Basic Concepts

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This introductory chapter discusses some of the basic concepts in the fascinating subject of structural dynamics.

1.1 Statics, dynamics and structural dynamics

Statics deals with the effect of forces on bodies at rest. Dynamics deals with the motion of nominally rigid bodies. The two aspects of dynamics are kinematics and kinetics. Kinematics is concerned only with the motion of bodies with geometric constraints, irrespective of the forces acting. So, for example, a body connected by a link so that it can only rotate about a fixed point is constrained by its kinematics to move in a circular path, irrespective of any forces that may be acting. On the other hand, in kinetics, the path of a particle may vary as a result of the applied forces. The term structural dynamics implies that, in addition to having motion, the bodies are non-rigid, i.e. ‘elastic’. ‘Structural dynamics’ is slightly wider in meaning than ‘vibration’, which implies only oscillatory behaviour.

1.2 Coordinates, displacement, velocity and acceleration

The word coordinate acquires a slightly different, additional meaning in structural dynamics. We are used to using coordinates, $x$, $y$ and $z$, say, when describing the location of a point in a structure. These are Cartesian coordinates (named after René Descartes), sometimes also known as ‘rectangular’ coordinates. However, the same word ‘coordinate’ can be used to mean the movement of a point on a structure from some standard configuration. As an example, the positions of the grid points chosen for the analysis of a structure could be specified as $x$, $y$ and $z$ coordinates from some fixed point. However, the displacements of those points, when the structure is loaded in some way, are often also referred to as coordinates.
Cartesian coordinates of this kind are not always suitable for defining the vibration behavior of a system. The powerful Lagrange method requires coordinates known as *generalized coordinates* that not only fully describe the possible motion of the system, but are also independent of each other. An often-used example illustrating the difference between Cartesian and generalized coordinates is the double pendulum shown in Fig. 1.1. The angles \( \theta_1 \) and \( \theta_2 \) are sufficient to define the positions of \( m_1 \) and \( m_2 \) completely, and are therefore suitable as generalized coordinates. All four Cartesian coordinates \( x_1, y_1, x_2 \) and \( y_2 \), taken together, are not suitable for use as generalized coordinates, since they are not independent of each other, but are related by the two *constraint equations*:

\[
x_1^2 + y_1^2 = r_1^2 \quad \text{and} \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = r_2^2
\]

This illustrates the general rule that the number of degrees of freedom, and the number of generalized coordinates required, is the total number of coordinates minus the number of constraint equations. In this case there can only be two generalized coordinates, but they do not necessarily have to be \( \theta_1 \) and \( \theta_2 \); for example, \( x_1 \) and \( x_2 \) also define the positions of the masses completely, and could be used instead.

Generalized coordinates are fundamentally displacements, but can also be differentiated, i.e. expressed in terms of velocity and acceleration. This means that if a certain displacement coordinate, \( z \), is defined as positive upwards, then its velocity, \( \dot{z} \), and its acceleration, \( \ddot{z} \), are also positive in that direction. The use of dots above symbols, as here, to indicate differentiation with respect to time is a common convention in structural dynamics.

### 1.3 Simple harmonic motion

Simple harmonic motion, more usually called ‘sinusoidal vibration’, is often encountered in structural dynamics work.
1.3.1 Time History Representation

Let the motion of a given point be described by the equation:

\[ x = X \sin \omega t \] (1.1)

where \( x \) is the displacement from the equilibrium position, \( X \) the displacement magnitude of the oscillation, \( \omega \) the frequency in rad/s and \( t \) the time. The quantity \( X \) is the single-peak amplitude, and \( x \) travels between the limits \( \pm X \), so the peak-to-peak amplitude (also known as double amplitude) is \( 2X \).

It appears to be an accepted convention to express displacements as double amplitudes, but velocities and accelerations as single-peak amplitudes, so some care is needed, especially when interpreting vibration test specifications.

Since \( \sin \omega t \) repeats every \( 2\pi \) radians, the period of the oscillation, \( T \), say, is \( 2\pi/\omega \) seconds, and the frequency in hertz (Hz) is \( 1/T = \omega/2\pi \). The velocity, \( \frac{dx}{dt} \), or \( \dot{x} \), of the point concerned, is obtained by differentiating Eq. (1.1):

\[ \dot{x} = \omega X \cos \omega t \] (1.2)

The corresponding acceleration, \( \frac{d^2x}{dt^2} \), or \( \ddot{x} \), is obtained by differentiating Eq. (1.2):

\[ \ddot{x} = -\omega^2 X \sin \omega t \] (1.3)

Figure 1.2 shows the displacement, \( x \), the velocity, \( \dot{x} \), and the acceleration \( \ddot{x} \), plotted against time, \( t \).

Since Eq. (1.2),

\[ \dot{x} = \omega X \cos \omega t \]

can be written as

\[ \dot{x} = \omega X \sin \left( \omega t + \frac{\pi}{2} \right) \]

Fig. 1.2 Displacement, velocity and acceleration time histories for simple harmonic motion.
or

$$\ddot{x} = \omega X \sin \left[ \omega \left( t + \frac{\pi}{2\omega} \right) \right]$$  \hspace{1cm} (1.4)

any given feature of the time history of $\ddot{x}$, for example the maximum value, occurs at a value of $t$ which is $\pi/2\omega$ less (i.e. earlier) than the same feature in the wave representing $x$. The velocity is therefore said to ‘lead’ the displacement by this amount of time. This lead can also be expressed as a quarter-period, $T/4$, a phase angle of $\pi/2$ radians, or 90°.

Similarly, the acceleration time history, Eq. (1.3),

$$\dddot{x} = -\omega^2 X \sin \omega t$$

can be written as

$$\dddot{x} = -\omega^2 X \sin (\omega t + \pi)$$  \hspace{1cm} (1.5)

so the acceleration ‘leads’ the displacement by a time $\pi/\omega$, a half-period, $T/2$, or a phase angle of $\pi$ radians or 180°. In Fig. 1.2 this shifts the velocity and acceleration plots to the left by these amounts relative to the displacement: the lead being in time, not distance along the time axis.

The ‘single-peak’ and ‘peak-to-peak’ values of a sinusoidal vibration were introduced above. Another common way of expressing the amplitude of a vibration level is the root mean square, or RMS value. This is derived, in the case of the displacement, $x$, as follows:

Squaring both sides of Eq. (1.1):

$$x^2 = X^2 \sin^2 \omega t$$  \hspace{1cm} (1.6)

The mean square value of the whole waveform is the same as that of the first half-cycle of $X \sin \omega t$, so the mean value of $x^2$, written $\langle x^2 \rangle$, is

$$\langle x^2 \rangle = X^2 \frac{2}{\pi} \int_0^{\pi/2} \sin^2 \omega t \cdot dt$$  \hspace{1cm} (1.7)

Substituting

$$t = \frac{1}{\omega} (\omega t), \quad dt = \frac{1}{\omega} d(\omega t), \quad T = \frac{2\pi}{\omega},$$

$$\langle x^2 \rangle = X^2 \frac{\omega}{\pi} \int_0^{\pi/\omega} \sin^2 \omega t \cdot d(\omega t) = \frac{X^2}{2}$$  \hspace{1cm} (1.8)

Therefore the RMS value of $x$ is $X/\sqrt{2}$, or about 0.707$X$. It can be seen that this ratio holds for any sinusoidal waveform: the RMS value is always $1/\sqrt{2}$ times the single-peak value.

