain point are found to be proportional to the elements of the square matrix

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & c \end{bmatrix}$$

Find the value of *c* for which there will be a traction-free plane passing through the given point. Compute the direction cosines of the normal to the traction-free plane.

Answer: $c = 0.4$, $l = 0.154$, $m = -0.617$, $n = 0.772$.

1.22 If *OP* represents the resultant stress vector across a plane passing through *O*, show that *P* will lie on the surface of an ellipsoid, known as stress ellipsoid, whose principal axes coincide with those of the stress at *O*. Prove that the given plane is parallel to the tangent plane of the stress director surface

$$\frac{x^2}{\sigma_1} + \frac{y^2}{\sigma_2} + \frac{z^2}{\sigma_3} = \text{const}$$

at the point where it is intersected by *OP*, the coordinate axes being taken through *O* along the principal stress axes.

1.23 The resultant stress at a given point *O* across an oblique plane is 135 MPa, acting in the direction (1/3, 2/3, −1/3) with respect to a set of rectangular axes. If the normal to the plane is inclined at 45° to the *x* axis, and makes equal acute angles with the *y* and *z* axes, find the normal and shear components of the stress. Assuming the state of stress at *O* to correspond to $\sigma_x = \sigma_y$, $\tau_{xy} = \tau_{yz}$, and $\tau_{zx} = 0$, determine the nonzero components of the stress tensor.

Answer: In units of MPa, $\sigma = 86.13$, $\tau = 103.95$, $\sigma_x = \sigma_y = 105.44$, $\sigma_z = -120.88$, $\tau_{xy} = \tau_{yz} = 30.88$.

1.24 The state of stress at a certain point in a material body is defined by the following rectangular components:

$$\sigma_x = 64\,\text{MPa} \qquad \sigma_y = -76\,\text{MPa} \qquad \sigma_z = 48\,\text{MPa}$$
$$\tau_{xy} = 30\,\text{MPa} \qquad \tau_{yz} = -25\,\text{MPa} \qquad \tau_{zx} = 55\,\text{MPa}$$

Determine the normal and shear stresses acting on a plane whose normal in inclined at 40 and 70° to the *x* and *y* axes respectively. Find also the direction cosines of the shear stress, assuming an acute angle between the normal and the *z* axis.

Answer: $\sigma = 95.12\,\text{MPa}$, $\tau = 52.42\,\text{MPa}$, $l_s = 0.312$, $m_s = -0.937$, $n_s = 0.152$.

1.25 In a prismatic beam subjected to combined bending and twisting, the components of the stress tensor at a given point are

$$\sigma_x = 72.5\,\text{MPa} \qquad \sigma_y = -12.8\,\text{MPa} \qquad \sigma_z = 0$$
$$\tau_{xy} = 62.3\,\text{MPa} \qquad \tau_{yz} = 0 \qquad \tau_{zx} = -45.4\,\text{MPa}$$

where the x axis is along the centroidal axis of the beam. Find the values of the principal stresses, the greatest shear stress, and the direction cosines of the largest principal stress.

Answer: $\sigma_1 = 119.2\,\text{MPa}$, $\sigma_2 = -4.0\,\text{MPa}$, $\sigma_3 = -55.5\,\text{MPa}$, $l_1 = 0.855$, $m_1 = 0.404$, $n_1 = -0.326$.

1.26 A strain rosette, consisting of three strain gauges OP, OQ, and OR (Fig. F), is constructed to measure simultaneously three extensional small strains ε_P, ε_Q, and ε_R along the surface of a strained body. Using the transformation formula for ε, show that the directions of the principal surface strains make angles α and $\pi/2 + \alpha$ with OQ in the counterclockwise sense, where

$$\tan 2\alpha = \frac{(\varepsilon_P - \varepsilon_R)\tan\psi}{\varepsilon_P + \varepsilon_R - 2\varepsilon_Q}$$

In the special case of an equiangular rosette ($\psi = \pi/3$), show that the principal values of the surface strain are

$$\tfrac{1}{3}(\varepsilon_P + \varepsilon_Q + \varepsilon_R) \pm \tfrac{1}{3}\sqrt{3(\varepsilon_P - \varepsilon_R)^2 + (\varepsilon_P + \varepsilon_R - 2\varepsilon_Q)^2}$$

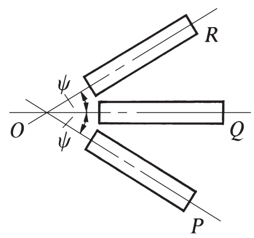


Figure F

1.27 A simple shear is a state of plane strain in which the final coordinates (x, y) of a typical particle are related to the initial coordinates (x_0, y_0) by the transformation

$$x = x_0 + y_0 \tan\phi \qquad y = y_0$$

where $\tan\phi$ is the amount of shear. Show that the straight lines which suffer the maximum extension and contraction are inclined to the x axis at angles $\pm\pi/4 - \alpha/2$ in the strained state, where

$$\alpha = \tan^{-1}\left(\tfrac{1}{2}\tan\phi\right)$$

Show also that the logarithms of the length ratios associated with maximum extension and contraction are $\pm\sinh^{-1}(\tan\alpha)$.