The waveforms considered here are assumed to have zero mean value, and it should be remembered that a steady component, if present, contributes to the RMS value.
Example 1.1

The sinusoidal vibration displacement amplitude at a particular point on an engine has a single-peak value of 1.00 mm at a frequency of 20 Hz. Express this in terms of single-peak velocity in m/s, and single-peak acceleration in both m/s² and g units. Also quote RMS values for displacement, velocity and acceleration.

Solution

Remembering Eq. (1.1),

\[ x = X \sin \omega t \]  \hspace{1cm} (A)

we simply differentiate twice, so,

\[ \dot{x} = \omega X \cos \omega t \] \hspace{1cm} (B)

and

\[ \ddot{x} = -\omega^2 X \sin \omega t \] \hspace{1cm} (C)

The single-peak displacement, \( X \), is, in this case, 1.00 mm or 0.001 m. The value of \( \omega = 2\pi f \), where \( f \) is the frequency in Hz. Thus, \( \omega = 2\pi(20) = 40\pi \) rad/s.

From Eq. (B), the single-peak value of \( \dot{x} \) is \( \omega X \), or \( (40\pi \times 0.001) = 0.126 \) m/s or 126 mm/s.

From Eq. (C), the single-peak value of \( \ddot{x} \) is \( \omega^2 X \) or \( [(40\pi)^2 \times 0.001] = 15.8 \) m/s² or \( (15.8/9.81) = 1.61 \) g.

Root mean square values are \( 1/\sqrt{2} \) or 0.707 times single-peak values in all cases, as shown in the Table 1.1.

<table>
<thead>
<tr>
<th>Peak and RMS Values, Example 1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single peak value</td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>Displacement</td>
</tr>
<tr>
<td>Velocity</td>
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<tr>
<td>Acceleration</td>
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</tbody>
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1.3.2 Complex Exponential Representation

Expressing simple harmonic motion in complex exponential form considerably simplifies many operations, particularly the solution of differential equations. It is based on Euler’s equation, which is usually written as:

\[ e^{i \theta} = \cos \theta + i \sin \theta \] \hspace{1cm} (1.9)

where \( e \) is the well-known constant, \( \theta \) an angle in radians and \( i \) is \( \sqrt{-1} \).

Multiplying Eq. (1.9) through by \( X \) and substituting \( \omega t \) for \( \theta \):
When plotted on an Argand diagram (where real values are plotted horizontally, and imaginary values vertically) as shown in Fig. 1.3, this can be regarded as a vector, of length $X$, rotating counter-clockwise at a rate of $\omega$ rad/s. The projection on the real, or $x$ axis, is $X \cos \omega t$ and the projection on the imaginary axis, $iy$, is $iX \sin \omega t$. This gives an alternate way of writing $X \cos \omega t$ and $X \sin \omega t$, since

$$X \sin \omega t = \text{Im}(X e^{i\omega t})$$  \hspace{1cm} (1.11)

where $\text{Im}(\ )$ is understood to mean ‘the imaginary part of ( )’, and

$$X \cos \omega t = \text{Re}(X e^{i\omega t})$$  \hspace{1cm} (1.12)

where $\text{Re}(\ )$ is understood to mean ‘the real part of ( )’. Figure 1.3 also shows the velocity vector, of magnitude $\omega X$, and the acceleration vector, of magnitude $\omega^2 X$, and their horizontal and vertical projections.

Equations (1.11) and (1.12) can be used to produce the same results as Eqs (1.1) through (1.3), as follows:

If

$$x = \text{Im}(X e^{i\omega t}) = \text{Im}(X \cos \omega t + iX \sin \omega t) = X \sin \omega t$$  \hspace{1cm} (1.13)

then

$$\dot{x} = \text{Im}(i\omega X e^{i\omega t}) = \text{Im}[i \omega(X \cos \omega t + iX \sin \omega t)] = \omega X \cos \omega t$$  \hspace{1cm} (1.14)
(since $i^2 = -1$) and
\[
\ddot{x} = \text{Im}(-\omega^2 X e^{i\omega t}) = \text{Im}[-\omega^2 (X \cos \omega t + iX \sin \omega t)] = -\omega^2 X \sin \omega t \quad (1.15)
\]

If the displacement $x$ had instead been defined as $x = X \cos \omega t$, then Eq. (1.12), i.e. $X \cos \omega t = \text{Re}(X e^{i\omega t})$, could have been used equally well.

The interpretation of Eq. (1.10) as a rotating complex vector is simply a mathematical device, and does not necessarily have physical significance. In reality, nothing is rotating, and the functions of time used in dynamics work are real, not complex.

### 1.4 Mass, stiffness and damping

The accelerations, velocities and displacements in a system produce forces when multiplied, respectively, by mass, damping and stiffness. These can be considered to be the building blocks of mechanical systems, in much the same way that inductance, capacitance and resistance (L, C and R) are the building blocks of electronic circuits.

#### 1.4.1 Mass and Inertia

The relationship between mass, $m$, and acceleration, $\ddot{x}$, is given by Newton’s second law. This states that when a force acts on a mass, the rate of change of momentum (the product of mass and velocity) is equal to the applied force:
\[
\frac{d}{dt} \left( m \frac{dx}{dt} \right) = F \quad (1.16)
\]
where $m$ is the mass, not necessarily constant, $dx/dt$ the velocity and $F$ the force. For constant mass, this is usually expressed in the more familiar form:
\[
F = m \ddot{x} \quad (1.17)
\]

If we draw a free body diagram, such as Fig. 1.4, to represent Eq. (1.17), where $F$ and $x$ (and therefore $\dot{x}$ and $\ddot{x}$) are defined as positive to the right, the resulting inertia force, $m \ddot{x}$, acts to the left. Therefore, if we decided to define all quantities as positive to the right, it would appear as $-m \ddot{x}$.

![Fig. 1.4 D'Alembert's principle.](image-url)
This is known as D'Alembert's principle, much used in setting up equations of motion. It is, of course, only a statement of the fact that the two forces, \( F \) and \( m\ddot{x} \), being in equilibrium, must act in opposite directions.

Newton’s second law deals, strictly, only with particles of mass. These can be ‘lumped’ into rigid bodies. Figure 1.5 shows such a rigid body, made up of a large number, \( n \), of mass particles, \( m_i \), of which only one is shown. For simplicity, the body is considered free to move only in the plane of the paper. Two sets of coordinates are used: the position in space of the mass center or ‘center of gravity’ of the body, \( G \), is determined by the three coordinates \( x_G, y_G \) and \( \theta_G \). The other coordinate system, \( \bar{x}, \bar{y} \), is fixed in the body, moves with it and has its origin at \( G \). This is used to specify the locations of the \( n \) particles of mass that together make up the body. Incidentally, if these axes did not move with the body, the moments of inertia would not be constant, a considerable complication.

The mass center, \( G \), is, of course, the point where the algebraic sum of the first moments of inertia of all the \( n \) mass particles is zero, about both the \( x \) and the \( y \) axes, i.e.,

\[
\sum_{i=1}^{n} m_i \bar{x}_i = \sum_{i=1}^{n} m_i \bar{y}_i = 0, \tag{1.18}
\]

where the \( n \) individual mass particles, \( m_i \) are located at \( \bar{x}_i, \bar{y}_i \) \((i = 1 \text{ to } n)\) in the \( \bar{x}, \bar{y} \) coordinate system.

External forces and moments are considered to be applied, and their resultants through and about \( G \) are \( F_x \), \( F_y \) and \( M_\theta \). These must be balanced by the internal inertia forces of the mass particles.
Thus in the $x$ direction, since $F_x$ acts at the mass center,

$$F_x + \sum_{i=1}^{n} -m_ix_i = 0; \quad (1.19a)$$

or

$$F_x = mx_G; \quad (1.19b)$$

where $m = \sum_{i=1}^{n} m_i = \text{total mass of body}$.  
Note that the negative sign in Eq. (1.19a) is due to D’Alembert’s principle.

Similarly in the $y$ direction,

$$F_y + \sum_{i=1}^{n} -m_\dot{y}_i = 0 \quad (1.20a)$$

or

$$F_y = m\dot{y}_G \quad (1.20b)$$

For rotation about $G$, the internal tangential force due to one mass particle, $m_i$ is $-m_i r_i \dot{\theta}_G$, the negative sign again being due to D’Alembert’s principle, and the moment produced about $G$ is $-m_i r_i^2 \ddot{\theta}_G$. The total moment due to all the mass particles in the body is thus $\sum_{i=1}^{n} (-m_i r_i^2 \ddot{\theta}_G)$, all other forces canceling because $G$ is the mass center. This must balance the externally applied moment, $M_\theta$, so

$$M_\theta + \sum_{i=1}^{n} (-m_i r_i^2 \ddot{\theta}_G) = 0 \quad (1.21a)$$

or

$$M_\theta = I_G \ddot{\theta}_G \quad (1.21b)$$

where

$$I_G = \sum_{i=1}^{n} m_i r_i^2 \quad (1.22)$$

Equation (1.22) defines the mass moment of inertia of the body about the mass center.

In matrix form Eqs (1.19b), (1.20b) and (1.21b) can be combined to produce

$$\begin{bmatrix} F_x \\ F_y \\ M_\theta \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_G \end{bmatrix} \begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\theta}_G \end{bmatrix} \quad (1.23)$$

The $(3 \times 3)$ matrix is an example of an inertia matrix, in this case diagonal, due to the choice of the mass center as the reference center. Thus the many masses making up the system of Fig. 1.5 have been ‘lumped’, and can now be treated as a single mass (with two freedoms) and a single rotational moment of inertia, all located at the mass center of the body, $G$.

In general, of course, for a three-dimensional body, the inertia matrix will be of size $(6 \times 6)$, and there will be six coordinates, three translations and three rotations.
1.4.2 Stiffness

Stiffnesses can be determined by any of the standard methods of static structural analysis. Consider the rod shown in Fig. 1.6(a), fixed at one end. Force $F$ is applied axially at the free end, and the extension $x$ is measured. As shown in Fig. 1.6(b), if the force $F$ is gradually increased from zero to a positive value, it is found, for most materials, that Hooke’s law applies; the extension, $x$, is proportional to the force, up to a point known as the elastic limit. This is also true for negative (compressive) loading, assuming that the rod is prevented from buckling. The slope $\delta F/\delta x$ of the straight line between these extremes, where $\delta F$ and $\delta x$ represent small changes in $F$ and $x$, respectively, is the stiffness, $k$. It should also be noted in Fig. 1.6(b) that the energy stored, the potential energy, at any value of $x$, is the area of the shaded triangle, $\frac{1}{2}Fx$, or $\frac{1}{2}kx^2$, since $F = kx$.

Calculating the stiffness of the rod, for the same longitudinal loading, is a straightforward application of elastic theory:

$$k = \frac{\delta F}{\delta x} = \frac{\delta \sigma}{\delta \varepsilon} \cdot \frac{a}{L} = \frac{Ea}{L}$$

where $a$ is the cross-sectional area of the rod, $L$ its original length and $\delta \sigma/\delta \varepsilon$ the slope of the plot of stress, $\sigma$, versus strain, $\varepsilon$, known as Young’s modulus, $E$.

The stiffness of beam elements in bending can similarly be found from ordinary elastic theory. As an example, the vertical displacement, $y$, at the end of the uniform built-in cantilever shown in Fig. 1.6(c), when a force $F$ is applied is given by the well-known formula:

$$y = \frac{FL^3}{3EI}$$

\[1.25\]
where $L$ is the length of the beam, $E$ Young’s modulus and $I$ the ‘moment of inertia’ or second moment of area of the cross-section applicable to vertical bending.

From Eq. (1.25), the stiffness $\delta F/\delta y$, or $k_y$, is

$$k_y = \frac{\delta F}{\delta y} = \frac{3EI}{L^3} \quad (1.26)$$

For a rod or beam in torsion, fixed at one end, with torque, $T$, applied at the other end, as shown in Fig. 1.6(d), the following result applies.

$$\frac{T}{J} = \frac{G\phi}{L} \quad (1.27)$$

where $T$ is the applied torque, $J$ the polar area ‘moment of inertia’ of the cross-section, $G$ the shear modulus of the material, $\phi$ the angle of twist at the free end and $L$ the length.

From Eq. (1.27), the torsional stiffness at the free end is

$$k_\phi = \frac{\delta T}{\delta \phi} = \frac{GJ}{L} \quad (1.28)$$

Example 1.3

Figure 1.7, where the dimensions are in mm., shows part of the spring suspension unit of a small road vehicle. It consists of a circular-section torsion bar, and rigid lever. The effective length of the torsion bar, $L_1$, is 900 mm, and that of the lever, $L_2$, is 300 mm. The bar is made of steel, having elastic shear modulus $G = 90 \times 10^9$ N/m$^2$ (= 90 GN/m$^2$ or 90 GPa), and its diameter, $d$, is 20 mm. Find the vertical stiffness, for small displacements, as measured at point B.