1.28 A state of uniform plane strain of arbitrary magnitude is given by the coordinate transformation

$$x = cx_0 \qquad y = dy_0$$

where c and d are positive constants. Assuming $c > 1 > d$, show that the straight lines whose lengths remain unchanged make angles $\pm\beta$ and $\pm\beta_0$ with the x-axis in the final and initial states respectively, where

$$\tan\beta = \frac{d}{c}\sqrt{\frac{c^2 - 1}{1 - d^2}} = \frac{d}{c}\tan\beta_0$$

Prove that the deformation is associated with a change in volume unless $cd = 1$, which corresponds to pure shear. Show also that the maximum engineering shear strain is $(c^2 - d^2)/(2cd)$, associated with lines that are inclined at $\pm\pi/4$ with the x axis in the unstrained state.

1.29 Find the relationship between the constants A, B, and C in the following expressions, which represent a possible deformation rate in a two-dimensional field:

$$\dot{\varepsilon}_x = Ax^2(x^2 + y^2) \qquad \dot{\varepsilon}_y = By^2(x^2 + y^2)$$
$$\dot{\gamma}_{xy} = Cxy(x^2 + y^2)$$

Show that the associated velocity field, to within a rigid-body motion, is given by

$$u = Cx^3\left(\tfrac{3}{5}x^2 + y^2\right) + Dy$$
$$v = Cy^3\left(x^2 + \tfrac{3}{5}y^2\right) - Dx$$

where D is an arbitrary constant. Obtain an expression for the component of spin in the xy plane, if a rigid body rotation of the material as a whole is excluded.

1.30 An element of material deforms in plane strain such that the principal axes of the strain rate remain fixed in the element as it rotates during its motion. The directions of $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ are assumed to be parallel to the x and y axes respectively in the initial state. Show that the principal natural strains produced by an arbitrary small deformation of the element are

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right\}$$
$$\varepsilon_2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

to second order, where u and v are the components of the displacement of the center of the element whose initial coordinates are x and y.

1.31 Let a_i and x_i be the initial and final coordinates of a typical particle P with respect to a fixed set of rectangular axes. Show that the ratio of the final and initial squared lengths of the material line elements through P, parallel to the coordinate axes in the initial state, are equal to the diagonal elements of the matrix of the tensor

$$g_{ij} = \frac{\partial x_k}{\partial a_i}\frac{\partial x_k}{\partial a_j}$$

Prove that the ratio of the initial and final densities of the material in the neighborhood of the considered particle is equal to the jacobian $|\partial x_i/\partial a_j|$ of the transformation of coordinates.

1.32 Green's strain tensor γ_{ij} at a typical particle in a finitely deformed body, having initial coordinates a_i, is defined as that whose scalar product with the tensor $2da_i\, da_j$ is equal to the difference between the final and initial squared lengths of a material line element emanating from the particle. Show that

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right)$$

where u_i is the displacement of the principle. Show also that the material rate of change of γ_{ij}, when the initial reference state coincides with the instantaneous state, is identical to the rate of deformation.

1.33 The curve obtained by plotting the nominal stress against the engineering strain in simple tension may be represented by the empirical equations $s = Be^m/(1 + e)$, where B and m are constants. If a specimen of the same material is loaded in simple compression, show that the relationship between the nominal stress and the engineering strain becomes

$$s = \frac{B}{1-e}\left(\frac{e}{1-e}\right)^m$$

Also show that the magnitude of the engineering strain at the point of inflection on the (s, e) curve is $e = [m(1+m)/2]^{1/2} - m$.

1.34 A compound bar is composed of a solid cylinder exactly fitting into a hollow cylinder of identical length and cross-sectional area, the two cylinders being firmly bonded at their interface. If the true stress–strain curves for the inner and outer cylinders are given by $\sigma = C_1\varepsilon^{n_1}$ and $\sigma = C_2\varepsilon^{n_2}$, respectively, show that the longitudinal true strain at the onset of instability, when the compound bar is subjected to axial tension, is given by

$$\left(\frac{\varepsilon - n_1}{n_2 - \varepsilon}\right)\varepsilon^{n_1 - n_2} = \frac{C_2}{C_1}$$

Assuming $n_1 = 0.2$, $n_2 = 0.3$, and $C_2/C_1 = 1.5$, compute the value of the instability strain.
Answer: $\varepsilon = 0.257$.

1.35 A thin-walled cylindrical tube with open ends in subjected to a gradually increasing internal pressure p, the initial thickness and mean radius of the tube being t_0 and r_0 respectively. Show that the condition for plastic instability, which occurs when the pressure attains a maximum, is given by $d\sigma/d\varepsilon = 3\sigma/2$, where σ and ε are the stress and strain in the circumferential direction. If the true stress–strain curve of the material is expressed by the power law $\sigma = C\varepsilon^n$, show that the instability or bursting pressure is given by

$$\frac{p}{C} = \frac{t_0}{r_0}\left(\frac{2n}{3}\right)^n \exp(-n)$$

1.36 Considering small elastic/plastic deformation of the vertical wires in the configuration of Fig. E, let the line of action of the load W be situated at a distance c from the vertical wall. Each wire is assumed to be of length l, cross-sectional area A, and made of a material whose stress–strain curve in the plastic range is given by $\sigma = Y(E\varepsilon/Y)^n$, where n is constant. Show that the relationship between the applied load W and the deflection δ of its point of application is given by

$$\frac{W}{AY} = \frac{a}{c}\left(\frac{Ea\delta}{Ycl}\right)^m + \frac{b}{c}\left(\frac{Eb\delta}{Ycl}\right)^n$$

where $m = 1$ for $cY/bE \leqslant \delta/l \leqslant cY/aE$, and $m = n$ for $\delta/l \geqslant cY/aE$. Taking $b = 2a$, $c = 3a$ and $n = 0.25$, obtain a graphical plot of W/AY against $E\delta/Yl \geqslant 1.5$.