Solution

From Eq. (1.30), the torsional stiffness, $k_\phi$, of the bar between point A and the fixed end, is

$$k_\phi = \frac{\delta T}{\delta \phi} = \frac{GJ}{L_1} \quad (A)$$

where $\delta T$ is a small change in torque applied to the bar at A and $\delta \phi$ the corresponding twist angle. $G$ is the shear modulus of the material, $J$ the polar moment of inertia of the bar cross-section and $L_1$ the length of the torsion bar. However, we require the linear stiffness $k_x$ or $\delta F/\delta x$, as seen at point B. Using Eq. (A), and the relationships, valid for small angles, that $\delta T = \delta F \cdot L_2$, and $\delta \phi = \delta x/L_2$, it is easily shown that

$$k_x = \frac{\delta F}{\delta x} = \frac{\delta T}{\delta \phi} \cdot \frac{1}{L_2} = \frac{GJ}{L_1L_2} \cdot \frac{1}{L_2} = k_\phi \cdot \frac{1}{L_2} \quad (B)$$

Numerically, $G = 90 \times 10^9$ N/m$^2$; $J = \pi d^4/32 = \pi(0.02)^4/32 = 15.7 \times 10^{-9}$ m$^4$; $L_1 = 0.90$ m; $L_2 = 0.30$ m giving $k_x = 17 400$ N/m or 17.4 k N/m.
1.4.3 Stiffness and Flexibility Matrices

A stiffness value need not be defined at a single point: the force and displacement can be at the same location, or different locations, producing a direct stiffness or a cross stiffness, respectively. An array of such terms is known as a stiffness matrix or a matrix of stiffness influence coefficients. This gives the column vector of forces or moments $f_1, f_2, \ldots$, required to be applied at all stations to balance a unit displacement at one station:

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \ldots \\ k_{21} & k_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

or

$$\{ f \} = [k] \{ x \}$$

The stiffness matrix, $[k]$, is square, and symmetric (i.e. $k_{ij} = k_{ji}$ throughout).

The mathematical inverse of the stiffness matrix is the flexibility matrix which gives the displacements $x_1, x_2, \ldots$, produced by unit forces or moments $f_1, f_2, \ldots$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots \\ \alpha_{21} & \alpha_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

Fig. 1.7 Torsion bar discussed in Example 1.3.

1.4.3 Stiffness and Flexibility Matrices

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$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \ldots \\ k_{21} & k_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

or

$$\{ f \} = [k] \{ x \}$$

The stiffness matrix, $[k]$, is square, and symmetric (i.e. $k_{ij} = k_{ji}$ throughout).

The mathematical inverse of the stiffness matrix is the flexibility matrix which gives the displacements $x_1, x_2, \ldots$, produced by unit forces or moments $f_1, f_2, \ldots$.
or
\[ \{x\} = [\alpha] \{f\} \quad (1.32) \]

The flexibility matrix, \([\alpha]\), is also symmetric, and its individual terms are known as *flexibility influence coefficients*.

Since \([\alpha]\) is the inverse of \([k]\), i.e. \([\alpha] = [k]^{-1}\),

\[ [\alpha][k] = [k]^{-1}[k] = [I] \quad (1.33) \]

where \([I]\) is the unit matrix, a square matrix with all diagonal terms equal to unity, and all other terms zero.

Since the stiffness matrix (or the flexibility matrix) relates forces (or moments) applied anywhere on a linear structure to the displacements produced anywhere, it contains all there is to know about the stiffness properties of the structure, provided there are sufficient coordinates.

### Example 1.4

(a) Derive the stiffness matrix for the chain of springs shown in Fig. 1.8.

(b) Derive the corresponding flexibility matrix.

(c) Show that one is the inverse of the other.

**Solution**

(a) The \((3 \times 3)\) stiffness matrix can be found by setting each of the coordinates \(x_1, x_2\), and \(x_3\) to 1 in turn, with the others at zero, and writing down the forces required at each node to maintain equilibrium:

when : \(x_1 = 1\)  
then : \(f_1 = k_1 x_1 + k_2 x_2\)  
\(f_2 = -k_2 x_1\)  
\(f_3 = 0\)  
\(x_2 = 0\)  
\(x_3 = 0\)  
\((A_1)\)

when : \(x_1 = 0\)  
then : \(f_1 = -k_2 x_2\)  
\(f_2 = k_2 x_2 + k_3 x_2\)  
\(f_3 = -k_3 x_2\)  
\(x_2 = 1\)  
\(x_3 = 0\)  
\((A_2)\)

when : \(x_1 = 0\)  
then : \(f_1 = 0\)  
\(f_2 = -k_3 x_3\)  
\(f_3 = k_3 x_3\)  
\(x_2 = 0\)  
\(x_3 = 1\)  
\((A_3)\)

![Fig. 1.8 Chain of spring elements discussed in Example 1.4.](image)
Equations (A_1) now give the first column of the stiffness matrix, Eqs (A_2) the second column, and so on, as follows:

\[
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix} = \begin{bmatrix}
(k_1 + k_2) & -k_2 & 0 \\
-k_2 & (k_2 + k_3) & -k_3 \\
0 & -k_3 & k_3 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\] \quad (B)

The required stiffness matrix is given by Eq. (B).

(b) The flexibility matrix can be found by setting \(f_1\), \(f_2\) and \(f_3\) to 1, in turn, with the others zero, and writing down the displacements \(x_1\), \(x_2\) and \(x_3\):

\[
\begin{align*}
f_1 &= 1 & f_2 &= 0 & f_3 &= 0 \\
x_1 &= 1/k_1 & x_2 &= 1/k_1 & x_3 &= 1/k_1
\end{align*}
\] \quad (C_1)

\[
\begin{align*}
f_1 &= 0 & f_2 &= 1 & f_3 &= 0 \\
x_1 &= 1/k_1 & x_2 &= 1/k_1 + 1/k_2 & x_3 &= 1/k_1 + 1/k_2
\end{align*}
\] \quad (C_2)

\[
\begin{align*}
f_1 &= 0 & f_2 &= 0 & f_3 &= 1 \\
x_1 &= 1/k_1 & x_2 &= 1/k_1 + 1/k_2 & x_3 &= 1/k_1 + 1/k_2 + 1/k_3
\end{align*}
\] \quad (C_3)

Equations (C_1) give the first column of the flexibility matrix, Eqs (C_2) the second column, and so on:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
1/k_1 & 1/k_1 & 1/k_1 \\
1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2) \\
1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2 + 1/k_3)
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
\end{bmatrix}
\] \quad (D)

Equation (D) gives the required flexibility matrix.

(c) To prove that the flexibility matrix is the inverse of the stiffness matrix, and vice versa, we multiply them together, which should produce a unit matrix, as is, indeed, the case.

\[
\begin{bmatrix}
(k_1 + k_2) & -k_2 & 0 \\
-k_2 & (k_2 + k_3) & -k_3 \\
0 & -k_3 & k_3
\end{bmatrix} \times \begin{bmatrix}
1/k_1 & 1/k_1 & 1/k_1 \\
1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2) \\
1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2 + 1/k_3)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] \quad (E)

### 1.4.4 Damping

Of the three ‘building blocks’ that make up the systems we deal with in structural dynamics, mass and stiffness are *conservative* in that they can only store, or conserve, energy. Systems containing only mass and stiffness are therefore known as *conservative systems*. The third quantity, damping, is different, in that it dissipates energy, which is lost from the system.
Some of the ways in which damping can occur are as follows.

(a) The damping may be inherent in a structure or material. Unfortunately, the term ‘structural damping’ has acquired a special meaning: it now appears to mean ‘hysteretic damping’, and cannot be used to mean the damping in a structure, whatever its form, as the name would imply. Damping in conventional jointed metal structures is partly due to hysteresis within the metal itself, but much more to friction at bolted or riveted joints, and pumping of the fluid, often just air, in the joints. Viscoelastic materials, such as elastomers (rubber-like materials), can be formulated to have relatively high damping, as well as stiffness, making them suitable for the manufacture of vibration isolators, engine mounts, etc.

(b) The damping may be deliberately added to a mechanism or structure to suppress unwanted oscillations. Examples are discrete units, usually using fluids, such as vehicle suspension dampers and viscoelastic damping layers on panels.

(c) The damping can be created by the fluid around a structure, for example air or water. If there is no relative flow between the structure and the fluid, only radiation damping is possible, and the energy loss is due to the generation of sound. There are applications where this can be important, but for normal structures vibrating in air, radiation damping can usually be ignored. On the other hand, if relative fluid flow is involved, for example an aircraft wing traveling through the air, quite large aerodynamic damping (and stiffness) forces may be developed.

(d) Damping can be generated by magnetic fields. The damping effect of a conductor moving in a magnetic field is often used in measuring instruments. Moving coils, as used in loudspeakers and, of particular interest, in vibration testing, in electromagnetic exciters, can develop surprisingly large damping forces.

(e) Figure 1.9 shows a discrete damper of the type often fitted to vehicle suspensions. Such a device typically produces a damping force, \( F \), in response to closure velocity, \( \dot{x} \), by forcing fluid through an orifice. This is inherently a square-law rather than a linear effect, but can be made approximately linear by the use of a special valve, which opens progressively with increasing flow. The damper is then known as an automotive damper, and the one shown in Fig. 1.9 will be assumed to be of this type. Then the force and velocity are related by:

\[
F = c\dot{x}
\]

where \( F \) is the external applied force and \( \dot{x} \) the velocity at the same point. The quantity \( c \) is a constant having the dimensions force/unit velocity. Equation (1.34) will apply for both positive and negative values of \( F \) and \( \dot{x} \), assuming that the device is double-acting.

---

**Fig. 1.9** Common form of damper used in vehicle suspensions.
Example 1.5

An automotive damper similar to that shown in Fig. 1.9 is stated by the supplier to produce a linear damping force, in both directions, defined by the equation $F = c\dot{x}$, where $F$ is the applied force in newtons, $\dot{x}$ is the stroking velocity, in m/s, and the constant $c$ is 1500 N/m/s. A test on the unit involves applying a single-peak sinusoidal force of $\pm 1000$ N at each of the frequencies, $f = 1.0$, 2.0 and 5.0 Hz. Calculate the expected single-peak displacement, and total movement, at each of these frequencies.

Solution

The applied force is $F = c\dot{x}$, where $F = 1000 \sin(2\pi ft)$. The expected velocity, $\dot{x}$, is therefore:

$$\dot{x} = \frac{F}{c} = \frac{1000 \sin (2\pi ft)}{1500}$$  \hspace{1cm} (A)

The displacement $x$ is given by integrating the velocity with respect to time:

$$x = -\frac{1000 \cos (2\pi ft)}{1500(2\pi f)} + x_0$$  \hspace{1cm} (B)

where the initial displacement $x_0$ is adjusted to mid-stroke. The single-peak displacement, $|x|$, measured from the mid-stroke position, is therefore:

$$|x| = \frac{1000}{1500 \times 2\pi f} = \frac{0.106}{f} \text{ m}$$  \hspace{1cm} (C)

The expected single-peak displacement and total movement at each test frequency are given in Table 1.2.

Table 1.2

| Frequency $f$ (Hz) | Single-peak displacement $|x|$ (m) | Total movement of damper piston (m) |
|-------------------|-----------------------------------|-----------------------------------|
| 1                 | 0.106                             | 0.212                             |
| 2                 | 0.053                             | 0.106                             |
| 5                 | 0.021                             | 0.042                             |

1.5 Energy methods in structural dynamics

It is possible to solve some problems in structural dynamics using only Newton’s second law and D’Alembert’s principle, but as the complexity of the systems analysed increases, methods based on the concept of energy, or work, become necessary. The terms ‘energy’ and ‘work’ refer to the same physical quantity, measured in the same units, but they are used in slightly different ways: work put into a conservative system, for example, becomes the same amount of energy when stored in the system.
Three methods based on work, or energy, are described here. They are (1) Rayleigh’s energy method; (2) the principle of virtual work (or virtual displacements), and (3) Lagrange’s equations. All are based on the principle of the conservation of energy. The following simple definitions should be considered first.

(a) Work is done when a force causes a displacement. If both are defined at the same point, and in the same direction, the work done is the product of the force and displacement, measured, for example, in newton-meters (or lbf-ft). This assumes that the force remains constant. If it varies, the power, the instantaneous product of force and velocity, must be integrated with respect to time, to calculate the work done. If a moment acts on an angular displacement, the work done is still in the same units, since the angle is non-dimensional. It is therefore permissible to mix translational and rotational energy in the same expression.

(b) The kinetic energy, $T$, stored in an element of mass, $m$, is given by $T = \frac{1}{2}m\dot{x}^2$, where $\dot{x}$ is the velocity. By using the idea of a mass moment of inertia, $I$, the kinetic energy in a rotating body is given by $T = \frac{1}{2}I\dot{\theta}^2$, where $\dot{\theta}$ is the angular velocity of the body.

(c) The potential energy, $U$, stored in a spring, of stiffness $k$, is given by $U = \frac{1}{2}kx^2$, where $x$ is the compression (or extension) of the spring, not necessarily the displacement at one end. In the case of a rotational spring, the potential energy is given by $U = \frac{1}{2}k\theta^2$, where $k\theta$ is the angular stiffness, and $\theta$ is the angular displacement.

1.5.1 Raleigh’s Energy Method

Raleigh’s method (not to be confused with a later development, the Raleigh–Ritz method) is now mainly of historical interest. It is applicable only to single-DOF systems, and permits the natural frequency to be found if the kinetic and potential energies in the system can be calculated. The motion at every point in the system (i.e. the mode shape in the case of continuous systems) must be known, or assumed. Since, in vibrating systems, the maximum kinetic energy in the mass elements is transferred into the same amount of potential energy in the spring elements, these can be equated, giving the natural frequency. It should be noted that the maximum kinetic energy does not occur at the same time as the maximum potential energy.

Example 1.6

Use Raleigh’s energy method to find the natural frequency, $\omega_1$, of the fundamental bending mode of the uniform cantilever beam shown in Fig. 1.10, assuming that the vibration mode shape is given by:

$$\frac{y}{y_T} = \left(\frac{x}{L}\right)^2$$

where $y_T$ is the single-peak amplitude at the tip; $y$ the vertical displacement of beam at distance $x$ from the root; $L$ the length of beam; $m$ the mass per unit length; $E$ the Young’s modulus and $I$ the second moment of area of beam cross-section.
Solution

From Eq. (A) it is assumed that

\[ y = y_T \frac{x^2}{L^2} \sin \omega t \quad \text{\textbf{(B)}} \]

Then,

\[ \dot{y} = \omega_1 y_T \frac{x^2}{L^2} \cos \omega t \quad \text{\textbf{(C)}} \]

The maximum kinetic energy, \( T_{\text{max}} \), occurs when \( \cos \omega t = 1 \), i.e. when \( \dot{y} = \omega_1 y_T x^2 / L^2 \), and is given by:

\[ T_{\text{max}} = \int_0^L \frac{1}{2} m y^2 = \frac{1}{2} \frac{m}{L^4} \omega_1^2 y_T^2 \int_0^L x^4 \cdot dx = \frac{1}{10} m L \omega_1^2 y_T^2 \quad \text{\textbf{(D)}} \]

To find the maximum potential energy, we need an expression for the maximum curvature as a function of \( x \).

From Eq. (A),

\[ y = y_T \frac{x^2}{L^2} \sin \omega t \]

so

\[ \frac{dy}{dx} = y_T \frac{2x}{L^2} \sin \omega t \]

and

\[ \frac{d^2y}{dx^2} = \frac{2y_T}{L^2} \sin \omega t \]

therefore

\[ \left( \frac{d^2y}{dx^2} \right)_{\text{max}} = \frac{2y_T}{L^2} \quad \text{\textbf{(E)}} \]

This is independent of \( x \) in this particular case, the curvature being constant along the beam. The standard expression for the potential energy in a beam due to bending is

\[ U = \frac{1}{2} \int_0^L EI \left( \frac{d^2y}{dx^2} \right)^2 \cdot dx \quad \text{\textbf{(F)}} \]
The maximum potential energy $A_{max}$ is given by substituting Eq. (E) into Eq. (F), giving

$$U_{max} = \frac{2EI \cdot y_1^2}{L^3}$$

(G)

Now the basis of the Raleigh method is that $T_{max} = A_{max}$. Thus, from Eqs (D) and (G) we have

$$\frac{1}{10} mL \omega_1^2 y_1^2 = \frac{2EI \cdot y_1^2}{L^3},$$

which simplifies to

$$\omega_1 = \sqrt{\frac{20}{L^2}} \sqrt{\frac{EI}{m}} = 4.47 \sqrt{\frac{EI}{m}}$$

(H)

The exact answer, from Chapter 8, is $3.52 \sqrt{\frac{EI}{m}}$, so the Raleigh method is somewhat inaccurate in this case. This was due to a poor choice of function for the assumed mode shape.

1.5.2 The Principle of Virtual Work

This states that in any system in equilibrium, the total work done by all the forces acting at one instant in time, when a small virtual displacement is applied to one of its freedoms, is equal to zero. The system being ‘in equilibrium’ does not necessarily mean that it is static, or that all forces are zero; it simply means that all forces are accounted for, and are in balance. Although the same result can sometimes be obtained by diligent application of Newton’s second law, and D’Alembert’s principle, the virtual work method is a useful time-saver, and less prone to errors, in the case of more complicated systems. The method is illustrated by the following examples.

Example 1.7

The two gear wheels shown in Fig. 1.11 have mass moments of inertia $I_1$ and $I_2$. A clockwise moment, $M$, is applied to the left gear about pivot $A$. Find the equivalent mass moment of the whole system as seen at pivot $A$.

Solution

Taking all quantities as positive when clockwise, let the applied moment, $M$, produce positive angular acceleration, $+\ddot{\theta}$, of the left gear. Then the counter-clockwise acceleration of the right gear must be $-(R/r)\ddot{\theta}$. By D’Alembert’s principle, the corresponding inertia moments are

Left gear: $-I_1 \ddot{\theta}$

Right gear: $+I_2 \frac{R}{r} \ddot{\theta}$
A virtual angular displacement $\delta \theta$ is now applied to the left gear. This will also produce a virtual angular displacement $-\left(\frac{R}{r}\delta \theta\right)$ of the right gear. Now multiplying all moments acting by their corresponding virtual displacements, summing and equating to zero:

$$M \cdot \delta \theta - I_1 \ddot{\theta} \cdot \delta \theta - \frac{R}{r} \delta \theta \cdot I_2 \frac{R}{r} \ddot{\theta} = 0$$  \hspace{1cm} (A)

or

$$M = \left[ I_1 + \left(\frac{R}{r}\right)^2 I_2 \right] \ddot{\theta}$$  \hspace{1cm} (B)

This is the equation of motion of the system, referred to point $A$. The effective mass moment at this point is $I_1 + (R/r)^2 I_2$.

In Example 1.7, the mass moment of inertia, $I_2$, was scaled by $(R/r)^2$, i.e. by the square of the velocity ratio. This result can easily be generalized for any linear devices, such as masses, springs and dampers, connected via a mechanical velocity ratio. It does not work for non-linear devices, however, as shown in the following example.

**Example 1.8**

Figure 1.12 shows part of an aircraft landing gear system, consisting of a lever and a double-acting square-law damper strut. The latter produces a force $\bar{F}$, given by $\bar{F} = C\dot{z}^2 \text{sgn}(\dot{z})$, where $\dot{z}$ is the closure velocity of the strut, and $C$ is a constant. The expression ‘$\text{sgn}(\dot{z})$’, meaning ‘sign of $(\dot{z})$’, is simply a way to allow the force $\bar{F}$ to change sign so that it always opposes the direction of the velocity. Thus, upward velocity, $\dot{z}$, at point $A$, produces a downward force $\bar{F}$ on the lever, and vice versa.
If $F$ is an external force applied at point $B$, and $\dot{y}$ is the velocity at that point, find an expression for the equivalent damper, as seen at point $B$, in terms of $F$ and $\dot{y}$.

**Solution**

Defining the forces and displacements at $A$ and $B$ as positive when upwards, let a virtual displacement $+\delta y$ be applied at $B$. Then the virtual displacement at $A$ is $+(r/R)\delta y$. The forces acting on the lever are $+F$ at $B$, and $-C((r/R)\dot{y})^2$ at $A$.

Multiplying the forces acting by the virtual displacements at corresponding points, and summing to zero, we have

$$\delta y \cdot F + \frac{r}{R} \delta y \left[-C \left(\frac{r}{R} \dot{y}\right)^2\right] = 0$$

or

$$F = \left(\frac{r}{R}\right)^3 C \dot{y}^2.$$  \hspace{1cm} (A)

So the effective square-law damping coefficient at point $B$ is the actual value $C$, multiplied by the cube of the velocity ratio $(r/R) = (\dot{z}/\dot{y})$, a useful result when dealing with square-law dampers.

---

1.5.3 Lagrange’s Equations

Lagrange’s equations were published in 1788, and remain the most useful and widely used energy-based method to this day, especially when expressed in matrix form. Their derivation is given in many standard texts [1.1, 1.2], and will not be repeated here.

The equations can appear in a number of different forms. The basic form, for a system without damping, is
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i
\]

where

- \( T \) is the total kinetic energy in the system;
- \( U \) is the total potential energy in the system;
- \( q_i \) are generalized displacements, as discussed in Section 1.2. These must meet certain requirements, essentially that their number must be equal to the number of degrees of freedom in the system; that they must be capable of describing the possible motion of the system; and they must not be linearly dependent.
- \( Q_i \) are generalized external forces, corresponding to the generalized displacements \( q_i \). They can be defined as those forces which, when multiplied by the generalized displacements, correctly represent the work done by the actual external forces on the actual displacements.

For most structures, unless they are rotating, the kinetic energy, \( T \), depends upon the generalized velocities, \( \dot{q}_i \), but not upon the generalized displacements \( q_i \), and the term \( \frac{\partial T}{\partial q_i} \) can often be omitted.

Usually, the damping terms are added to a structural model after it has been transformed into normal coordinates, and do not need to appear in Lagrange’s equations. However, if the viscous damping terms can be defined in terms of the generalized coordinates, a dissipation function \( D \) can be introduced into Lagrange’s equations which, when partially differentiated with respect to the \( \dot{q}_i \) terms, produces appropriate terms in the final equations of motion. Thus, just as we have \( T = \frac{1}{2} m \dot{x}^2 \) for a single mass, and \( U = \frac{1}{2} k x^2 \) for a single spring, we can invent the function \( D = \frac{1}{2} c \dot{x}^2 \) for a single damper \( c \), where \( c \) is defined by Eq. (1.34), i.e. \( F = c \dot{x} \). Lagrange’s equations, with this addition, become

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i, \quad i = 1, 2, 3, \ldots, n.
\]

Lagrange’s equations, without damping terms, may sometimes be seen written in the form:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i
\]

where the kinetic potential, \( L \) is defined as:

\[
L = T - U
\]

By substituting Eq. (1.38) into Eq. (1.37), the latter can be seen to be exactly the same as Eq. (1.35), since \( \frac{\partial U}{\partial \dot{q}_i} = 0 \).

Many examples showing the practical application of Lagrange’s equations appear in Chapters 2, 6 and 8.
1.6 Linear and non-linear systems

The linearity of a mechanical system depends upon the linearity of its components, which are mass, stiffness and damping. Linearity implies that the response of each of these components (i.e., acceleration, displacement, and velocity, respectively) bears a straight line or linear relationship to an applied force. The straight line relationship must extend over the whole range of movement of the component, negative as well as positive.

Nearly all the mathematical operations used in structural dynamics, such as superposition, Fourier analysis, inversion, eigensolutions, etc., rely on linearity, and although analytic solutions exist for a few non-linear equations, the only solution process always available for any seriously non-linear system is step-by-step solution in the time domain.

Fortunately, with the possible exception of damping, significant non-linearity in vibrating structures, as they are encountered in industry today, is actually quite rare. This may seem a strange statement to make, in view of the large number of research papers produced annually on the subject, but this is probably due to the fact that it is an interesting and relatively undeveloped field, with plenty of scope for original research, rather than to any great need from industry. In practical engineering work, it is quite likely that if very non-linear structural behaviour is found, it may well be indicative of poor design. It may then be better to look for, and eliminate, the causes of the non-linearity, such as backlash and friction, rather than to spend time modelling the non-linear problem.

1.7 Systems of units

In dynamics work, the four main dimensions used are mass, force, length and time, and each of these is expressed in units. The unit of time has always been taken as the second (although frequencies in the aircraft industry were still being quoted in cycles/minute in the 1950s), but the unit of length can be feet, meters, centimeters and so on. The greatest cause of confusion, however, is the fact that the same unit (usually the pound or the kilogram) has traditionally been used for both mass and force. Any system of units that satisfies Newton's second law, Eq. (1.17), will work in practice, with the further proviso that the units of displacement, velocity and acceleration must all be based on the same units for length and time.

The pound (or the kilogram) cannot be taken as the unit of force and mass in the same system, since we know that a one pound mass acted upon by a one pound force produces an acceleration of \( g \), the standard acceleration due to gravity assumed in defining the units, not of unity, as required by Eq. (1.17). This problem is used to be solved in engineering work (but not in physics) by writing \( W/g \) for the mass, \( m \), where \( W \) is the weight of the mass under standard gravity. The same units, whether pounds or kilograms, could then, in effect, be used for both force and mass. Later practice was to choose a different unit for either mass or force, and incorporate the factor \( g \) into one of the units, as will be seen below. The very latest system, the SI, is, in theory, independent of earth gravity. The kilogram (as a mass unit), the meter, and the second are fixed, more or less arbitrarily, and the force unit, the newton, is then what follows from Newton's second law.
1.7.1 Absolute and Gravitational Systems

The fact that the pound, kilogram, etc. can be taken as either a mass or a force has given rise to two groups of systems of units. Those in which mass is taken as the fundamental quantity, and force is inferred, are known as absolute systems. Some examples are the following:

(a) the cogs system, an early metric system, used in physics, but now obsolete. Here the gram was the unit of mass, and acceleration was measured in centimeters/s\(^2\). The resulting force unit was called the ‘dyne’.

(b) an obsolete system based on British units, also used in physics, where the pound was taken as the unit of mass, acceleration was in ft/s\(^2\), and the resulting force unit was called the pounded.

(c) the current SI system (from System International daunts), in which the mass unit is the kilogram, acceleration is in m/s\(^2\), and the force unit is the newton.

The other group, gravitational systems, take force as the fundamental quantity, and infer mass. The acceleration due to gravity, even at the earth’s surface, varies slightly, so a standard value has to be assumed. This is nominally the value at sea level at latitude 45\(^\circ\) north, and is taken as 9.806 65 m/s\(^2\), or about 32.1740 ft/s\(^2\). In practice, the rounded values of 9.81 m/s\(^2\) or 32.2 ft/s\(^2\) will do for most purposes. Some gravitational systems are the following:

(a) A system using the pound force (lbf) as the fundamental quantity, with acceleration in ft/s\(^2\). The unit of mass is given by Eq. (1.17) as \( m = F/x \). Since a unit of \( F \) is one lbf, and a unit of \( x \) is one (ft/s\(^2\)), then a unit of \( m \) is one (lbf/ft/s\(^2\)), more correctly expressed as one (lbf ft\(^{-1}\)s\(^2\)). This is known as the ‘slug’. Its weight, \( W \), the force that would be exerted on it by standard gravity, is equal to the value of \( F \) given by Eq. (1.17), with \( m = 1 \) and \( x = g \), the acceleration due to gravity, about 32.2 ft/s\(^2\). Therefore, \( W = F = mg = (1 \times 32.2) = 32.2 \) lbf.

(b) A system, as above, except that the inch is used instead of the foot. The mass unit is one (lbf in.\(^{-1}\) s\(^2\)) sometimes known as the ‘mug’. This has a weight of about 386 lbf, as can be confirmed by applying the method used in (a) above.

(c) A system, formerly used in some European countries, taking the kilogram as a unit of force (kef), and measuring acceleration in m/s\(^2\). Using the same arguments as above, the unit of mass, the kilopond (kip), is seen to be one (kef m\(^{-1}\) s\(^2\)), and has a weight of 9.81 kef. In this system the kilogram-force (kef) is very close to one decanewton (daN) in the SI system, a fact sometimes used (confusingly) when presenting data.

Table 1.3 summarizes the six systems of units mentioned above. Some are now completely obsolete, but are included here not only because they can be found in old books, and are of historical interest, but also because they help to illustrate the principles involved.

Although the SI system will eventually replace all others, some engineering work still appears to be being carried out in the two British gravitational systems, and many engineers will have to use both SI and British units for some time to come. Also, of course, archived material, in obsolete units, may be kept for many years. The best way to deal with this situation is to be familiar with, and able to work in, all systems. The examples in this book therefore use both SI units and the British units still in use.
<table>
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<th>Table 1.3</th>
<th>Some Current and Obsolete Systems of Units</th>
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<td>second (s)</td>
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<tr>
<td>Acceleration due to gravity (approx)</td>
<td>9.807 m/s$^2$</td>
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</table>
To summarize, the rules, when working in an unfamiliar system, are simply the following:

(a) The mass and acceleration units must satisfy Newton’s second law, Eq. (1.17); that is, one unit of force acting on a unit mass must produce one unit of acceleration.

(b) Acceleration, velocity and displacement must all use the same basic unit for length. This may seem obvious, but care is needed when the common practice of expressing acceleration in g units is used.

1.7.2 Conversion between Systems

When converting between SI units and British units, it is helpful to remember that one inch = 25.4 mm exactly. Therefore, if the number of inches in a meter is required, it can be remembered as (1000/25.4) inches exactly. Similarly, a foot expressed in meters is (12 × 25.4)/1000 m. Unfortunately, there is no easy way with force and mass, and we just have to know that one pound force (lbf) = 4.448 22 N, and one pound mass = 0.453 592 k.

Example 1.6

(a) The actual force output of a vibration exciter is being measured by using it to shake a rigid mass of 2 kg, to which an accurately calibrated accelerometer is attached. The exciter specification states that when supplied with a certain sinusoidal current, it should produce a force output of ±100 N. When this current is actually supplied, the accelerometer reads ±5.07g. Calculate the actual force output of the exciter. The stiffness of the mass suspension, and gravity, may be ignored.

(b) Repeat the calculation using the British pound–inch–second system.

Solution

(a) It is clear from the data given that the SI system is being used. The required force output is given by Eq. (1.17):

\[ F = m\ddot{x} \]

We note that in this equation, the force, \( F \), must be in newtons, the mass, \( m \), must be in kilograms, and the acceleration \( \ddot{x} \), in m/s\(^2\). The acceleration, given as ±5.07g, must therefore first be converted to m/s\(^2\) by multiplying by 9.807, giving ±(5.07 × 9.807) = ±49.72 m/s\(^2\). Then the force output of the exciter in newtons is \( F = m\ddot{x} = ±(2 × 49.72) = ±99.4 \) N, slightly less than the nominal value of ±100 N.

(b) In the stated British gravitational system:

The test mass = 2 kg = (2/0.4536) = 4.409 lb. wt = (4.410/386.1) = 0.01142 lb in.\(^{-1}\).s.

The measured acceleration = ± 5.07g = ± (5.07 × 386.1) in./s\(^2\) = ± 1958 in./s\(^2\).

The required force output in lbf. is, as before, given by Eq. (1.17):

\[ F = m\ddot{x} = ±(0.01142 × 1958) = ±22.36 \text{ lbf} \]
Converting back to SI units: \( \pm 22.36 \text{ ibf} = \pm (22.36 \times 4.448) \text{ N} = \pm 99.4 \text{ N} \), agreeing with the original calculation in SI units. The conversion factors used in this case were accurate to four figures in order to achieve three figure accuracy overall.

1.7.3 The SI System

The main features of the SI system have already been described, but the following points should also be noted.

Prefixes are used before units to avoid very large or very small numbers. The preferred prefixes likely to be used in structural dynamics work are given in Table 1.4.

Prefixes indicating multiples and sub-multiples other than in steps of \( 10^3 \) are frowned upon officially. The prefixes shown in Table 1.5 should therefore be used only when the preferred prefixes are inconvenient.

It should be noted that the centimeter, cm, is not a preferred SI unit.

Only one prefix should be used with any one unit; for example, 1000 kg should not be written as 1 kkg, but 1 Mg is acceptable. When a unit is raised to a power, the power applies to the whole unit, including the prefix; for example, \( \text{mm}^3 \) is taken as \( (10^{-3} \text{ m})^3 = 10^{-9} \text{ m}^3 \), not \( 10^{-3} \text{ m}^3 \).

The SI system, unlike some other systems, has units for pressure or stress. The pascal (Pa) is defined as one N m\(^{-2}\). Young’s modulus, \( E \), may therefore be found specified in pascals, for example a steel sample may have \( E = 200 \text{ GPa} \) (200 giga-pascals), equal to \( (200 \times 10^9) \text{ N m}^{-2} \). The pascal may also be encountered in defining sound pressure levels.

### Table 1.4

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<tr>
<th>Prefix</th>
<th>Symbol</th>
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<tbody>
<tr>
<td>giga</td>
<td>G</td>
<td>( 10^9 )</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>( 10^6 )</td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>( 10^3 )</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>( 10^{-3} )</td>
</tr>
<tr>
<td>micro</td>
<td>( \mu )</td>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>

### Table 1.5

<table>
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<th>Prefix</th>
<th>Symbol</th>
<th>Factor</th>
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</thead>
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<tr>
<td>hecto</td>
<td>h</td>
<td>( 10^2 )</td>
</tr>
<tr>
<td>deca</td>
<td>da</td>
<td>10</td>
</tr>
<tr>
<td>deci</td>
<td>d</td>
<td>( 10^{-1} )</td>
</tr>
<tr>
<td>centi</td>
<td>c</td>
<td>( 10^{-2} )</td>
</tr>
</tbody>
</table>
The bar (b), equal to $10^5$ Pa, is sometimes used for fluid pressure. This is equivalent to 14.503 lbf/in$^2$, roughly one atmosphere.

Units generally use lower case letters, e.g. m for meter, including those named after people, when spelt out in full, e.g. newton, ampere, watt, but the latter use upper case letters when abbreviated, i.e. N for newtons, A for amperes, W for watts.

A major advantage of the SI system is that the units for all forms of energy and power are the same. Thus, one newton-meter (N m) of energy is equal to one joule (J), and one J/s is a power of one watt. All other systems require special factors to relate, say, mechanical energy to heat or electrical energy.

References

**Chapter No: 1**

<table>
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<th>Contents</th>
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</tr>
<tr>
<td>AU2</td>
<td>“The maximum kinetic energy, $T_{ax}$” has been changed to “The maximum kinetic energy, $T_{\text{max}}$“. Is this OK?</td>
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